A causality analysis of the linearized relativistic Navier-Stokes equations

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Abstract

It is shown by means of a simple analysis that the linearized system of transport equations for a relativistic, single component ideal gas at rest obeys the *antecedence principle*, which is often referred to as causality principle. This task is accomplished by examining the roots of the dispersion relation for such a system. This result is important for recent experiments performed in relativistic heavy ion colliders, since it suggests that the Israel-Stewart like formalisms may be unnecessary in order to describe relativistic fluids.

1 Introduction

In 1940 C. Eckart published three papers entitled *The thermodynamics of irreversible processes* [1], the third one addressing the problem of a relativistic simple (single component ideal gas) fluid. In that paper, Eckart proceeded following the basic ideas of classical irreversible thermodynamics [2], except for the fact that he introduced relativistic terms in the energy-momentum tensor. As part of his phenomenological approach, he proposed constitutive equations with relativistic corrections. Since Eckart's theory apparently leads to results that violate causality and involves undesirable unstable modes [3], it has been patched up in several ways using formalisms introduced by Israel and coworkers [4] [5] [6] and sometimes using extended irreversible thermodynamics [7] [8]. Recently, it has been shown that the unphysical behavior of the unstable modes is due to the coupling between heat an acceleration proposed by Eckart [9]. Indeed, it has been shown that such a relation is not sustained by kinetic theory [10].

The so-called *causality problem* of heat conduction, which should be more precisely stated as *antecedence problem*, remains still a controversial issue which suggests the need of extended theories. We wish to point out that eventhough the term causality is the most favoured, the problem of faster that light propagation of fluctuations is not striclty a cause-effect issue but rather the prediction of an unphysical behavior concerning arrival times. However, as is shown in the following sections, relativistic classical linear irreversible thermodynamics, as obtained from relativistic kinetic theory features both stability for the equilibrium state [10] and satisfaction of the antecedence principle.

To accomplish this task we divide the rest of the paper as follows. In Sect. 1.2 we recall the Navier-Stokes equations for a simple relativistic fluid [11] and introduce the appropriate constitutive equation for the heat flux. The linearized set of transport equations is thoroughly analyzed in Sect. 1.3. Conclusions and final remarks are included in Sect. 1.4.

2 Transport equations for the relativistic single component ideal gas

The starting point are the balance equations for a relativistic fluid which are obtained from the conservation of the particle density flow

$$N^{\nu} = nu^{\nu} \tag{1}$$

and the energy-momentum tensor which, following Eckart[1] reads

$$T^{\mu}_{\nu} = \frac{n\varepsilon}{c^2} u^{\mu} u_{\nu} + p h^{\mu}_{\nu} + \pi^{\mu}_{\nu} + \frac{1}{c^2} q^{\mu} u_{\nu} + \frac{1}{c^2} u^{\mu} q_{\nu} \tag{2}$$

In Eqs. (1) and (2), n is the particle number density, u^{ν} the hydrodynamic velocity four vector, c the speed of light, p the hydrostatic pressure and $h^{\mu}_{\nu} = \delta^{\mu}_{\nu} + u^{\mu} u_{\nu}/c^2$ the spatial projector. The internal energy per particle, ε , includes the rest energy since it is given by [14]

$$\varepsilon = mc^2 \left(3z + \frac{\mathcal{K}_1\left(\frac{1}{z}\right)}{\mathcal{K}_2\left(\frac{1}{z}\right)} \right) \sim mc^2 + \frac{3}{2} kT + ... \tag{3}$$

where $z = \frac{kT}{mc^2}$ is the relativistic parameter and $\mathcal{K}_n\left(\frac{1}{z}\right)$ are the modified Bessel function of the second kind. The dissipative fluxes are the Navier tensor π^{μ}_{ν} and the heat flux q^{ν} . The conservation equations $N^{\nu}_{;\nu} = 0$ and $T^{\mu}_{\nu;\mu} = 0$ for the quantities defined above yield the Navier-Stokes equations for the relativistic simple fluid namely,

$$\dot{n} + n\theta = 0 \tag{4}$$

$$\begin{aligned} &\left(\frac{n\varepsilon}{c^2} + \frac{p}{c^2} \right) \dot{u}_{\nu} + \left(\frac{n\dot{\varepsilon}}{c^2} + \frac{p}{c^2}\theta \right) u_{\nu} + p_{,\mu} h^{\mu}_{\nu} + \pi^{\mu}_{\nu,\mu} \\ &\quad + \frac{1}{c^2} \left(q^{\mu}_{,\mu} u_{\nu} + q^{\mu} u_{\nu,\mu} + \theta q_{\nu} + u^{\mu} q_{\nu,\mu} \right) = 0 \end{aligned} \tag{5}$$

$$nC_n\dot{T} + \left(\frac{T\beta}{\kappa_T}\right)\theta + u^{\nu}_{,\mu}\pi^{\mu}_{\nu} + q^{\mu}_{;\mu} + \frac{1}{c^2}\dot{u}^{\nu}q_{\nu} = 0 \qquad (6)$$

where κ_T is the isothermal compressibility, β the thermal expansion coefficient and C_n the heat capacity at constant particle density. As it will be remarked below, the tensor π^{μ}_{ν} can be further decomposed in a traceless symmetric part and the trace multiplied by the spatial projector.

