A geometric invariant measuring the deviation from Kerr data

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A geometrical invariant for regular asymptotically Euclidean data for the vacuum Einstein field equations is constructed. This invariant vanishes if and only if the data corresponds to a slice of the Kerr black hole spacetime —thus, it provides a measure of the "non-Kerrness" of generic data. In order to proceed with the construction of the geometric invariant, we introduce the notion of approximate Killing spinors.

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Introduction.— It is widely expected that the late time behaviour of a dynamical black hole spacetime will approach, in some suitable sense, the Kerr spacetime. Making sense of this expectation is one of the outstanding challenges of modern General Relativity. In particular, clarifying what it means that a spacetime is close to the Kerr spacetime is of great relevance for the problem of the non-linear stability of the Kerr spacetime and for the numerical evolution of black holes. Due to the coordinate freedom in General Relativity it is, in general, difficult to measure how much two spacetimes differ from each other. Nevertheless, invariant characterisations of spacetimes provide a way of bridging this difficulty.

Most analytical and numerical studies of the Einstein field equations make use of a 3+1 decomposition of the equations and the unknowns. Thus, it is important to have a characterisation of the Kerr solution which is amenable to this type of splitting. Most known invariant characterisations of the Kerr spacetime have problems in this or other respects. For example, the characterisation of the Kerr spacetime in terms of the so-called *Mars-Simon tensor* —see [1, 2]— requires the *a priori* existence of a Killing vector in the spacetime —thus, it is of more relevance for the problem of uniqueness of stationary black holes. An invariant characterisation in terms of concomitants of the Weyl tensor —see [3]— produces very involved expressions when performing a 3+1 split —[4]. Furthermore, the above characterisations are local by construction, and it is not clear how they could be used to produce a global characterisation of initial data sets. In this letter we discuss an alternative characterisation of the Kerr spacetime and show how it can be used to obtain a global geometrical invariant of asymptotically Euclidean slices of a spacetime. This geometric invariant has the key property of vanishing if and only if the hypersurface is a slice of the Kerr spacetime. In this sense, our invariant is analogous to the invariant characterising time symmetric slices of static spacetimes discussed in [5].

Killing spinors and Petrov type D spacetimes.— Let $(\mathcal{M}, g_{\mu\nu})$ be an orientable and time orientable globally hyperbolic spacetime. A valence-2 Killing spinor is a symmetric spinor $\kappa_{AB} = \kappa_{(AB)}$ satisfying the equation

$$\nabla_{A'(A}\kappa_{BC)} = 0, \qquad (1)$$

where $\nabla_{AA'}$ denotes the spinorial counterpart of the Levi-Civita connection of the metric $g_{\mu\nu}$. Here, and in what follows, A, B, $\cdots$ denote abstract spinorial indices, while $\mathbf{A}$, $\mathbf{B}$, $\cdots$ denote indices with respect to a specific frame. The spinorial conventions of [6] are used. Killing spinors offer a way of relating properties of the curvature with properties of the symmetries of the spacetime. Given a Killing spinor κ_{AB}, the concomitant $\xi_{AA'} = \nabla^{B}{}_{A'}\kappa_{AB}$ is a complex Killing vector of the spacetime.

We note a local characterisation of the Kerr spacetime in terms of valence-2 Killing spinors based on the following results: (i) a vacuum spacetime admits a valence-2 Killing spinor, if and only if it is of Petrov type D, N or O [7, 8]; (ii) the Killing vector $\xi_{AA'}$ is real only in the case of the Kerr-NUT spacetime [9, 10]; (iii) The Petrov type of Kerr is always D —there are no points where it degenerates to N or O [1, 2]. Let Ψ_{ABCD} denote the Weyl spinor of the spacetime. One has:

Theorem 1.—*Let $\mathcal{N}$ be an open subset of $(\mathcal{M}, g_{\mu\nu})$ where $\Psi_{ABCD} \neq 0$ and $\Psi_{ABCD}\Psi^{ABCD} \neq 0$. Then $\mathcal{N}$ is a portion of the Kerr-NUT spacetime if and only if there exists a Killing spinor in $\mathcal{N}$ such that the associated Killing vector is real.*

