EXTREMES OF LÉVY PROCESSES WITH LIGHT TAILS

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Abstract. Let $X(t)\, t \geq 0$, $X(0) = 0$, be a Lévy process with spectral Lévy measure ρ. Assuming that $\rho((-\infty, 0)) < \infty$ and the right tail of ρ is light, we show that in the presence of Brownian component

$$P\left(\sup_{0\leq t\leq 1} X(t) > u\right) \sim P\left(X(1) > u\right)$$

as $u \to \infty$. In the absence of Brownian component these tails are not always comparable. An example of Lévy process of the type $X(t) = B(t) + Z(t)$, where $B(t)$ is a Brownian motion and $Z(t)$ is a compound Poisson process with positive jumps, for which these tails are incomparable is also given.

1. Introduction

The problem of finding asymptotics of the probabilities $P(\sup_{t\in T} X(t) > u)$ as $u \to \infty$, where $X(t)$ is a stochastic process, is a classical one. It was intensively studied, but many unsolved questions still remain.

In what follows $T = [0, 1]$ and $X(t)$ is a Lévy process, $X(0) = 0$. Its characteristic function is given by the well known Lévy–Khintchin formula

$$E \exp\left(isX(t)\right) = \exp\left(t\psi(s)\right)\,,$$

where

$$\psi(s) = -ibt - \frac{\sigma^2 s^2}{2} + \int_{-\infty}^{\infty} \left(e^{isx} - 1 - isx\mathbf{1}(|x| \leq 1)\right) \rho(dx)\,. \tag{1.1}$$

Here $b \in \mathbf{R}, \sigma \geq 0$ and ρ is a Borel measure such that $\int_{-\infty}^{\infty} \min\{1, x^2\}\rho(dx) < \infty$ (the Lévy measure).

If σ is strictly positive, then the process can be represent as a sum of independent Brownian motion $B(t)$ and another Lévy process $X_1(t)$. In the case $\rho(\mathbf{R}) < \infty$ the last process is a compound Poisson. So, if the Lévy measure is finite, we can write

$$X(t) = \sigma B(t) + Z(t) - bt,\ t \geq 0, \tag{1.2}$$

where $Z(t)$ is a compound Poisson process with the parameter $\lambda = \rho(\mathbf{R})$. It means that

$$Z(t) = \sum_{k=1}^{N(t)} X_k\,, \tag{1.3}$$

where $N(t)$ is a Poisson process with parameter λ independent of iid random variables $\{X_k\}_{k=1}^{\infty}$ (the *jumps* of the process).

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One of the approaches to the mentioned problem is to establish a relation

$$P\left(\sup_{0\le t\le 1} X(t) > u\right) \sim aP\left(X(1) > u\right) \quad \text{as } u \to \infty \tag{1.4}$$

where a is a constant. Then Lévy-Khinchin formula allows to derive the asymptotics of the right hand side probabilities by powerful analytical tools.

The first result of type (1.4) is Lévy theorem, which states that for Brownian motion $B(t)\,, t \ge 0$ the following holds:

$$P\left(\sup_{0\le t\le 1} B(t) > u\right) = 2P\left(B(1) > u\right) \tag{1.5}$$

for all $u > 0$. During recent years (1.4) was established for various classes of Lévy processes (see [1]–[6], [9], [11], [12]). One of the methods used in these studies is to represent the process in the form $X(t) = Y(t) + Z(t)$, where $Y(t)$ and $Z(t)$ are independent, $Z(t)$ is a compound Poisson process and $Y(t)$ is a Lévy process for which $E\exp(c|Y(t)|) < \infty$ for each $c > 0$. Assuming the distribution of the jumps of $Z(t)$ to be *heavy*, (subexponential or exponential), one first establishes (1.4) for this process. Such distributions possess the following property: if X and Y are independent random variables, the tail of X is heavy and $P(Y > u) = o(P(X > u))$ as $u \to \infty$, then $P(X + Y > u) \sim bP(X > u)$ as $u \to \infty$, where b is a constant. Using it, on can pass from $Z(t)$ to $X(t)$ (see, for example, [6] and [11] and references threin).

But such approach does not work if jumps have a *light tail* in the sense of [4]. So, other methods are called for.

In what follows C denotes a generic constant which value may vary from line to line. As usually, F_Y stands for the disrtibution of a random variable Y. Througout the paper $\{X_k\}_{k=1}^{\infty}$ are iid random variables, $S_k = X_1 + \cdots + X_k$, $k \ge 1$, $S_0 = 0$.

2. Results

We say that the distribution of a random variable X has *light* right tail if one of the following conditions holds:

$$P(X_1 > u) > 0 \quad \text{for all } u > 0 \ \text{ and } \quad \lim_{u\to\infty} \frac{P(X_1 > u)}{P(X_1 + X_2 > u)} = 0\,, \tag{2.1}$$

where X_1 and X_2 are independent copies of X, or

$$X \le A \quad \text{a.s. and } P(X > \alpha) > 0 \tag{2.2}$$

for positive constants A and α.

It is known that X has a light tail if and only if $X^+ := \max\{X, 0\}$ has it (see [4], Lemma 2).

In what follows we assume that

$$\rho((0,\infty)) > 0 \tag{2.3}$$

and

$$\rho((-\infty,0)) < \infty\,. \tag{2.4}$$

Clearly, (2.3) implies $\rho((a,\infty)) > 0$ for some positive a. The third assumption is:

$$\text{for some } a > 0 \text{ the disribution function} \quad F_\rho(x) = 1 - \frac{\rho((\min\{x,a\},\infty))}{\rho((a,\infty))} \quad \text{has light tail.} \tag{2.5}$$

Our main result is the following

Theorem 2.1. *Let $\sigma > 0$ and (2.3)–(2.5) hold. Then for each $b \in \mathbf{R}$*

$$\lim_{u\to\infty} \frac{P(\sup_{0\le t\le 1} X(t) > u)}{P(X(1) > u)} = 1\,. \tag{2.6}$$

It should be mentioned that under the additional condition:

$$\lim_{u\to\infty} \frac{1 - F_\rho(x+c)}{1 - F_\rho(x)} = e^{-\alpha c} \quad \text{for any real } c \text{ and a constant } \alpha > 0,$$

and whithout assumption (2.4), this statement was proved in [5] and [1].

In the case $\sigma = 0$ and $\rho(\mathbf{R}) < \infty$ the process is compound Poisson with drift. It is known that (2.6) holds for such processes with $b \le 0$, but this limit not always exists if $b > 0$ (see [4]). Our next result gives a condition under which this relation holds for $b > 0$ in the absence of Brownian component.

Assume that $P(X > u) > 0$ for all $u > 0$ and

$$\lim_{u\to\infty} \frac{P(X > u+a)}{P(X > u)} = 0 \quad \text{for a constant } a > 0. \tag{2.7}$$

For independent copies X_1 and X_2 of X we have $P(X_1 + X_2 > u) \ge P(X_1 > u-a)P(X_2 > a)$, which implies (2.1). Hence the right tail of X is light.

If the tail of X is given in the form

$$1 - F_X = \exp\left(-\int_0^u h(v)dv\right)\,,\ u > u_0\,, \tag{2.8}$$

where $u_0 \ge 0$ is a constant and h is a positive function on (u_0, ∞) such that

$$h(v) \to \infty \qquad \text{as } v \to \infty, \tag{2.9}$$

then (2.7) holds and, therefore, X has a light tail.

Theorem 2.2. *Assume (2.3)–(2.5) hold, $\sigma = 0$ and the function F_ρ from (2.5) can be represented in the form (2.8) with (2.9). Assume also that the function h is continuous, increasing and satisfies the condition*

$$h(v+b) \le \exp\left(\frac{bh(v)}{8}\right) \tag{2.10}$$

for v large enough. Then (2.6) holds for each $b \in \mathbf{R}$.

Condition (2.10) means that the function $h(v)$ cannot grow too fast as $v \to \infty$. If $h(v) = \exp(g(v))$ and $g(v+a) \le C(a)g(v)$ for positive a and v, then (2.10) holds. Another examples are $h(v) = v^c$, and $h(v) = [\log(v+1)]^c$, where c is a positive constant. In can be easily verified that X_1 with a normal distiribution satisfies the conditions of the theorem. Therefore, (2.6) holds for compound Poisson processes with normal jumps and negative drifts.

