Stability of thin-shell wormholes supported by ordinary matter in Einstein-Maxwell-Gauss-Bonnet gravity

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Abstract

Recently in (Phys. Rev. D 76, 087502 (2007) and Phys. Rev. D 77, 089903(E) (2008)) a thin-shell wormhole has been introduced in 5-dimensional Einstein-Maxwell-Gauss-Bonnet (EMGB) gravity which was supported by ordinary matter. We wish to consider this solution and investigate its stability. Our analysis shows that for the Gauss-Bonnet (GB) parameter $\alpha < 0$, stability regions form for a narrow band of finely-tuned mass and charge. For the case $\alpha > 0$, we iterate once more that no stable, ordinary matter thin-shell wormhole exists.

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To construct a thin-shell wormhole by cutting and pasting we take two copies of the region $r \geq a$, ($a > r_{\min}$ to avoid singularities in the geometry of wormhole) to obtain a geodesically new manifold with a matter shell at the surface $r = a$, where the throat of the wormhole is located. The static, spherically symmetric 5D metric of space time is adapted by

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2\right) \tag{1}$$

in which $f(r)$ is a function of r to be determined appropriately. Following the Darmois-Israel formalism [1, 2], in terms of the original coordinates $x^\gamma = (t, r, \theta, \phi, \psi)$, we define $\xi^i = (\tau, \theta, \phi, \psi)$, with τ the proper time. The Gauss-Bonnet (GB) extension of the thin-shell Einstein-Maxwell (EM) theory requires further modification. For this purpose in the present study we adopt the generalized Darmois-Israel boundary conditions [3], where the surface energy momentum tensor is expressed by $S_i^j = \text{diag}(\sigma, p_\theta, p_\phi, p_\psi)$. Through this formalism, Richarte and Simeone [4] have constructed such a thin shell wormhole in Einstein-Maxwell-Gauss-Bonnet (EMGB) gravity that is supported by ordinary matter, not exotic. For the exotic matter case the stability problem for the EMGB theory has been considered before [5]. Our strategy in this note is to analyze the stability and therefore explore the reality of such a thin-shell wormhole.

In order to study the radial perturbation of the wormhole we take the throat radius, a function of the proper time, i.e., $a = a(\tau)$ (note that we use $a(\tau)$ instead of $b(\tau)$ in Ref. [4]; our other notations follow those of Ref. [4]). Based on the generalized Birkhoff theorem, for $r > a(\tau)$ the geometry will be given by (1). For a metric function $f(r)$ one finds the energy density and pressures as [4]

$$\sigma = -S_\tau^\tau = -\frac{1}{4\pi}\left[\frac{3\Delta}{a} - \frac{4\alpha}{a^3}\left(\Delta^3 - 3\left(1+\dot{a}^2\right)\Delta\right)\right], \tag{2}$$

$$S_{\hat{\theta}\hat{\theta}} = S_{\hat{\phi}\hat{\phi}} = S_{\hat{\psi}\hat{\psi}} = p = \frac{1}{4\pi}\left[\frac{2\Delta}{a} + \frac{\ell}{\Delta} - \frac{4\alpha}{a^2}\left(\ell\Delta - \frac{\ell}{\Delta}\left(1+\dot{a}^2\right) - 2\ddot{a}\Delta\right)\right], \tag{3}$$

where $\ell = \ddot{a} + f'(a)/2$ and $\Delta = \sqrt{f(a) + \dot{a}^2}$ in which

$$f(a) = 1 + \frac{a^2}{4\alpha}\left(1 - \sqrt{1 + \frac{8\alpha}{a^4}\left(\frac{2M}{\pi} - \frac{Q^2}{3a^2}\right)}\right). \tag{4}$$

We note that in our notation a 'dot' denotes derivative with respect to the proper time τ and a 'prime' implies differentiation with respect to the argument of the function. For simplicity, we set the cosmological constant to zero. By a simple substitution one can show that, the conservation equation

$$\frac{d}{d\tau}\left(\sigma a^3\right)+p\frac{d}{d\tau}\left(a^3\right)=0. \tag{5}$$

is satisfied. The static configuration of radius a_0 reads

$$\sigma_0 = -\frac{\sqrt{f(a_0)}}{4\pi}\left[\frac{3}{a_0}-\frac{4\alpha}{a_0^3}\left(f(a_0)-3\right)\right], \tag{6}$$

$$p_0 = \frac{\sqrt{f(a_0)}}{4\pi}\left[\frac{2}{a_0}+\frac{f'(a_0)}{2f(a_0)}-\frac{2\alpha}{a_0^2}\frac{f'(a_0)}{f(a_0)}\left(f(a_0)-1\right)\right]. \tag{7}$$