Asymptotically Euclidean slices.— Let $(\mathcal{S}, h_{ab}, K_{ab})$ denote a smooth initial data set for the vacuum Einstein field equations —that is, (h_{ab}, K_{ab}) satisfy the vacuum constraint equations on $\mathcal{S}$. In what follows, the 3-manifold $\mathcal{S}$ will be assumed to be asymptotically Euclidean with two asymptotic ends, i_1, i_2. An asymptotic end is an open set diffeomorphic to the complement of an open ball in $\mathbb{R}^3$. The fall off conditions of the various fields will be expressed in terms of weighted Sobolev spaces H^s_β, where s is a non-negative integer and β is a real number. We say that $\eta \in H^\infty_\beta$ if $\eta \in H^s_\beta$ for all s. In what follows we use the theory for these spaces developed in [11] written in the conventions of [12]. Thus, the functions in H^∞_β are smooth over $\mathcal{S}$ and have a fall off at infinity such that $\partial^l \eta = o(r^{\beta-|l|})$. We will often write $\eta = o_\infty(r^\beta)$ for $\eta \in H^\infty_\beta$ at an asymptotic end.

We assume that on each end it is possible to introduce asymptotically Cartesian coordinates $x^i_{(k)}$, $k = 1, 2$, with $r = ((x^1_{(k)})^2 + (x^2_{(k)})^2 + (x^3_{(k)})^2)^{1/2}$, such that the intrinsic metric and extrinsic curvature of $\mathcal{S}$ satisfy

$$h_{ij} = -\left(1 + 2m_{(k)} r^{-1}\right)\delta_{ij} + o_\infty(r^{-3/2}), \qquad (2)$$

$$K_{ij} = o_\infty(r^{-5/2}), \qquad (3)$$

where i, j are coordinate indices —in contrast to a, b which are taken to be abstract ones. In view of the mass positivity theorem [13, 14], we assume that $m_{(k)} > 0$. For simplicity we have excluded from our analysis boosted slices —this will be discussed elsewhere. Note, however, that the slices considered allow a non-vanishing ADM angular momentum.

Killing spinor initial data.— A set of necessary and sufficient conditions for the development $(\mathcal{M}, g_{\mu\nu})$ of the data $(\mathcal{S}, h_{ab}, K_{ab})$ to be endowed with a Killing spinor was obtained in [8]. Let $\tau_{AA'}$ be the spinor counterpart of the normal to $\mathcal{S}$, with normalisation given by $\tau_{AA'}\tau^{AA'} = 2$. The spinor $\tau_{AA'}$ allows to introduce a space spinor formalism —see e.g. [8, 15] for details. In particular, the covariant derivative $\nabla_{AA'}$ can be split according to $\nabla_{AA'} = \frac{1}{2}\tau_{AA'}\nabla - \tau^Q{}_{A'}\nabla_{AQ}$, where $\nabla \equiv \tau^{AA'}\nabla_{AA'}$ and $\nabla_{AB} \equiv \tau_{(A}{}^{A'}\nabla_{B)A'}$ is the Sen connection. The Sen connection is not intrinsic to the hypersurface $\mathcal{S}$, however, it can be expressed in terms of the spinorial Levi-Civita connection of h_{ab}, D_{AB}, and of the spinorial counterpart of K_{ab}, $K_{ABCD} = K_{(AB)(CD)} = K_{CDAB}$. One has, for example, that $\nabla_{AB}\pi_C = D_{AB}\pi_C + \frac{1}{2}K_{ABC}{}^D\pi_D$. Given a spinor π_A, we define its Hermitian conjugate via $\hat{\pi}_A \equiv \tau_A{}^{E'}\bar{\pi}_{E'}$. The Hermitian conjugate can be extended to higher valence symmetric spinors in the obvious way. The spinors ν_{AB} and ξ_{ABCD} are said to be real if $\hat{\nu}_{AB} = -\nu_{AB}$ and $\hat{\xi}_{ABCD} = \xi_{ABCD}$. It can be verified that $\nu_{AB}\hat{\nu}^{AB}$, $\xi_{ABCD}\hat{\xi}^{ABCD} \geq 0$. If the spinors are real, then there exist real tensors ν_a, ξ_{ab} such that ν_{AB} and ξ_{ABCD} are their spinorial counterparts. Notice that $\hat{D}_{AB} = -D_{AB}$. The Killing vector $\xi_{AA'} = \nabla^B{}_{A'}\kappa_{AB}$ can be decomposed in terms of its lapse, ξ, and shift, ξ_{AB}, according to $\xi_{AA'} = \frac{1}{2}\tau_{AA'}\xi - \tau^Q{}_{A'}\xi_{AQ}$, where