As it was shown in [4], relation (1.4) does not hold if $X(t)$ is a compound Poisson process with negative drift and jumps having a lattice distribution bounded from above. The following result shows that the condition of boundedness can be ommited.

Theorem 2.3. *Let (1.2) hold with $\sigma = 0$, and jumps X_k having a lattice distribution with a minimal step a. Assume that*

$$P(X_1 > na) > 0 \quad \textit{for all } n \in \mathbf{N} \quad \textit{and} \quad \lim_{n\to\infty} \frac{P(X_1 > (n+1)a)}{P(X_1 > na)} = 0\,. \tag{2.11}$$

Then for each $b > 0$

$$\limsup_{u\to\infty} \frac{P\left(\sup_{0\le t\le 1} X(t) > u\right)}{P(X(1) > u)} = \infty \tag{2.12}$$

and

$$\liminf_{u\to\infty} \frac{P\left(\sup_{0\le t\le 1} X(t) > u\right)}{P(X(1) > u)} = 1\,. \tag{2.13}$$

Remark 2.4. One can obtain a lattice distribution by a "discretization". Namely, for a random variable X and a fixed $a > 0$ put

$$X^{(a)} = \sum_{n=-\infty}^{\infty} naI_{(na\le X<(n+1)a)}\,.$$

Assume now that the distribution of the jumps X_k satisfies (2.8)–(2.10). Denote by $X_k^{(a)}$ the discretizations of X_k, and by $Z^{(a)}(t)$ the corresponding compound Poisson process given by (1.3). Let $b > 0$. Then for the process $X(t) = Z(t) - bt$ we have (2.6), while for the process $X^{(a)}(t) = Z^{(a)}(t) - bt$ relations (2.14) and (2.15) hold. For example, it is true if the jumps X_k are normal.

The situation is different when the tail of jumps is "heavy", i.e. if

$$\lim_{u\to\infty} \frac{P(X_1 > u + a)}{P(X_1 > u)} = 1$$

for any $a > 0$. It is known that under this assumption (2.6) holds for the process $X(t)$ (see [12]). Because in this case the tail of "discretized" jumps $X_k^{(a)}$ is also heavy, (2.6) holds for the process $X^{(a)}(t) = Z^{(a)}(t) - bt$ also.

Theorems 2.1 and 2.3 show that sometimes the process $Z(t) - bt$ does not satisfy (1.4), while for the process (1.2) with $\sigma > 0$ relation (2.6) holds. Our last result states that a compound Poisson process $Z(t)$ may satisfy (2.6), while for the process $X(t) = \sigma B(t) + Z(t)$ relation (1.4) does not hold. Clearly, if the jumps of $Z(t)$ are positive, then its supremum over $[0, 1]$ is $Z(1)$.

Theorem 2.5. *There is a compound Poisson process $Z(t)$ with positive jumps such that for the process (1.2) with $\sigma > 0$ and $b = 0$*

$$\limsup_{u\to\infty} \frac{P(\sup_{0\le t\le 1} X(t) > u)}{P(X(1) > u)} > 1 \tag{2.14}$$

and

$$\liminf_{u\to\infty} \frac{P(\sup_{0\le t\le 1} X(t) > u)}{P(X(1) > u)} = 1\,. \tag{2.15}$$

3. Auxiliary statements

Here we prove some statements that are used later .

Lemma 3.1. *Let Z and W be random variables, $P(Z > u) > 0$ and $P(W > u) > 0$ for all positive u, and one of the following conditions holds:*

$$\lim_{u\to\infty} \frac{P(Z > u)}{P(W > u)} = 1\,, \tag{3.1}$$

or

$$\lim_{u\to\infty} \frac{P(Z>u)}{P(W>u)} = 0\,. \tag{3.2}$$

Let a random variable Y satisfy condition (2.7). If Y is independent of Z and W, then

$$\lim_{u\to\infty} \frac{P(Y+Z>u)}{P(Y+W>u)} = 1$$

if (3.1) holds, and

$$\lim_{u\to\infty} \frac{P(Y+Z>u)}{P(Y+W>u)} = 0$$

if (3.2) holds.

Proof. If (3.1) holds, then for a fixed $\epsilon > 0$ we can find $u_0 > 0$ such that

$$P(Z>u) \le (1+\epsilon)P(W>u)$$

for all $u \ge u_0$. Hence

$$P(Y+Z>u) \le (1+\epsilon)\int_{-\infty}^{u-u_0} P(W>u-t)F_Y(dt) + P(Y>u-u_0)$$

$$\le (1+\epsilon)P(Y+W>u) + P(Y>u-u_0)\,.$$

We also have $P(Y+W>u) \ge P(Y>u-u_0-a)P(W>u_0+a)$. From here and (2.7)

$$\limsup_{u\to\infty} \frac{P(Y+Z>u)}{P(Y+W>u)} \le (1+\epsilon)\,.$$

But by the same way

$$\limsup_{u\to\infty} \frac{P(Y+W>u)}{P(Y+Z>u)} \le (1+\epsilon)\,.$$

Letting $\epsilon \to 0$ we get the first needed relation. The second one can be obtained similarly. □

It is known that for compound Poisson process with light tail relation (1.4) holds with $a = 1$ (see Theorem 1 from [4]). Because the random variable $Y = B(1)$ satisfies (2.7), we come to the following statement.

Corollary 3.2. *If the jumps of a compound Poisson process $Z(t)$ have a light tail, and $B(1)$ is independent of this process, then*

$$\lim_{u\to\infty} \frac{P(B(1)+\sup_{0\le t\le 1} Z(t)>u)}{P(B(1)+Z(1)>u)} = 1\,. \tag{3.3}$$

Lemma 3.3. *Let random variables $\{X_k\}_{k=1}^{\infty}$ and Y be independent. Assume that the tail of X_k is light and Y satisfies (2.7). Assume also that $a \le \alpha$ in the case (2.2). Then*

$$\lim_{u\to\infty} \frac{P(Y+S_k>u)}{P(Y+S_{k+1}>u)} = 0$$

for all $k = 1, 2, \ldots$.

Proof. Fist we consider case (2.1). Then, according to Lemma 4 from [4]

$$\lim_{u\to\infty}\frac{P(S_k>u)}{P(S_{k+1}>u)}=0 \tag{3.4}$$

and Lemma 3.1 leads to the needed conclusion.

Turn to case (2.2). Then $S_k \le Ak$ and

$$P(Y+S_k>u)=\int_{-\infty}^{Ak}P(Y>u-t)F_{S_k}(dt)\,,$$

$$P(Y+S_{k+1}>u)=\int_{-\infty}^{Ak}P(Y+X_1>u-t)F_{S_k}(dt).$$

We have $P(Y+X_1>u-t)\ge P(Y>u-t-a)P(X_1>a)$, and $P(X_1>a)>0$ because $a\le\alpha$. It follows from this estimate and (2.7) that for a fixed $\epsilon>0$ there is $u_0>0$ such that if $u-t>u_0$, then

$$\frac{P(Y>u-t)}{P(Y+X_1>u-t)}\le\epsilon\,.$$

But in the last integrals $t\le Ak<u-u_0$, i.e. $u-t>u_0$ for u large enough. For such u

$$\frac{P(Y+S_k>u)}{P(Y+S_{k+1}>u)}\le\epsilon\,.$$

Letting $u\to\infty$ and then $\epsilon\to0$ we obtain the lemma. □

The next statement plays an important role in the proof of Theorem 2.1.

Lemma 3.4. *Assume random variables X and Y are independent and Y is symmetric. Then*

$$P(X+|Y|>u)=2P(X+Y>u)-P(X>u+|Y|)\,.$$

for all $u>0$.

Proof. We have

$$P(X+|Y|>u)=P(X+Y>u)+P(X+Y\le u\,,\,X+|Y|>u)=P(X+Y>u)$$
$$+P(X>u\,,\,X+Y\le u)+P(X\le u\,,\,X+Y\le u\,,\,X+|Y|>u)\,.$$

Because of symmetry and independence

$$P(X>u\,,\,X+Y\le u)=P(X>u\,,\,Y\le u-X)=P(X>u\,,\,Y\ge X-u)\,.$$

By the same reasons

$$P(X\le u\,,\,X+Y\le u\,,\,X+|Y|>u)=P(X\le u\,,\,Y\le u-X\,,\,|Y|>u-X)$$
$$=P(X\le u\,,\,Y\le u-X\,,\,-Y>u-X)=P(X\le u\,,\,Y>u-X)$$
$$=P(X\le u\,,\,X+Y>u)=P(X+Y>u)-P(X>u\,,\,X+Y>u)\,.$$

Inserting the last two relation in the first one we get

$$P(X+|Y|>u)=2P(X+Y>u)+P(X>u\,,\,Y\ge X-u)-P(X>u\,,\,Y>u-X)$$
$$=2P(X+Y>u)-P(X>u\,,\,u-X<Y<X-u)\,.$$

Since the last probability is

$$P(X>u,\,|Y|<X-u)=P(X>u+|Y|)\,,$$

the lemma follows. □

The following lemma will allow us to reduce the proofs of Theorems 2.1 and 2.2 to the case of processes of type (1.2).