In what follows we shall study small radial perturbations around a radius of equilibrium a_0. To this end we adapt a linear relation between p and σ as

$$p=p_0+\beta^2\left(\sigma-\sigma_0\right). \tag{8}$$

Here since we are only interested in the wormholes which are supported by ordinary matter, β^2 is the speed of sound. By virtue of the Eq.s (5) and (8) we find the energy density in the form

$$\sigma(a)=\left(\frac{\sigma_{0+}p_0}{\beta^2+1}\right)\left(\frac{a_0}{a}\right)^{3\left(\beta^2+1\right)}+\frac{\beta^2\sigma_{0-}p_0}{\beta^2+1}. \tag{9}$$

This, together with (2) lead us to the equation of motion for the radius of throat, which reads

$$-\frac{\sqrt{f(a)+\dot{a}^2}}{4\pi}\left[\frac{3}{a}-\frac{4\alpha}{a^3}\left(f(a)-3-2\dot{a}^2\right)\right]=\left(\frac{\sigma_{0+}p_0}{\beta^2+1}\right)\left(\frac{a_0}{a}\right)^{3\left(\beta^2+1\right)}+\frac{\beta^2\sigma_{0-}p_0}{\beta^2+1}. \tag{10}$$

After some manipulations this can be cast into

$$\dot{a}^2+V(a)=0, \tag{11}$$

where

$$V(a)=f(a)-\left(\left[\sqrt{A^2+B^3}-A\right]^{1/3}-\frac{B}{\left[\sqrt{A^2+B^3}-A\right]^{1/3}}\right)^2 \tag{12}$$

in which

$$A = \frac{\pi a^3}{4\alpha}\left[\left(\frac{\sigma_{0+}p_0}{\beta^2+1}\right)\left(\frac{a_0}{a}\right)^{3\left(\beta^2+1\right)}+\frac{\beta^2\sigma_{0-}p_0}{\beta^2+1}\right], \tag{13}$$

$$B = \frac{a^2}{8\alpha}+\frac{1-f(a)}{2}. \tag{14}$$

We notice that $V(a)$, and more tediously $V'(a)$, both vanish at $a = a_0$. The stability requirement for equilibrium reduces therefore to the determination of $V''(a_0) > 0$. Of course, $V(a)$ is complicated enough for an immediate analytical result. For this reason we shall proceed through numerical calculation to see whether stability regions/ islands develop or not. Since the hopes for obtaining thin-shell wormholes with ordinary matter when $\alpha > 0$, have already been dashed [4], we shall investigate only the case for $\alpha < 0$.

In order to analyze the behavior of $V(a)$ (and its double derivative) we introduce new parameterization as follows

$$\tilde{a}^2 = -\frac{a^2}{\alpha},\ m = -\frac{16M}{\pi\alpha},\ q^2 = \frac{8Q^2}{3\alpha^2},\ \tilde{\sigma}_0 = \sqrt{-\alpha}\sigma_0,\ p_0 = \sqrt{-\alpha}p_0 \tag{15}$$

Accordingly, our new variables $f(\tilde{a})$, $\tilde{\sigma}_0$, $\tilde{p}_0$, A and B take the forms

$$f(\tilde{a}) = 1 - \frac{\tilde{a}^2}{4} + \frac{\tilde{a}^2}{4}\sqrt{1 - \frac{m}{\tilde{a}^4} + \frac{q^2}{\tilde{a}^6}} \tag{16}$$

and

$$\tilde{\sigma}_0 = -\frac{\sqrt{f(\tilde{a}_0)}}{4\pi}\left[\frac{3}{\tilde{a}_0} + \frac{4}{\tilde{a}_0^3}\left(f(\tilde{a}_0) - 3\right)\right], \tag{17}$$

$$\tilde{p}_0 = \frac{\sqrt{f(\tilde{a}_0)}}{4\pi}\left[\frac{2}{\tilde{a}_0} + \frac{f'(\tilde{a}_0)}{2f(\tilde{a}_0)} + \frac{2}{\tilde{a}_0^2}\frac{f'(\tilde{a}_0)}{f(\tilde{a}_0)}\left(f(\tilde{a}_0) - 1\right)\right], \tag{18}$$

$$A = -\frac{\pi\tilde{a}^3}{4}\left[\left(\frac{\tilde{\sigma}_{0+\tilde{p}_0}}{\beta^2+1}\right)\left(\frac{\tilde{a}_0}{\tilde{a}}\right)^{3\left(\beta^2+1\right)} + \frac{\beta^2\tilde{\sigma}_{0-\tilde{p}_0}}{\beta^2+1}\right], \tag{19}$$

$$B = -\frac{\tilde{a}^2}{8} + \frac{1 - f(\tilde{a})}{2}. \tag{20}$$

Following this parametrization our Eq. (11) takes the form

$$\left(\frac{d\tilde{a}}{d\tau}\right)^2 + \tilde{V}(\tilde{a}) = 0, \tag{21}$$

where

$$\tilde{V}(\tilde{a}) = -\frac{V(\tilde{a})}{\alpha}. \tag{22}$$

We explore now all possible constraints on our parameters that they must satisfy.