In order to close the system of equations, constitutive relations for the heat flux and Navier tensor must be introduced. The equation for the heat flux has been recently established by means relativistic kinetic theory and reads

$$q^{\ell} = -L_T\frac{T^{,\ell}}{T} + L_n\frac{n^{,\ell}}{n} \qquad (7)$$

where L_T and L_n are transport coefficients[10]. A detailed discussion on this equations can be found elsewhere[10]. The equations for π^{μ}_{ν} in Eq. (2) are well-known namely,

$$\pi^{(s)}_{\mu\nu} = -2\eta\sigma_{\mu\nu} \qquad (8)$$

$$tr\left(\pi\right) = -\xi\nabla\cdot\vec{u} \qquad (9)$$

where $\pi^{(s)}_{\mu\nu}$ is the symmetric and traceless part of π^{μ}_{ν}, $tr\left(\pi\right)$ its trace and $\sigma_{\mu\nu}$ is the symmetric and taceless part of the velocity gradient. The transport coefficients in Eqs. (8) and (9) are the shear and bulk viscosities respectively.

3 Linearized relativistic hydrodynamics

In order to linearize the set of equations (4-6) we consider $n = n_0 + \delta n$, $T = T_0 + \delta T$ and $u^{\nu} = \delta u^{\nu}$ where naught subscripts denote equilibrium quantities and the δ prefix indicates small perturbations around it. With this hypothesis, the linearized transport equations for a simple, relativistic fluid in the absence of external fields are

$$\delta\dot{n} + n_0\delta\theta = 0 \qquad (10)$$

$$\begin{aligned}&\frac{1}{c^2}\left(n_0\varepsilon_0 + p_0\right)\delta\dot{u}_{\nu} + \frac{1}{n\kappa_T}\delta n_{,\nu} + \frac{\beta}{\kappa_T}\delta T_{,\nu}\\ &-\zeta\delta\theta_{,\nu} - 2\eta\left(\delta\sigma^{\mu}_{\nu}\right)_{,\mu} - \frac{L_T}{c^2}\delta\dot{T}_{,\nu} - \frac{L_n}{c^2}\delta\dot{n}_{,\nu} \quad = 0\end{aligned} \qquad (11)$$

$$nC_n\delta\dot{T} + \left(\frac{T_0\beta}{\kappa_T}\right)\delta\theta - \left(L_T\delta T^{,k} + L_n\delta n^{,k}\right)_{;k} = 0 \qquad (12)$$

where we have defined $\theta = u^{\nu}_{;\nu}$. It is important to point out that the transport coefficients in general depend on the state variables. However, since they only appear as factors of derivatives of the corresponding fluctuations, considering fluctuations on them would induce higher order terms, which are neglected in the linear approximation.

It is crucial at this point to make the following observation. The so-called causality violation of the transport equations to first order in the gradients, given by linear irreversible thermodynamics, can be easily spotted by observing that, considering $u_0^\ell = 0$ and linearizing, Eq. (6) leads to a parabolic equation for T. This clearly admits arbitrary propagation speeds for the corresponding signals. However, the hypothesis of a fluid at rest or the fact that calculations can be performed in the comoving frame should not be translated into a vanishing hydrodynamic velocity, but in $u_0^\ell = 0$ as considered above. That is, δu^ν should not vanish even for the fluid at rest or in the comoving frame; *only the mean or equilibrium velocity can be zero.* This fact has already been pointed out in the analysis of the linearized relativistic Euler regime[12].

The analysis of the dynamics given the system of equations (10-12) can be found in detail in Section 4 of Ref. [11] where we discussed the modifications to the Rayleigh-Brillouin spectrum. Here we only quote the results needed in order to address the problem at hand namely, the *causality* of the system. We start by calculating the divergence of Eq. (11). The transverse mode is then uncoupled from the system and a set of three scalar differential equations for δn, $\delta\theta$ and δT is obtained. A Fourier-Laplace transform is then performed, leading to a system of algebraic equations depending on the time and space variables, s and q respectively, whose associated determinant reads

$$\begin{vmatrix} s & n_0 & 0 \\ -\frac{1}{n_0\kappa_T}q^2 + \frac{L_n}{c^2}sq^2 & \tilde{\rho}_0 s + Aq^2 & \frac{L_T}{c^2}q^2 s - \frac{\beta}{\kappa_T}q^2 \\ \frac{L_n}{n_0 c_n}q^2 & \frac{T_0\beta}{n_0 c_n \kappa_T} & s + \frac{L_{TT}}{n_0 c_n}q^2 \end{vmatrix} = 0 \tag{13}$$