Lemma 3.5. *Let*

$$X(t) = X_1(t) + X_2(t)\,, \tag{3.5}$$

where Lévy processes $X_1(t)$ and $X_2(t)$ are independent, $X_2(t)$ is a subordinator with Lévy measure ρ_2 such that $\rho_2((a_2,\infty)) = 0$, where $a_2 > 0$ is a constant. Assume that $\rho_1((a_1,\infty)) > 0$ for $a_1 > a_2$, where ρ_1 is the Lévy measure of $X_1(t)$. Assume also that $X_1(t)$ satisfies (2.6). Then this relation holds for the process $X(t)$.

Proof. Since $X_2(t)$ is a subordinator, then

$$\begin{aligned} P\left(\sup_{0\le t\le 1} X(t) > u\right) &\le P\left(\sup_{0\le t\le 1} X_1(t) + X_2(1) > u\right) \\ &\le \int_{-\infty}^{u-A} P\left(\sup_{0\le t\le 1} X(t) > u - v\right) F_{X_2(1)}(dv) + P(X_2(1) > u - A)\,, \end{aligned} \tag{3.6}$$

where A is a positive constant. Because $X_1(t)$ satisfies (2.6), for a fixed $\epsilon > 0$ there is A such that the integral does not exceed

$$(1+\epsilon)P(X_1(1) + X_2(1) > u) = (1+\epsilon)P(X(1) > u)\,.$$

It is well known that the conditions $\rho_1((a_1,\infty)) > 0$ and $\rho_2((a_2,\infty)) = 0$ for $a_1 > a_2$ implies

$$P(X_2(1) > u - A) = o\left(P(X_1(1) > u)\right)$$

for any positive A as $u \to \infty$ (see [8]). From here and (3.6)

$$\limsup_{u\to\infty} \frac{P(\sup_{0\le t\le 1} X(t) > u)}{P(X(1) > u)} \le (1+\epsilon)$$

for each $\epsilon > 0$, and the lemma follows. □

Let $b > 0$ and $Z(t)$ be defined by (1.3). Denote by Γ_k, $k \ge 1$, the arrival times of $Z(t)$ and put $\Gamma_0 = 0$. Let

$$\tau = \max\{k : \Gamma_k < 1\}\,. \tag{3.7}$$

Let

$$m = \min\{k : P(S_k > b) > 0\}\,, \tag{3.8}$$

and

$$a_k = \max\left\{1 - \frac{(m+1)\log k}{k}\,,\, 0\right\}. \tag{3.9}$$

Put

$$Q(u) = P\left(Z(1) > u + b\Gamma_\tau\,, \Gamma_\tau \le a_\tau\right) \tag{3.10}$$

Lemma 3.6. *If X_k have a light tail, then for any $b > 0$*

$$\lim_{u\to\infty} \frac{Q(u)}{P(Z(1) > u + b)} = 0\,. \tag{3.11}$$

Proof. It can be easily verified that

$$Q(u) = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \int_0^{a_k} \frac{(\lambda t)^{k-1}}{(k-1)!} P(S_k > u + bt) dt\,.$$

Fix an index M and denote

$$Q_M(u) = \lambda e^{-\lambda} \sum_{k=1}^{M} \int_0^{a_k} \frac{(\lambda t)^{k-1}}{(k-1)!} P(S_k > u + bt) dt\,, \quad Q^{(M)}(u) = Q(u) - Q_M(u)\,.$$

It is clear that

$$P(Z(1) > u + b) = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} P(S_k > u + b)$$

and, therefore, for each k

$$P(Z(1) > u + b) > \frac{\lambda^{k+m}}{(k+m)!} P(S_{k+m} > u + b) > \frac{\lambda^{k+m}}{(k+m)!} P(S_m > b) P(S_k > u)\,. \tag{3.12}$$

Since

$$Q_M(u) \le e^{-\lambda} \sum_{k=1}^{M} \frac{\lambda^k}{k!} P(S_k > u),$$

the last estimate and (3.4) imply

$$\lim_{u\to\infty} \frac{Q_M(u)}{P(Z(1) > u + b)} = 0\,.$$

Further, denoting

$$\delta(k, u) = \frac{\lambda \int_0^{a_k} \frac{(\lambda t)^{k-1}}{(k-1)!} P(S_k > u + bt) dt}{\frac{\lambda^{k+m}}{(k+m)!} P(S_{k+m} > u + b)}\,,$$

we see that

$$\frac{Q^{(M)}(u)}{P(X(1) > u)} \le \sup_{k>M} \delta(k, u)\,,$$

and

$$\delta(k, u) \le \frac{\frac{(\lambda a_k)^k}{k!} P(S_k > u)}{\frac{\lambda^{k+m}}{(k+m)!} P(S_k > u) P(S_m > b)} = \frac{(k+1)\cdots(k+m) a_k^k}{\lambda^m P(S_m > b)}\,.$$

Hence

$$\limsup_{u\to\infty} \frac{Q(u)}{P(X(1) > u)} = \limsup_{u\to\infty} \frac{Q^{(M)}(u)}{P(X(1) > u)} \le \sup_{k>M} \frac{(k+1)\cdots(k+m) a_k^k}{\lambda^m P(S_m > b)}\,.$$

According to (3.9) $(k+1)\cdots(k+m)a_k^k \to 0$ as $k \to \infty$. Hence, letting $M \to \infty$ we come to (3.11). □

We also will use the following well known estimate for the normal distribution. If Y is normal with mean zero and variance one, then for all $x > 1$

$$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{x} - \frac{1}{x^3} \right) \exp\left(-\frac{x^2}{2} \right) \le P(Y > x) \le \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left(-\frac{x^2}{2} \right)\,. \tag{3.13}$$

4. Proof of Theorem 2.1

According to (2.4) we can represent our process in the form (3.5), and Lemma 3.5 shows that it is enough to prove the theorem for the process of type (1.2).

If $b < 0$, then

$$P\left(\sup_{0\le t\le 1} X(t) > u\right) \le P\left(\sup_{0\le t\le 1} [B(t) + Z(t)] > u + b\right).$$

So, (2.6) for $b = 0$ implies the same relation for $b < 0$. Hence, we may assume $b \ge 0$ in the sequel. Without loss of generality $\sigma = 1$.

Let τ be given by (3.7). Then

$$P\left(\sup_{0\le t\le 1} X(t) > u\right) \le P\left(\sup_{0\le t<\Gamma_\tau} X(t) > u\right) + P\left(\sup_{\Gamma_\tau\le t\le 1} X(t) > u\right)$$
$$= A(u) + C(u).$$

The theorem will follow from the next two equalities:

$$\lim_{u\to\infty} \frac{C(u)}{P(X(1) > u)} = 1 \tag{4.1}$$

and

$$\lim_{u\to\infty} \frac{A(u)}{P(X(1) > u)} = 0. \tag{4.2}$$

4.1. **Proof of (4.1).** Let $\widetilde{B}(t)$ be a Brownian motion independent of $X(t)$. We have

$$C(u) = P\left(\sup_{\Gamma_\tau\le t\le 1} \left[B(\Gamma_\tau) + S_\tau - bt + \widetilde{B}(1-t)\right] > u\right)$$
$$\le P\left(B(\Gamma_\tau) + S_\tau - b\Gamma_\tau + |\widetilde{B}(1-\Gamma_\tau)| > u\right)$$

because Lévy formula (1.5) can be written in the form

$$\sup_{0\le t\le 1} B(t) \overset{\mathrm{d}}{=} |B(1)|. \tag{4.3}$$

Applying Lemma 3.4 conditionally on Γ_τ and taking into account the relations

$$B(\Gamma_\tau) + \widetilde{B}(1-\Gamma_\tau) \overset{\mathrm{d}}{=} B(1) \quad \text{and} \quad S_\tau = Z(\Gamma_\tau) = Z(1),$$

we conclude that

$$C(u) \le 2P\left(B(1) + Z(1) > u + b\Gamma_\tau\right) - P\left(B(\Gamma_\tau) + Z(1) > u + b\Gamma_\tau + |\widetilde{B}(1-\Gamma_\tau)|\right). \tag{4.4}$$