$$\xi \equiv \tau^{AA'}\xi_{AA'} = \nabla^{AB}\kappa_{AB}, \qquad (4)$$

$$\xi_{AB} \equiv \tau_{(A}{}^{A'}\xi_{B)A'} = \tfrac{3}{2}\nabla^P{}_{(A}\kappa_{B)P}. \qquad (5)$$

Some extensive computer algebra calculations carried out in the suite xAct [16] show that the conditions found in [8] for the existence of a Killing spinor in the development of $(\mathcal{S}, h_{ab}, K_{ab})$ are equivalent to:

$$\nabla_{(AB}\kappa_{CD)} = 0, \qquad (6)$$

$$\Psi_{(ABC}{}^F\kappa_{D)F} = 0, \qquad (7)$$

$$3\kappa_{(A}{}^E\nabla_B{}^F\Psi_{CD)EF} + \Psi_{(ABC}{}^F\xi_{D)F} = 0, \qquad (8)$$

where ξ_{AB} is used as a shorthand for $\frac{3}{2}\nabla^P{}_{(A}\kappa_{B)P}$. The restriction of Ψ_{ABCD} to the initial hypersurface $\mathcal{S}$ can be expressed in terms of its electric and magnetic parts as $\Psi_{ABCD} = E_{ABCD} + \mathrm{i}B_{ABCD}$, where

$$E_{ABCD} = \tfrac{1}{6}\Omega_{ABCD}K - \tfrac{1}{2}\Omega_{(AB}{}^{PQ}\Omega_{CD)PQ} - r_{(ABCD)}, \qquad (9)$$

$$B_{ABCD} = \mathrm{i}D^Q{}_{(A}K_{BCD)Q}, \qquad (10)$$

where $\Omega_{ABCD} \equiv K_{(ABCD)}$ and $K \equiv K^{AB}{}_{AB}$. The spinor r_{ABCD} is the spinorial representation of the Ricci tensor of h_{ab}. All these quantities can be computed from the initial data. From the analysis in [8] one has the following result:

Theorem 2.— *The development $(\mathcal{M}, g_{\mu\nu})$ of an initial data set for the vacuum Einstein field equations, $(\mathcal{S}, h_{ab}, K_{ab})$, has a Killing spinor if and only if there exists a symmetric spinor κ_{AB} on $\mathcal{S}$ satisfying equations* (6)-(8).

Equations (6)-(8) will be collectively referred to as the *Killing spinor initial data equations*. Equation (6) will be called the spatial Killing spinor equation whereas (7) and (8) will be known as the *algebraic conditions*. A solution to equations (6)-(8) will be called a *Killing spinor data*, while a solution to only equation (6) will be known as a *Killing spinor candidate*.