where for convenience we have introduced the following notation[11]

$$\tilde{\rho}_0 = \frac{1}{c^2}\left(n_0\varepsilon_0 + p_0\right) \tag{14}$$

$$A = \zeta + 4\eta/3 \tag{15}$$

The dispersion relation is thus given by

$$s^3 + d_2 s^2 q^2 + s\left(d_3 q^4 + d_4 q^2\right) + d_5 q^4 = 0 \tag{16}$$

where the coefficients d_2 to d_5 have been specified in an earlier work[11]. The physical interpretation of the three roots of Eq. (16) is well known. The dynamics of the perturbations in the fluid are characterized by a strictly dissipative component which decays in time depending on the value of the real root while the other wave-like component propagates at a speed given by the imaginary parts of the conjugate roots, damped by a coefficient which depends on a Stokes-Kirchhoff like factor. Moreover, a plot of the dynamic structure factor as a function of s for a fixed $\vec{q}$, will feature three peaks. In this work, we are interested in the location of the symmetric Brillouin peaks[13], which are given by the imaginary part of the conjugate roots, that is $\omega = \pm\sqrt{d_4}q$. Thus, in this case

$$\omega = \pm\sqrt{\frac{\gamma}{\kappa_T\tilde{\rho}_0}}q \tag{17}$$

such that, the distance between the peaks, i. e. the speed of propagation of the wave-like component of the fluctuations, and the origin is bounded by

$$c\sqrt{\frac{\gamma}{\kappa_T\left(n_0\varepsilon_0+p_0\right)}} \tag{18}$$

Notice that, in the non-relativistic case, the fluctuations propagate at the speed of sound, i. e.

$$c_s^2=\frac{\gamma}{\kappa_T\rho_0} \tag{19}$$

As an example, for an ideal gas $\gamma=5/3$ and $\kappa_T=1/p$, such that

$$c_s^2=\frac{5}{3}\frac{kT}{m} \tag{20}$$

which is clearly unbounded and can be increasingly large for high temperatures. However, the speed of propagation in the relativistic calculation reads

$$c_R^2=\frac{\gamma}{\kappa_T\left(n_0\varepsilon_0+p_0\right)}c^2 \tag{21}$$

Using the expression for the internal energy density given by Eq. (3) we now obtain

$$c_R^2=\left\{\frac{5}{3}\frac{z}{\left[3z+\frac{\mathcal{K}_1\left(\frac{1}{z}\right)}{\mathcal{K}_2\left(\frac{1}{z}\right)}\right]+1}\right\}c^2 \tag{22}$$

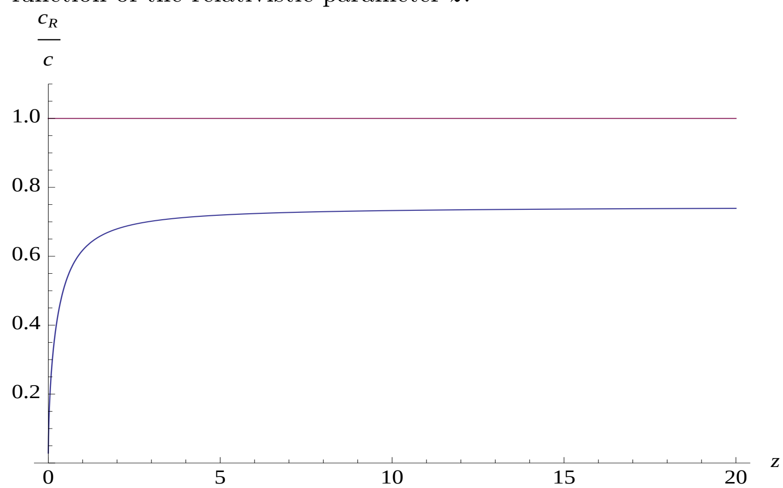
As can be seen in Fig. 1, the expression in curly brackets never exceeds the unity which finally shows that the propagation speed for signals in the transport equations for the relativistic fluid does not violate the antecedence principle.

4 Final remarks

The transport equations derived from Eckart's modified theory show, in its linear version, no problems regarding stability and causality features. This fact implies that there is no real motivation to introduce the so-called second order theories, which introduce non-fundamental adjustable parameters. A simple first order formalism is desirable to describe the fluids which are formed in RHIC type experiments[15].

In the non-linear case the relativistic Navier-Stokes equations here employed are, by far, more complex. In this context, very little can be said regarding the problems of stability and causality with the techniques included in this paper. It is desirable to perform further work in this direction.

Figure 1: The ratio of the speed of propagation to the speed of light, as a function of the relativistic parameter z.



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