To obtain (4.1) it is enough to prove the following

Lemma 4.1. *For each* $b \ge 0$*:*

$$\liminf_{u\to\infty} \frac{P\left(B(\Gamma_\tau) + Z(1) > u + b\Gamma_\tau + |\widetilde{B}(1-\Gamma_\tau)|\right)}{P(B(1) + Z(1) > u + b)} \ge 1, \tag{4.5}$$

and

$$\lim_{u\to\infty}\frac{P(B(1)+Z(1)>u+b\Gamma_\tau)}{P(B(1)+Z(1)>u+b)}=1\,. \tag{4.6}$$

Proof of (4.5). Because

$$P\left(B(\Gamma_\tau)+Z(1)>u+b\Gamma_\tau+|\widetilde{B}(1-\Gamma_\tau)|\right)\geq P\left(B(\Gamma_\tau)+Z(1)>u+b+|\widetilde{B}(1-\Gamma_\tau)|\right)\,,$$

it is enough to establish (4.5) for $b=0$.

Step 1. Fix a $\delta\in(0,1)$. There is a positive constant D such that

$$P(|B(1)|<D)>1-\delta\,.$$

Since

$$\widetilde{B}(1-t)\stackrel{\mathrm{d}}{=}\sqrt{1-t}B(1)\,,\;0<t<1\,,$$

we get for each $t\in(0,1)$ and $k\in\mathbf{N}$

$$P\left(B(t)+S_k>u+|\widetilde{B}(1-t)|\right)\geq P\left(B(t)+S_k>u+D\sqrt{1-t}\right)P\left(|\widetilde{B}(1-t)|<D\sqrt{1-t}\right)$$
$$\geq(1-\delta)P\left(B(t)+S_k>u+D\sqrt{1-t}\right)\,.$$

Integrating with respect to Γ-densities and summing up over k's we obtain

$$\begin{aligned} &P(X(\Gamma_\tau)>u+|\widetilde{B}(1-\Gamma_\tau)|) \\ &\geq(1-\delta)e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}\int_0^1 P\left(B(t)+S_k>u+D\sqrt{1-t}\right)t^{k-1}dt \\ &=(1-\delta)P(X(\Gamma_\tau)>u+D\sqrt{1-\Gamma_\tau}):=(1-\delta)H(u)\,. \end{aligned}\tag{4.7}$$

To prove (4.5) for $b=0$ it is enough to establish that for each $D>0$

$$\liminf_{u\to\infty}\frac{H(u)}{P(X(1)>u)}\geq 1\,. \tag{4.8}$$

Step 2. From now on α is a positive constant for which

$$P(X_1>\alpha)>0\,, \tag{4.9}$$

and a is a constant such that

$$a>\max\left\{1,\frac{1}{\alpha}\right\}\,. \tag{4.10}$$

For fixed $T\in\mathbf{N}$ and $u>0$, where $2\leq T<au$, we divide $\mathbf{N}$ into three parts:

$$\begin{aligned} &\mathbf{N}_1(T,u)=\{k:k\leq T\}\,,\;\mathbf{N}_2(T,u)=\{k:T<k\leq[au]\}, \\ &\mathbf{N}_3(T,u)=\{k:k>[au]\}\,. \end{aligned}\tag{4.11}$$

Using (4.36) and denoting by $G_i(u)\,,i=1,2,3,$ the sums of summands over $\mathbf{N}_i(T,u)$ we may write

$$P(X(1)>u)=e^{-\lambda}P(B(1)>u)+G_1(u)+G_2(u)+G_3(u)\,. \tag{4.12}$$

It follows from Lemma 3.3 that for each $T\in\mathbf{N}$

$$\lim_{u\to\infty}\frac{e^{-\lambda}P(B(1)>u)+G_1(u)}{P(X(1)>u)}=0\,. \tag{4.13}$$

Now we show that

$$\lim_{u\to\infty} \frac{G_3(u)}{P(X(1)>u)} = 0\,. \tag{4.14}$$

Indeed, according to Stirling formula

$$G_3(u) \le \frac{\lambda^{[au]+1}}{([au]+1)!} = \exp\left(-a(u\log u)(1+g(u))\right)\,,$$

where $g(u)\to 1$ as $u\to\infty$.

On the other hand, for

$$k(u) = \max\left\{[u]\,, \left[\frac{u}{\alpha}\right]\right\}$$

we have, once again applying Stirling formula,

$$P(X(1)>u) \ge e^{-\lambda}\frac{\lambda^{k(u)}}{k(u)!}P(B(1)+S_{k(u)}>u)$$

$$\ge e^{-\lambda}\frac{\lambda^{k(u)}}{k(u)!}P(B(1)>0)P(X_j>\alpha\,, 1\le j\le k(u))$$

$$= \exp\left(-k(u)\log k(u)(1+g_1(u))\right)\,,$$

where, as above, $g_1(u)\to 1$ as $u\to\infty$. According to (4.10)

$$k(u) \le u\max\left\{1\,, \frac{1}{\alpha}\right\} < au\,,$$

and (4.14) follows from here and the last two estimates.

So, for each $T\in\mathbf{N}$

$$\lim_{u\to\infty} \frac{G_2(u)}{P(X(1)>u)} = 1\,. \tag{4.15}$$

Step 3. Here we represent $G_2(u)$ as a sum of two quantities, such that the first of them is small relative to $P(X(1)>u)$. Denote

$$g_a(k,u) = u - a\log(\min\{k,u\}) \tag{4.16}$$

and

$$I(k,u) = P\left(B(1)+S_k>u\,, S_k\le g_a(k,u)\right)\,. \tag{4.17}$$

Put

$$G_{21}(u) = e^{-\lambda}\sum_{k=T+1}^{[au]}\frac{\lambda^k}{k!}I(k,u)\,, \quad G_{22}(u) = G_2(u)-G_{21}(u)\,. \tag{4.18}$$

Proposition 4.2. *For $u>T$ the following inequality holds:*

$$\frac{G_{21}(u)}{P(X(1)>u)} \le \frac{2aCe^{\alpha^2/2}}{\lambda P(X_1>\alpha)}T^{1-\alpha a}\,, \tag{4.19}$$

where C is an absolute constant.

Proof. We have, using (4.36),

$$\frac{G_{21}(u)}{P(X(1)>u)} \le \max_{T+1\le k\le au} \frac{\frac{\lambda^k}{k!}I(k,u)}{\frac{\lambda^{k+1}}{(k+1)!}P(B(1)+S_{k+1}>u)} = \max_{T+1\le k\le au} \frac{(k+1)I(k,u)}{\lambda P(B(1)+S_{k+1}>u)}\,. \tag{4.20}$$

Further,

$$P(B(1)+S_{k+1}>u) \ge P(B(1)+S_k>u-\alpha)P(X_1>\alpha)\,,$$

which yields

$$\frac{I(k,u)}{P(B(1)+S_{k+1}>u)} \le \frac{1}{P(X_1>\alpha)}\frac{I(k,u)}{P(B(1)+S_k>u-\alpha)}$$
$$\le \frac{1}{P(X_1>\alpha)}\frac{\int_{-\infty}^{g_a(k,u)}P(B(1)>u-y)F_{S_k}(dy)}{\int_{-\infty}^{g_a(k,u)}P(B(1)>u-y-\alpha)F_{S_k}(dy)}$$
$$\le \frac{1}{P(X_1>\alpha)}\max_{y\le g_a(k,u)}\frac{P(B(1)>u-y)}{P(B(1)>u-y-\alpha)}\,.$$

If $y\le g_a(k,u)$, then $u-y\ge u-g_a(k,u)=a\min\{\log k,\log u\}$. Since $k,u>T$, we obtain using (3.13) and elementary computations,

$$\frac{P(B(1)>u-y)}{P(B(1)>u-y-\alpha)} \le C\exp\left(-\alpha(u-y)+\frac{\alpha^2}{2}\right) \le Ce^{\alpha^2/2}\exp\left(-\alpha a\min\{\log k,\log u\}\right)\,,$$

where C is a constant independent of k and u. Because $\alpha a>1$, this inequality jointly with previous ones give us (4.19). □

Step 4. Define

$$J(k,u)=e^{-\lambda}\frac{\lambda^k}{(k-1)!}\int_0^1 P\left(B(t)+S_k>u+D\sqrt{1-t}\,,\,S_k>g_a(k,u)\right)t^{k-1}dt\,. \tag{4.21}$$

The following statement is the main part of our proof.