As a consequence of the characterisation of the Kerr spacetime discussed in Theorem 1, equations (6)-(8) are known to have a non-trivial solution if and only if the initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ is data for the Kerr/Schwarzschild spacetime. For Kerr initial data satisfying the asymptotic conditions (2)-(3), one can always choose asymptotically Cartesian coordinates (x^1, x^2, x^3) and orthonormal frames on the asymptotic ends such that

$$\kappa_{\mathbf{AB}} = \mp\frac{\sqrt{2}}{3}x_{\mathbf{AB}} \mp \frac{2\sqrt{2}m}{3r}x_{\mathbf{AB}} + o_\infty(r^{-1/2}), \qquad (11)$$

with

$$x_{\mathbf{AB}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -x^1 + \mathrm{i}x^2 & x^3 \\ x^3 & x^1 + \mathrm{i}x^2 \end{pmatrix}. \qquad (12)$$

Using (11) one finds that $\xi = \pm\sqrt{2} + o_\infty(r^{-1/2})$, $\xi_{\mathbf{AB}} = o_\infty(r^{-1/2})$. In other words, the Killing spinor of the Kerr spacetime gives rise to its stationary Killing vector.

Approximate Killing spinors.— Equation (6) constitutes an overdetermined condition for the 3 complex components of the spinor κ_{AB}. One would like to replace it by an equation which always has a solution. For this, one notes that the operator defined by the left hand side of equation (6) sending valence-2 symmetric spinors to valence-4 totally symmetric spinors has a formal adjoint whose action on a valence-4 totally symmetric spinor, ξ_{ABCD}, is given by $\nabla^{AB}\xi_{ABCD} - 2\Omega^{ABF}{}_{(C}\xi_{D)ABF}$. The composition of these two operators renders the equation

$$L(\kappa_{CD}) \equiv \nabla^{AB}\nabla_{(AB}\kappa_{CD)} - \Omega^{ABF}{}_{(C}\nabla_{|AB|}\kappa_{D)F} - \Omega^{ABF}{}_{(C}\nabla_{D)F}\kappa_{AB} = 0. \qquad (13)$$

A calculation reveals that the operator defined by the left hand side of this last equation is elliptic. Moreover, it can be verified that under the asymptotic conditions (2)-(3) the operator is asymptotically homogeneous [11, 17]. This is the standard assumption on the coefficients for elliptic operators on weighted Sobolev spaces. It follows that the operator is a linear bounded operator with finite dimensional Kernel and closed range [11, 18].

Clearly, any solution to the spatial Killing equation (6) is also a solution to equation (13). Equation (13) arises as the Euler-Lagrange equation of the functional

$$J = \int_{\mathcal{S}} \nabla_{(AB}\kappa_{CD)}\widehat{\nabla^{AB}\kappa^{CD}}\mathrm{d}\mu, \qquad (14)$$

where $\mathrm{d}\mu$ denotes the volume element of the metric h_{ab}. We note the following identity:

$$\int_{\mathcal{U}} \nabla^{AB}\kappa^{CD}\hat{\xi}_{ABCD}\mathrm{d}\mu - \int_{\mathcal{U}} \kappa^{AB}\widehat{\nabla^{CD}\xi_{ABCD}}\mathrm{d}\mu + \int_{\mathcal{U}} 2\kappa^{AB}\Omega^{CDF}{}_{A}\widehat{\xi_{BCDF}}\mathrm{d}\mu = \int_{\partial\mathcal{U}} n^{AB}\kappa^{CD}\hat{\xi}_{ABCD}\mathrm{d}S, \qquad (15)$$

with $\mathcal{U} \subset \mathcal{S}$, and where $\mathrm{d}S$ denotes the area element of $\partial\mathcal{U}$, n_{AB} its outward pointing normal, and ξ_{ABCD} is a symmetric spinor.