i) Starting from the metric function we must have

$$1 - \frac{m}{\tilde{a}_0^4} + \frac{q^2}{\tilde{a}_0^6} \geq 0. \tag{23}$$

ii) In the potential, the reality condition requires also that

$$A^2 + B^3 \geq 0. \tag{24}$$

At the location of the throat this amounts to

$$\left(-\frac{\pi \tilde{a}_0^3}{4}\tilde{\sigma}_0\right)^2 + \left(-\frac{\tilde{a}_0^2}{8} + \frac{1 - f(\tilde{a}_0)}{2}\right)^3 \geq 0 \tag{25}$$

or after some manipulations it yields

$$f(\tilde{a}_0) - 2 + \frac{\tilde{a}_0^2}{2} \leq 0. \tag{26}$$

This is equivalent to

$$0 \leq 1 - \frac{m}{\tilde{a}_0^4} + \frac{q^2}{\tilde{a}_0^6} \leq \left(\frac{4}{\tilde{a}_0^2} - 1\right)^2. \tag{27}$$

iii) Our last constraint condition concerns, having a wormhole supported by ordinary matter, which means that

$$\tilde{\sigma}_0 > 0. \tag{28}$$

This implies, from (17) that

$$\left[\frac{3}{\tilde{a}_0} + \frac{4}{\tilde{a}_0^3}\left(f(\tilde{a}_0) - 3\right)\right] < 0 \tag{29}$$

or equivalently

$$0 \leq 1 - \frac{m}{\tilde{a}_0^4} + \frac{q^2}{\tilde{a}_0^6} < 4\left(\frac{4}{\tilde{a}_0^2} - 1\right)^2. \tag{30}$$

It is remarkable to observe that the foregoing constraints $(i - iii)$ on our parameters can all be expressed in a single constraint, namely

$$0 \leq 1 - \frac{m}{\tilde{a}_0^4} + \frac{q^2}{\tilde{a}_0^6} \leq \left(\frac{4}{\tilde{a}_0^2} - 1\right)^2. \tag{31}$$

We plot $\tilde{V}''(\tilde{a})$ from (12) for various fixed values of mass and charge, as a projection into the plane with coordinates β and $\tilde{a}_0$. In other words, we search and identify the regions for which $\tilde{V}''(\tilde{a}) > 0$, in 3−dimensional figures considered as a projection in the $(\beta, \tilde{a}_0)$ plane. The metric function $f(r)$ and energy density $\tilde{\sigma}_0 > 0$, behavior also are given in Fig.s 1-4. It is evident from Fig.s 1-4 that for increasing charge the stability regions shrink to smaller domains and tends ultimately to disappear completely. For smaller $\tilde{a}_0$ bound we obtain fluctuations in $\tilde{V}''(\tilde{a})$, which is smooth otherwise.

In conclusion, our numerical analysis reveals that for $\alpha < 0$, and specific ranges of mass and charge the $5-$dimensional EMGB thin-shell wormholes can be made stable against linear, radial perturbations. Let us note that in the Gauss-Bonnet extension of Einstein's theory, $\alpha > 0$ has always been the prime choice, neglecting the $\alpha < 0$ branch as less significant. Now, it becomes clear, within the realm of stable, physically realistic thin-shell wormholes, that this class ($\alpha < 0$) finds application.

Another point of interest is that the magnitude of $\alpha < 0$ is irrelevant in the foregoing stability analysis. This reflects, as for the black holes, the universality of the thin-shell wormholes which arises at each scale.

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Figure Caption:

Fig. 1: $\tilde{V}''(\tilde{a}) > 0$ region ($m = 0.5$, $q = 1.0$) for various ranges of β and $\tilde{a}_0$. The lower and upper limits of the parameters are evident in the figure. The metric function $f(\tilde{r})$ and $\tilde{\sigma}_0 > 0$, are also indicated in the smaller figures.

Fig. 2: $\tilde{V}''(\tilde{a}) > 0$ plot for $m = 1.0$, $q = 1.5$. The stability region is seen clearly to shrink with the increasing charge. This effect reflects also to the $\tilde{\sigma}_0 > 0$, behavior.

Fig. 3: The stability region for $m = 1.0$, $q = 2.0$, is seen to shift outward and get smaller.

Fig. 4: For fixed mass $m = 1.0$ but increased charge $q = 2.5$ it is clearly seen that the stability region and the associated energy density both get further reduced.