Proposition 4.3. *For each $\epsilon\in(0,1)$ and $D>0$ there are $T_0\in\mathbf{N}$ and $u_0>0$ such that*

$$\gamma_k(u):=\frac{J(k,u)e^{\lambda}k!}{P(B(1)+S_k>u\,,\,S_k>g_a(k,u))\lambda^k}>1-\epsilon \tag{4.22}$$

for all $k>T_0$ and $u>u_0$.

Proof. We can write

$$J(k,u)=e^{-\lambda}\frac{\lambda^k}{(k-1)!}\int_{g_a(k,u)}^{\infty}\int_0^1 P\left(B(t)>u-y+D\sqrt{1-t}\right)t^{k-1}dtF_{S_k}(dy)$$

and

$$P(B(1)+S_k>u\,,\,S_k>g_a(k,u))=\int_{g_a(k,u)}^{\infty}P\left(B(t)>u-y\right)F_{S_k}(dy)\,,$$

which yields that

$$\gamma_k(u)\ge k\min_{y>g_a(k,u)}\frac{\int_0^1 P\left(B(t)>u-y+D\sqrt{1-t}\right)t^{k-1}dt}{P(B(1)>u-y)}\,. \tag{4.23}$$

We estimate the expression in the right hand side dividing the area $[g_a(k,u),\infty)$ into three parts: $[g_a(k,u),u-\beta)$, $[u-\beta,u+\beta_1)$ and $[u+\beta_1,\infty)$, where positive constants β and β_1 will be choosen later. We also denote by $\gamma_k^{(1)}(u)$, $\gamma_k^{(2)}(u)$ and $\gamma_k^{(3)}(u)$ the minima over these parts correspondingly.

Case 1: $g_a(k,u)\le y<u-\beta$. We assume $\beta>1$. Estimate (3.13) implies that

$$\begin{aligned}\nu(t,u-y) &:= \frac{P\left(B(t)>u-y+D\sqrt{1-t}\right)}{P(B(1)>u-y)}\\ &\ge \frac{\beta}{\beta+D}\left(1-\frac{1}{\beta^2}\right)\exp\left(\frac{(u-y)^2}{2}-\frac{(u-y+D\sqrt{1-t})^2}{2t}\right)\\ &= \frac{\beta^2-1}{\beta(\beta+D)}\exp\left(-\frac{(u-y)^2(1-t)}{2t}-\frac{(u-y)D\sqrt{1-t}}{t}-\frac{D^2(1-t)}{2t}\right).\end{aligned}$$

Fix $b>0$. If $1-\frac{b}{k}<t<1$, then

$$\nu(t,u-y)\ge\frac{\beta^2-1}{\beta(\beta+D)}\exp\left(-\frac{(u-y)^2b}{2(k-b)}-\frac{D(u-y)\sqrt{bk}}{k-b}-\frac{D^2b}{2(k-b)}\right).$$

Denoting

$$\xi_b(x)=\exp\left(-\frac{a^2b(\log x)^2}{2(x-b)}-\frac{a\sqrt{b}D\sqrt{x}\log x}{x-b}-\frac{bD^2}{2(x-b)}\right) \tag{4.24}$$

and taking into account that $\beta<u-y<a\log(\min\{k,u\})$, we obtain

$$\nu(t,u-y)\ge\frac{\beta^2-1}{\beta(\beta+D)}\xi_b\left(\min\{k,u\}\right).$$

Restriction of the area of integration in (4.23) to $1-\frac{b}{k}\le t<1$ yields

$$\gamma_k^{(1)}(u)\ge\left[1-\left(1-\frac{b}{k}\right)^k\right]\frac{\beta^2-1}{\beta(\beta+D)}\xi_b\left(\min\{k,u\}\right). \tag{4.25}$$

Case 2: $u-\beta\le y<u+\beta_1$. Now for $1-\frac{b}{k}\le t<1$

$$\left|\frac{u-y+D\sqrt{1-t}}{\sqrt{t}}-(u-y)\right|\le\max\{\beta\,,\,\beta_1\}\frac{b}{2k\left(1-\frac{b}{k}\right)^{3/2}}+D\sqrt{\frac{b}{k}}:=\chi_b(k) \tag{4.26}$$

Hence

$$\begin{aligned}\nu(t,u-y)&\ge\frac{P\left(B(1)>u-y+\chi_b(k)\right)}{P(B(1)>u-y)}=1-\frac{\int_{u-y}^{u-y+\chi_b(k)}e^{-t^2/2}dt}{\int_{u-y}^{\infty}e^{-t^2/2}dt}\\ &\ge 1-\frac{\chi_b(k)}{\int_\beta^\infty e^{-t^2/2}dt}=1-\frac{\chi_b(k)}{\sqrt{2\pi}P(B(1)>\beta)}.\end{aligned}$$

because $u-y<\beta$. From here, as above

$$\gamma_k^{(2)}(u)\ge\left[1-\left(1-\frac{b}{k}\right)^k\right]\left(1-\frac{\chi_b(k)}{\sqrt{2\pi}P(B(1)>\beta)}\right). \tag{4.27}$$

Case 3: $y \ge u+\beta_1$. Now

$$\gamma_k^{(3)}(u) \ge k\int_0^1 P(B(t) > -\beta_1 + D)t^{k-1}dt\,.$$

Choose $\beta_1 > D$. Then

$$P(B(t) > -\beta_1 + D) = P\left(B(1) > \frac{-\beta_1 + D}{\sqrt{t}}\right) > P(B(1) > -\beta_1 + D)$$

for all $0 < t < 1$. Hence

$$\gamma_k^{(3)}(u) \ge P(B(1) > -\beta_1 + D)\,. \tag{4.28}$$

Now we are able to finish the proof of the proposition. Fix $\delta \in (0,1)$ and choose $b > 0$ such that $e^{-b} < \delta$, and $k_0 \in \mathbf{N}$ for which

$$1 - \left(1 - \frac{b}{k}\right)^k > (1-\delta)^2 \quad \text{for } k > k_0\,.$$

Choose now β under the condition

$$\frac{\beta^2}{\beta(\beta + D)} > 1 - \delta\,.$$

According to (4.24), $\xi_b(x) \to 1$ as $x \to \infty$ for each $b > 0$. Hence, for choosen b there is $u_0 > 0$ such that $\xi_b(\min\{k,u\}) > 1-\delta$ if $\min\{k,u\} > u_0$. So, (4.25) implies

$$\gamma_k^{(1)}(u) > (1-\delta)^4 \quad \text{for } k > \max\{k_0, u_0\} \text{ and } u > u_0\,.$$

Further, (4.28) allows us to find β_1 such that $\gamma_k^{(3)}(u) > (1-\delta)$. According to (4.27) and (4.26) for choosen b, β and β_1 there exists $k_1 > k_0$ such that $\gamma_k^{(2)}(u) > (1-\delta)^3$ for all $k > k_1$. Finally,

$$\gamma_k(u) \ge \min\left\{\gamma_k^{(1)}(u)\,,\, \gamma_k^{(2)}(u)\,,\, \gamma_k^{(3)}(u)\right\} > (1-\delta)^4$$

for $k > T_0 = \max\{k_1, u_0\}$ and $u > u_0$, and the needed statement follows. □

Corollary 4.4. *Let*

$$H_{22}(u) = \sum_{k=T+1}^{[au]} J(k,u)\,. \tag{4.29}$$

For each $\epsilon > 0$ there are $T_0 \in \mathbf{N}$ and $u_0 > T_0/a$ such that

$$\frac{H_{22}(u)}{G_{22}(u)} > 1 - \epsilon$$

for all $u > u_0$ and $T_0 \le T < [au]$.