We shall call a solution, κ_{AB}, to equation (13) an *approximate Killing spinor*. If one assumes the decay $\xi = \pm\sqrt{2} + o_\infty(r^{-1/2})$ (each sign is associated to a different end), $\xi_{AB} = o_\infty(r^{-1/2})$, $\kappa_{AB} = o_\infty(r^{3/2})$, and $\nabla_{(AB}\kappa_{CD)} = o_\infty(r^{-3/2})$ at an asymptotic end, one can always obtain a solution of the form (11) at that end. This computation is lengthy and will be presented elsewhere. The solution can then be smoothly cut off so it is zero outside the asymptotic end. Repeating this for the other asymptotic end and adding the solutions yields a real spinor $\mathring{\kappa}_{AB}$ on the entire slice such that $\nabla_{(AB}\mathring{\kappa}_{CD)} \in H^\infty_{-3/2}$ with asymptotic behaviour (11) at both ends.

We write the following Ansatz for the solution to equation (13):

$$\kappa_{AB} = \mathring{\kappa}_{AB} + \theta_{AB}, \quad \theta_{AB} \in H^\infty_{-1/2}. \qquad (16)$$

One has the following result:

Theorem 3. *Given an asymptotically Euclidean initial data set* $(\mathcal{S}, h_{ab}, K_{ab})$ *satisfying the asymptotic conditions* (2) *and* (3), *there exists a smooth unique solution to equation* (13) *with asymptotic behaviour given by* (16).

Proof of Theorem 3.— Substitution of Ansatz (16) into equation (13) renders the following equation for the spinor θ_{AB}:

$$L(\theta_{CD}) = -L(\mathring{\kappa}_{CD}). \qquad (17)$$

First, it is noticed that due to elliptic regularity, any $H^2_{-1/2}$ solution to the previous equation is in fact a $H^\infty_{-1/2}$ solution, so that if θ_{AB} exists, then it must be smooth —see e.g. [11]. By construction it follows that $\nabla_{(AB}\mathring{\kappa}_{CD)} \in H^\infty_{-3/2}$, so that $F_{CD} \equiv -L(\mathring{\kappa}_{CD}) \in H^\infty_{-5/2}$.

We make use of the Fredholm alternative for weighted Sobolev spaces to discuss the existence of solutions to equation (17) —see e.g. [17, 18]. In the particular case of equation (17) there exists a unique $H^2_{-1/2}$ solution if

$$\int_{\mathcal{S}} F_{AB}\hat{\nu}^{AB}\mathrm{d}\mu = 0 \qquad (18)$$

for all ν_{AB} satisfying

$$\nu_{AB} \in H^2_{-1/2}, \quad L(\nu_{CD}) = 0. \qquad (19)$$

It will be shown in the sequel that such ν_{AB} must be trivial. Using the identity (15) with $\xi_{ABCD} = \nabla_{(AB}\nu_{CD)}$ and assuming that $L(\nu_{CD}) = 0$, one obtains

$$\int_{\mathcal{S}} \nabla^{AB}\nu^{CD}\widehat{\nabla_{(AB}\nu_{CD)}}\mathrm{d}\mu = \int_{\partial\mathcal{S}_\infty} n^{AB}\nu^{CD}\widehat{\nabla_{(AB}\nu_{CD)}}\mathrm{d}S, \qquad (20)$$

where ∂S_∞ denotes the sphere at infinity. As $\nu_{AB} \in H^2_{-1/2}$ by assumption, it follows that $\nabla_{(AB}\nu_{CD)} \in H^\infty_{-3/2}$ and furthermore that $n^{AB}\nu^{CD}\widehat{\nabla_{(AB}\nu_{CD)}} = o(r^{-2})$. An integral over a finite sphere will then be of type $o(1)$. Thus, the integral over ∂S_∞ vanishes. Consequently,

$$\int_{\mathcal{S}} \nabla^{AB}\nu^{CD}\widehat{\nabla_{(AB}\nu_{CD)}}\mathrm{d}\mu = 0. \qquad (21)$$