Step 5. Now we can proof (4.8). Indeed, (4.12) allows us to write

$$\frac{H(u)}{P(X(1)>u)} \ge \frac{H_{22}(u)}{P(X(1)>u)} = \frac{H_{22}(u)}{G_{22}(u)} \times \frac{1}{\frac{G_{21}(u)}{G_{22}(u)} + 1} \times \frac{1}{\frac{e^{-\lambda}P(B(1)>u) + G_1(u)}{G_2(u)} + 1 + \frac{G_3(u)}{G_2(u)}}\,.$$

Fix $\epsilon \in (0,1)$. Applying Corollary 4.4 we see that the first fraction in the right hand side is greater then $1-\epsilon$ for $u > u_0$ and $T_0 \le T < [au]$, where u_0 and T_0 are constants. It follows from (4.19), (4.18) and (4.15) that there is $T_1 \in \mathbf{N}$ such that $G_{21}(u)/G_{22}(u) < \epsilon$ for $u > T_1$. Choose now $T > \max\{T_0, T_1\}$. For such T, according to (4.13) and (4.14), there is $u_1 > 0$, $au_1 > T$, such that $[e^{-\lambda}P(B(1) > u) + G_1(u)]/G_2(u) < \epsilon$ and $G_3(u)/G_2(u) < \epsilon$ for $u > u_1$. So,

$$\frac{H(u)}{P(X(1) > u)} > \frac{1-\epsilon}{(1+\epsilon)(1+2\epsilon)}$$

for $u > \max\{T,\, u_0,\, u_1\}$. Letting $u \to \infty$ and then $\epsilon \to 0$ we get (4.8). □

Proof of (4.6). Put

$$\widetilde{Q}(u) = P\left(B(1) + Z(1) > u + b\Gamma_\tau\,, \Gamma_\tau \le a_\tau\right), \tag{4.30}$$

$$R(u) = P\left(B(1) + Z(1) > u + b\Gamma_\tau)\right) - \widetilde{Q}(u),$$

where the numbers a_k are given by (3.9).

Let $Q(u)$ be given by (3.10). Denote $C = \lim_{u\to-\infty} Q(u)$ and $F(u) = 1 - C^{-1}Q(u)$. Let W be a random variable with distribution function F, independent of $B(1)$. It can be easily verified, using formulas for Γ-densities, that $P(B(1) + W > u) = C^{-1}\widetilde{Q}(u)$. Now (3.11) and Lemma 3.1 imply that

$$\widetilde{Q}(u) = o\left(P(B(1) + Z(1) > u + b)\right) \tag{4.31}$$

as $u \to \infty$.

Next we show that

$$\limsup_{u\to\infty} \frac{R(u)}{P(B(1) + Z(1) > u + b)} \le 1. \tag{4.32}$$

Fix a constant $a > 0$ and denote

$$R_1(u) = P\left(B(1) + Z(1) > u + b\Gamma_\tau\,,\, \Gamma_\tau > a_\tau\,,\, S_\tau > u - a\log\tau\right), \tag{4.33}$$

$$R_2(u) = R(u) - R_1(u).$$

Estimate for $R_1(u)$. We show here that

$$\limsup_{u\to\infty} \frac{R_1(u)}{P(B(1) + Z(1) > u + b)} \le 1. \tag{4.34}$$

As above, one can easily check the formula

$$R_1(u) = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \int_{a_k}^{1} \frac{(\lambda t)^{k-1}}{(k-1)!} P(B(1) + S_k > u + bt\,,\, S_k > u - a\log k)dt\,, \tag{4.35}$$

and it is clear that

$$P(B(1) + Z(1)) = e^{-\lambda}P(B(1) > u) + e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}P(B(1) + S_k > u)\,. \tag{4.36}$$

Denote

$$\gamma_k(u) = \frac{k\int_{a_k}^1 P(B(1) + S_k > u + bt\,,\, S_k > u - a\log k)t^{k-1}dt}{P(B(1) + S_k > u + b)}\,.$$

Representing the probabilities in this fraction as integrals with respect to F_{S_k}, we obtain the estimate

$$\gamma_k(u) \le \sup_{y > u - a\log k} \frac{P(B(1) > u + a_k b - y)}{P(B(1) > u + b - y)}\,. \tag{4.37}$$

Step 1. Fix $\epsilon > 0$ and choose $A > 0$ such that

$$\frac{x^{-1}}{x^{-1} - x^{-2}} = \frac{x}{x-1} < 1 + \epsilon$$

for $x > A$. Then, using (3.13) we see that if $y < u - A$, then

$$\begin{aligned}\frac{P(B(1) > u + a_k b - y)}{P(B(1) > u + b - y)} &\le (1+\epsilon)\exp\left(-\frac{(u + a_k b - y)^2}{2} + \frac{(u+b-y)^2}{2}\right) \\ &= (1+\epsilon)\exp\left(b(1-a_k)(u-y) + \frac{b^2(1-a_k^2)}{2}\right) \\ &\le (1+\epsilon)\exp\left(b(1-a_k)a\log k + \frac{b^2(1-a_k^2)}{2}\right),\end{aligned}$$

because $u - y < a\log k$. Taking into account that $1 - a_k \le mk^{-1}\log k$, we conclude that there is an index k_0 such that

$$\gamma_k^{(1)} := \sup_{u - a\log k < y < u - A} \frac{P(B(1) > u + a_k b - y)}{P(B(1) > u + b - y)} \le (1+\epsilon)^2. \tag{4.38}$$

Step 2. If $y > u - A$, then

$$\frac{P(B(1) > u + a_k b - y)}{P(B(1) > u + b - y)} = 1 + \frac{\int_{u+a_k b - y}^{u+b-y} e^{-x^2/2}dx}{\int_{u+b-y}^{\infty} e^{-x^2/2}dx} \le \frac{b(1-a_k)}{\int_{A+b}^{\infty} e^{-x^2/2}dx}.$$

So, we can find k_1 such that for choosen A the last expression is less that $1 + \epsilon$ for $k > k_1$. Therefore, for given $\epsilon > 0$ there is an index k_2 such that $\gamma_k(u) < (1+\epsilon)^2$ for all $k > k_2$ and $u > 0$.

Denote by $\widetilde{R}_1(u)$ the sum of summands from (4.35) over $k > k_2$. Then

$$\limsup_{u\to\infty} \frac{\widetilde{R}_1(u)}{P(B(1) + Z(1) > u + b)} \le (1+\epsilon)^2. \tag{4.39}$$

We have

$$R_1(u) - \widetilde{R}_1(u) \le e^{-\lambda}\sum_{k=1}^{k_2} \frac{\lambda^k}{k!} P(B(1) + S_k > u).$$

On the other hand, for each j

$$\begin{aligned}P(B(1) + Z(1) > u + b) &> e^{-\lambda}\frac{\lambda^{k_2+j+1}}{(k_2+j+1)!} P(B(1) + S_{k_2+j+1} > u + b) \\ &\ge e^{-\lambda}\frac{\lambda^{k_2+j+1}}{(k_2+j+1)!} P(B(1) + S_{k_2+1} > u) P(S_j > b).\end{aligned}$$

Choosing j under the condition $P(S_j > b) > 0$ and applying Lemma 3.3 we conclude that

$$\lim_{u\to\infty} \frac{R_1(u) - \widetilde{R}_1(u)}{P(B(1) + Z(1) > u + b)} = 0.$$

Now (4.34) follows from here and (4.39).
Estimate for $R_2(u)$. We show here that

$$\lim_{u\to\infty} \frac{R_2(u)}{P(B(1) + Z(1) > u + b)} = 0. \tag{4.40}$$

The probability $R_2(u)$ admittes a representation similar to (4.35). Denoting by $g_k(u)$ the corresponding summands we get

$$g_k(u) \le e^{-\lambda} \frac{\lambda^k}{k!} P(B(1) + S_k > u \, , \, S_k \le u - a \log k) \, .$$

Hence

$$\delta_k(u) := \frac{g_k(u)}{e^{-\lambda} \frac{\lambda^{k+m}}{(k+m)!} P(B(1) + S_{k+m} > u + b)}$$
$$\le \frac{(k+1)\cdots(k+m)}{\lambda^m} \frac{P(B(1) + S_k > u \, , \, S_k \le u - a \log u)}{P(B(1) + S_k > u - 1) P(S_m > b + 1)} \, ,$$

where m is choosen under the condition $P(S_m > b + 1) > 0$. Once again representing the probabilities as integrals with respect to F_{S_k} and using (3.13) we can find an index k_1 such that

$$\frac{P(B(1) + S_k > u \, , \, S_k \le u - a \log u)}{P(B(1) + S_k > u - 1)} \le 2 \sup_{y \le u - a \log k} \exp\left(-(u - y) + \frac{1}{4} \right)$$
$$\le 2e^{\frac{1}{4}} \exp(-a \log k) = 2e^{\frac{1}{4}} k^{-a}$$

for $k > k_1$ and $u > 0$, which yields $\delta_k(u) \le C k^{m-a}$. So,

$$\frac{\sum_{k=k_k+1}^{\infty} g_k(u)}{P(B(1) + Z(1) > u + b)} \le C k_1^{m-a} .$$

As in previous case,

$$\lim_{u \to \infty} \frac{\sum_{k=1}^{k_1} g_k(u)}{P(B(1) + Z(1) > u + b)} = 0 \, .$$

Therefore,

$$\limsup_{u \to \infty} \frac{R_2(u)}{P(B(1) + Z(1) > u + b)} \le C k_1^{m-a} \, .$$

Since m does not depend on a, we may choose $a > m$. Then letting $k_1 \to \infty$ we obtain (4.40). Now (4.32) follows from (4.33), (4.34) and (4.40).

According (4.30)–(4.32)

$$\limsup_{u \to \infty} \frac{P(B(1) + Z(1) > u + b\Gamma_\tau)}{P(B(1) + Z(1) > u + b)} \le 1 \, .$$

Since $P(B(1) + Z(1) > u + b\Gamma_\tau) \ge P(B(1) + Z(1) > u + b)$, (4.6) follows. □

4.2. **Proof of (4.2).** Because

$$\sup_{0 \le t < \Gamma_\tau} X(t) \le \sup_{0 \le t \le 1} B(t) + \sup_{0 \le t < \Gamma_\tau} [Z(t) - bt] \, ,$$

we get, once again applying Lévy formula (4.3) and Lemma 3.4,

$$A(u) \le 2P\left(B(1) + \sup_{0 \le t < \Gamma_\tau} [Z(t) - bt] > u \right) .$$

It was shown in [4], pp. 149–151, that

$$P\left(\sup_{0 \le t < \Gamma_\tau} [Z(t) - bt] > u \right) = o\left(P(Z(\Gamma_\tau) - b\Gamma_\tau > u) \right)$$

as $u \to \infty$. Hence, according to Lemma 3.1 $A(u) = o\left(P(B(1)+Z(\Gamma_\tau) - b\Gamma_\tau > u)\right)$. Since $Z(\Gamma_\tau) = Z(1)$, (4.6) yields now (4.2). □

5. Proof of Theorem 2.2.

As above, (2.4) and Lemma 3.5 allow us to prove the theorem for processes of the type (1.2). Relation (2.6) holds for compound Poisson processes with non-negative drifts and light tails (see Theorem 1 from [4]). Hence we may assume that $b > 0$. The proof is divided into a series of lemmas.

Lemma 5.1. *Assume (2.7) holds for iid random variables X_k and put*

$$G(u) = \sum_{k=2}^{\infty} \frac{\lambda^k P(S_{k-1} > u)}{k!}\,. \tag{5.1}$$

Let

$$Z = \sum_{k=1}^{N} X_k\,. \tag{5.2}$$

where N is a Poisson random variable with parameter λ, independent of X_k. Then

$$\lim_{u\to\infty} \frac{G(u)}{P(Z > u+b)} = 0$$

for each $b > 0$.

Proof. Because $P(X_1 + X_2 > u + b) \geq P(X_1 > u - a)P(X_2 > a + b)$, relation (2.7) implies

$$\lim_{u\to\infty} \frac{P(X_1 > u)}{P(X_1 + X_2 > u + b)} = 0\,. \tag{5.3}$$

Fix a constant $A > 0$. Then

$$G(u) \leq \lambda P(X_1 > u) + \sum_{k=2}^{\infty} \frac{\lambda^k P(S_{k-1} > u\,,\, S_{k-2} \leq u - A)}{k!} \tag{5.4}$$

$$+ \sum_{k=2}^{\infty} \frac{\lambda^k P(S_{k-2} > u - A)}{k!} := \lambda P(X_1 > u) + G_1(u) + G_2(u)\,.$$

Further,

$$h(A,k,u) := \frac{P(S_{k-1} > u\,,\, S_{k-2} \leq u - A)}{P(S_k > u + b)}$$

$$\leq \frac{\int_{-\infty}^{u-A} P(X_1 > u - y) F_{S_{k-2}}(dy)}{\int_{-\infty}^{u-A} P(X_1 + X_2 > u - y + b) F_{S_{k-2}}(dy)} \leq \sup_{y \leq u-A} \frac{P(X_1 > u - y)}{P(X_1 + X_2 > u - y + b)}\,,$$

and it follows from (5.3), that for a fixed $\epsilon > 0$ one can find A such that $h(A,k,u) < \epsilon$ for all $k \geq 3$ and positive u. This yields the estimate $G_1(u) < \epsilon P(Z > u + b)\,,\, u > 0$.

Turn now to $G_2(u)$. We have for a fixed index $M > 2$

$$G_2(u) \leq \frac{1}{P(X_1 > A + b)} \left(\sum_{k=2}^{M} \frac{\lambda^k P(S_{k-1} > u + b)}{k!} + \sum_{k=M+1}^{\infty} \frac{\lambda^k P(S_{k-1} > u + b)}{k!} \right)\,,$$

and (3.4) implies that the first sum is $o(P(Z > u+b))$ as $u \to \infty$. The second sum is bounded from above by

$$\frac{1}{M+1}\sum_{k=M+1}^{\infty}\frac{\lambda^k P(S_{k-1} > u+b)}{(k-1)!} \le \frac{\lambda e^{\lambda}}{M+1}P(Z > u+b)\,.$$

So, letting first $u \to \infty$ and then $M \to \infty$ we conclude that $G_2(u) = o(P(Z > u+b))$ as $u \to \infty$ for each $A > 0$. From here and the previous

$$\limsup_{u\to\infty}\frac{G(u)}{P(Z > u+b)} \le \epsilon$$

for each $\epsilon > 0$, which yields the lemma. □

Because the function h is increasing and continuous, there exists the inverse function h^{-1}. Put for $s > 1$

$$\psi(s) = h^{-1}\left(\frac{4}{b}\log s\right) \tag{5.5}$$

and

$$g(k,u) = u - \psi(k)\,. \tag{5.6}$$

Lemma 5.2. *There is an index k_0 such that*

$$\gamma(k,u) := \frac{\frac{\lambda^k}{k!}P(S_k > u + ba_k\,,\ S_{k-1} \le g(k,u))}{\frac{\lambda^{k+1}}{(k+1)!}P(S_{k+1} > u+b)} \le \frac{C}{\lambda k}$$

for all $k > k_0$ and $u > 0$, where the constant C is independent of these parameters.

Proof. We have

$$\begin{aligned}\gamma(k,u) &\le \frac{k+1}{\lambda}\frac{P(S_k > u+ba_k\,,\ S_{k-1}\le g(k,u))}{P(S_k > u\,,\ X_{k+1} > b)}\\ &\le \frac{k+1}{\lambda P(X_1 > b)}\frac{\int_{-\infty}^{g(k,u)}P(X_1 > u+ba_k-y)F_{S_{k-1}}(dy)}{\int_{-\infty}^{g(k,u)}P(X_1 > u-y)F_{S_{k-1}}(dy)}\\ &\le \frac{k+1}{\lambda P(X_1 > b)}\sup_{y\le g(k,u)}\frac{P(X_1 > u+ba_k-y)}{P(X_1 > u-y)}\,.\end{aligned}$$

Because

$$u - y \ge u - g(k,u) = \psi(k) > 0\,, \tag{5.7}$$

we may apply (2.8). Hence for $y \le g(k,u)$

$$\frac{P(X_1 > u+ba_k-y)}{P(X_1 > u-y)} = \exp\left(-\int_{u-y}^{u+ba_k-y}h(v)dv\right) \le \exp\left(-ba_k h(u-y)\right)\,.$$

Taking into account (5.7) and (5.5) we see that $h(u-y) \ge h(\psi(k)) = 4\log k/b$. Formula (2.8) implies $m = 1$ in (3.8). So $a_k = 1 - \frac{2\log k}{k} > \frac{1}{2}$ for k large enough, and, therefore, $ba_k h(u-y) \ge 2\log k$. From here and the previous estimates

$$\gamma(k,u) \le \frac{k+1}{\lambda P(X_1 > b)}\frac{1}{k^2}$$

for k large enough, and the lemma follows. □

Denote

$$I(k,u)=\lambda\int_{a_k}^{1}\frac{(\lambda t)^{k-1}}{(k-1)!}P(S_k>u+bt\,,\,S_{k-1}\le g(k,u))dt\,. \tag{5.8}$$

Since

$$I(k,u)\le\frac{\lambda^k}{k!}P(S_k>u+ba_k\,,\,S_{k-1}\le g(k,u))\,,$$

we immediately obtain the following statement.