Therefore one concludes that $\nabla_{(AB}\nu_{CD)} = 0$. That is, ν_{AB} has to be a Killing spinor candidate. Using the methods devised in [19] to prove that there are no non-trivial Killing vectors of a 3-dimensional manifold that go to zero at infinity, one can prove that if $\nu_{AB} \in H^\infty_{-1/2}$ is a solution to the spatial Killing spinor equation (6) then $\nu_{AB} \equiv 0$ on $\mathcal{S}$. The proof of this last result relies on the fact that

$$\nabla_{AB}\nabla_{CD}\nabla_{EF}\nu_{GH} = H_{ABCDEFGH}, \qquad (22)$$

where $H_{ABCDEFGH}$ is a homogeneous expression of ν_{AB}, $\nabla_{AB}\nu_{CD}$ and $\nabla_{AB}\nabla_{CD}\nu_{EF}$ —this expression is obtained out of a lengthy computer algebra calculation. Consequently, the Kernel of equation (13) with decay in $H^2_{-1/2}$ is trivial. Accordingly, the Fredholm alternative imposes no restriction. Thus, there exists a unique solution to equation (13) with asymptotic decay given by (16). This completes the proof of Theorem 3.

The geometric invariant.— We use the functional (14) and the algebraic conditions (7) and (8) to construct the geometric invariant measuring the deviation of $(\mathcal{S}, h_{ab}, K_{ab})$ from Kerr initial data. To this end, let κ_{AB} be a solution to equation (13) as given by Theorem 3, and furthermore, let $\xi_{AB} \equiv \frac{3}{2}\nabla^P{}_{(A}\kappa_{B)P}$. Define

$$I_1 \equiv \int_{\mathcal{S}} \Psi_{(ABC}{}^F\kappa_{D)F}\hat{\Psi}^{ABC}{}_G\hat{\kappa}^{DG}\mathrm{d}\mu, \tag{23}$$

$$I_2 \equiv \int_{\mathcal{S}} \left(3\kappa_{(A}{}^E\nabla_B{}^F\Psi_{CD)EF} + \Psi_{(ABC}{}^F\xi_{D)F}\right) \times \left(3\hat{\kappa}^A{}_P\widehat{\nabla^B{}_Q\Psi^{CDPQ}} + \hat{\Psi}^{ABC}{}_P\hat{\xi}^{DP}\right)\mathrm{d}\mu. \tag{24}$$

The geometric invariant is then defined by

$$I \equiv J + I_1 + I_2. \tag{25}$$

By construction I is coordinate independent. We have that $\nabla_{(AB}\kappa_{CD)} \in H^\infty_{-3/2}$, which because of our conventions means that $\nabla_{(AB}\kappa_{CD)} \in L^2$. Consequently, $J < \infty$. From the form of the metric (2) we have $\Psi_{ABCD} \in H^\infty_{-3+\varepsilon}$, $\varepsilon > 0$. By the multiplication lemma in [11] and $\kappa_{AB} \in H^\infty_{1+\varepsilon}$ we have $\Psi_{(ABC}{}^F\kappa_{D)F} \in H^\infty_{-3/2}$. Thus, again one finds that $I_1 < \infty$. A similar argument shows $I_2 < \infty$. Hence, the invariant (25) is finite and well defined. Clearly $I \geq 0$.

Due to our smoothness assumptions, if $I = 0$ it follows that equations (6)-(8) are satisfied on the whole of $\mathcal{S}$. Thus, the development of $(\mathcal{S}, h_{ab}, K_{ab})$ is, at least in a slab, of Petrov type D, N or O. The types N and O can be excluded by requiring $\Psi_{ABCD} \neq 0$, $\Psi_{ABCD}\Psi^{ABCD} \neq 0$ everywhere on $\mathcal{S}$. Finally, if $I = 0$ one has that the pair (ξ, ξ_{AB}) gives rise to a (possibly complex) spacetime Killing vector, $\xi_{AA'}$. As a consequence of our decay assumptions, $\xi - \hat{\xi} = o_\infty(r^{-1/2})$ and $\xi_{AB} + \hat{\xi}_{AB} = o_\infty(r^{-1/2})$, corresponding to the imaginary part of the Killing data (ξ, ξ_{AB}), give rise to a Killing vector that goes to zero at infinity. However, there are no non-trivial Killing vectors of this type [19, 20]. Thus, $\xi_{AA'}$, is a real Killing vector. Hence, one has our main result:

Theorem 4.— *Let $(\mathcal{S}, h_{ab}, K_{ab})$ be an asymptotically Euclidean initial data set for the Einstein vacuum field equations satisfying in every asymptotic end the decay conditions* (2) *and* (3)*, and such that $\Psi_{ABCD} \neq 0$ and $\Psi_{ABCD}\Psi^{ABCD} \neq 0$ everywhere on $\mathcal{S}$. Let I be the invariant defined by equations* (14)*,* (23)*,* (24) *and* (25)*, where κ_{AB} is given as the only solution to equation* (13) *with asymptotic behaviour given by* (16)*. The invariant I vanishes if and only if $(\mathcal{S}, h_{ab}, K_{ab})$ is an initial data set for the Kerr spacetime.*

Applications and generalisations.— Given the invariant of theorem 4, a natural question to be asked is how it behaves under time evolution. Addressing this question requires an analysis of the spinor $\nabla\kappa_{AB}$, which can be seen to satisfy an elliptic equation of the form (13). In this letter we have restricted our attention to asymptotically Euclidean slices, however, a similar analysis can be carried out on hyperboloidal and asymptotically cylindrical slices. If some type of constancy or monotonicity property could be established, this would be a useful tool for studying non-linear stability of the Kerr spacetime and also in the numerical evolutions of black hole spacetimes. For example, it could be the case that the invariant I remains constant along the leaves of a foliation of asymptotically Euclidean slices, while monotonicity holds only if one considers a foliation intersecting null infinity —like in the case of the ADM and Bondi masses.

The decay and regularity assumptions used are certainly not optimal —we have used these for the ease of the presentation. Full arguments and generalisations, including the discussion of boosted slices will be discussed elsewhere.

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[1] M. Mars, Class. Quantum Grav. **16**, 2507 (1999).
[2] M. Mars, Class. Quantum Grav. **17**, 3353 (2000).
[3] J. J. Ferrando and J. A. Sáez, Class. Quantum Grav. **26**, 075013 (2009).
[4] A. García-Parrado and J. A. Valiente Kroon, In preparation (2010).
[5] S. Dain, Phys. Rev. Lett. **93**, 231101 (2004).
[6] R. Penrose and W. Rindler, *Spinors and space-time. Volume 1. Two-spinor calculus and relativistic fields* (Cambridge University Press, 1984).
[7] B. P. Jeffryes, Proc. Roy. Soc. Lond. A **392**, 323 (1984).
[8] A. García-Parrado and J. A. Valiente Kroon, J. Geom. Phys. **58**, 1186 (2008).
[9] R. Penrose and W. Rindler, *Spinors and space-time. Volume 2. Spinor and twistor methods in space-time geometry* (Cambridge University Press, 1986).
[10] J. J. Ferrando and J. A. Sáez, J. Math. Phys. **48**, 102504 (2007).
[11] Y. Choquet-Bruhat and D. Christodolou, Acta Mathematica **146**, 129 (1981).
[12] R. Bartnik, Comm. Pure Appl. Math. p. 661 (1986).
[13] R. Schoen and S. T. Yau, Comm. Math. Phys. **65**, 45 (1979).
[14] R. Schoen and S. T. Yau, Comm. Math. Phys. **79**, 231 (1981).
[15] P. Sommers, J. Math. Phys. **21**, 2567 (1980).
[16] J. M. Martín-García, http://metric.iem.csic.es/Martin-Garcia/xAct/.
[17] M. Cantor, Bull. Am. Math. Soc. **5**, 235 (1981).
[18] R. B. Lockhart, Duke Math. Journal **48**, 289 (1981).
[19] D. Christodoulou and N. O'Murchadha, Comm. Math. Phys. **80**, 271 (1981).
[20] R. Beig and P. T. Chruściel, J. Math. Phys. **37**, 1939 (1996).