Corollary 5.3. *There is an index k_0 such that*

$$\frac{I(k,u)}{\frac{\lambda^{k+1}}{(k+1)!}P(S_{k+1}>u+b)}\le\frac{C}{\lambda k}$$

for all $k>k_0$ and $u>0$, where the constant C is independent of these parameters.

Denote

$$J(k,u)=\lambda\int_{a_k}^{1}\frac{(\lambda t)^{k-1}}{(k-1)!}P(S_k>u+bt\,,\,g(k,u)<S_{k-1}\le u)dt\,. \tag{5.9}$$

Lemma 5.4. *For each positive ϵ there is an index k_1 such that*

$$\beta(k,u):=\frac{J(k,u)}{\frac{\lambda^k}{k!}P(S_k>u+b)}\le 1+\epsilon$$

for all $k>k_1$ and $u>0$.

Proof. We have

$$\beta(k,u)\le\frac{\lambda\int_{g(k,u)}^{u}\int_{a_k}^{1}\frac{(\lambda t)^{k-1}}{(k-1)!}P(X_1>u+bt-y)dtF_{S_{k-1}}(dy)}{\frac{\lambda^k}{k!}\int_{g(k,u)}^{u}P(X_1>u+b-y)F_{S_{k-1}}(dy)} \tag{5.10}$$

$$\le\sup_{g(k,u)<y\le u}\frac{k\int_{a_k}^{1}P(X_1>u+bt-y)t^{k-1}dt}{P(X_1>u+b-y)}:=\alpha(k,u)\,.$$

It follows from (2.8) that

$$\nu(k,u,y):=\frac{k\int_{a_k}^{1}P(X_1>u+bt-y)t^{k-1}dt}{P(X_1>u+b-y)} \tag{5.11}$$

$$=k\int_{a_k}^{1}\exp\left(\int_{u+bt-y}^{u+b-y}h(v)dv\right)t^{k-1}dt=k\int_{a_k}^{1}\exp\left(b(1-t)h(v(u,y,t)\right)t^{k-1}dt\,,$$

where

$$u+bt-y<v(u,y,t)<u+b-y\,.$$

Because $u+b-y<u+b-g(k,u)=\psi(k)+b$, we get, using (2.10),

$$h(v(u,y,t))\le h(\psi(k)+b)=h\left(h^{-1}\left(\frac{4\log k}{b}\right)+b\right)\le\exp\left(\frac{b}{8}\frac{4\log k}{b}\right)=\sqrt{k}$$

for k large enough. Since $a_k\le t<1$, we see, taking into account (3.9) that

$$b(1-t)h(v(u,y,t))\le\frac{2b\log k}{k}\sqrt{k}\ \to 0$$

as $k \to \infty$. So, (5.11) and the last estimates imply that there is an index k' such that $\nu(k,u,y) < 1+\epsilon$ for all $k > k'$, $u > 0$ and $g(k,u) < y < u$, which yields $\alpha(k,u) < 1+\epsilon$ for $k > k'$ and $u > 0$, and the lemma follows.

□

Proof of Theorem 2.2. According to Theorem 1 from [4] it is enough to show that

$$\lim_{u\to\infty} \frac{P(Z(1) > u + b\Gamma_\tau)}{P(X(1) > u)} = 1\,. \tag{5.12}$$

Applying (3.10), (5.1), (5.8) and (5.9) we may write

$$P(Z(1) > u + b\Gamma_\tau) \le Q(u) + e^{-\lambda}\sum_{k=1}^{\infty} I(k,u) + e^{-\lambda}\sum_{k=1}^{\infty} J(k,u) + e^{-\lambda}G(u)\,. \tag{5.13}$$

We show first that

$$\lim_{u\to\infty} \frac{\sum_{k=1}^{\infty} I(k,u)}{P(X(1) > u)} = 0\,. \tag{5.14}$$

Since $I(k,u) \le \lambda^k P(S_k > u)/k!$, relations (3.12) and (3.4) yield

$$\lim_{u\to\infty} \frac{\sum_{k=1}^{M} I(k,u)}{P(X(1) > u)} = 0$$

for each fixed index M. Choosing $M > k_0$, where k_0 is from Corollary 5.3, we conclude that

$$\frac{\sum_{k=M+1}^{\infty} I(k,u)}{P(X(1) > u)} \le \frac{C}{\lambda(M+1)}$$

for all $u > 0$. Letting first $u \to \infty$ and then $M \to \infty$ we come to (5.14).

Now we show that

$$\limsup_{u\to\infty} \frac{e^{-\lambda}\sum_{k=1}^{\infty} J(k,u)}{P(X(1) > u)} \le 1. \tag{5.15}$$

For a fixed $\epsilon > 0$ Lemma 5.4 and (3.12) provide us with an index k_1 such that

$$\frac{e^{-\lambda}\sum_{k=k_1+1}^{\infty} J(k,u)}{P(X(1) > u)} < 1+\epsilon$$

for all $u > 0$. As above, (3.12) and (3.4) lead to the equality

$$\lim_{u\to\infty} \frac{\sum_{k=1}^{k_1} J(k,u)}{P(X(1) > u)} = 0.$$

The last two relations imply (5.15).

Now (5.13), (5.14) , (5.15) and Lemmas 3.6 and 5.1 give us

$$\limsup_{u\to\infty} \frac{P(Z(1) > u + b\Gamma_\tau)}{P(X(1) > u)} \le 1\,. \tag{5.16}$$

Obviously, $P(Z(1) > u + b\Gamma_\tau) \ge P(X(1) > u)$ for all $u > 0$, and we come to (5.12). □

6. Proof of Theorem 2.3.

First we prove the following

Proposition 6.1. *If for X_k the condition (2.7) holds, then it also holds for random variable (5.2).*

The proof is based on the next statement.

Lemma 6.2. *Assume X_k satisfy (2.7). Then for each $\epsilon > 0$ there is $B > 0$ such that*

$$P(S_k > u + a) \leq \epsilon P(S_k > u) + \frac{P(S_{k-1} > u)}{P(X_1 > B)}$$

for all $k \geq 3$ and $u > 0$.

Proof. Fix $A > 0$. Then

$$P(S_k > u + a) \leq \int_{-\infty}^{u-A} P(X_1 > u + a - t) F_{S_{k-1}}(dt) + P(S_{k-1} > u - A)\,.$$

Because of (2.7), there is $A_0 > 0$ such that $P(X_1 > u + a - t)/P(X_1 > u - t) < \epsilon/2$ for all $A > A_0$ and $t < u - A$. Then

$$P(S_k > u + a) \leq \frac{\epsilon}{2} P(S_k > u) + P(S_{k-1} > u - A)\,. \tag{6.1}$$

Fix now $A > A_0$. We have for a positive B:

$$P(S_{k-1} > u - A) \leq \int_{-\infty}^{u-B} P(X_1 > u - A - t) F_{S_{k-2}}(dt) + P(S_{k-2} > u - B)\,. \tag{6.2}$$

Further, $P(X_1 + X_2 > u - t) \geq P(X_1 > u - t - 2A)P(X_1 > 2A)$, and $t \leq u - B$ implies $u - A - t \geq B - A$. Hence, once again applying (2.7), we can choose B so large that $P(X_1 > u - A - t)/P(X_1 + X_2 > u - t) < \epsilon/2$ if $t \leq u - B$. Therefore,

$$\int_{-\infty}^{u-B} P(X_1 > u - A - t) F_{S_{k-2}}(dt) \leq \frac{\epsilon}{2} \int_{-\infty}^{\infty} P(X_1 + X_2 > u - t) F_{S_{k-2}}(dt) = \frac{\epsilon}{2} P(S_k > u)\,.$$

Since $P(S_{k-2} > u - B) \leq P(S_{k-1} > u)/P(X_1 > B)$, the lemma follows from here, (6.2) and (6.1). □

Proof of Proposition 6.1. According to Lemma 6.2, for a fixed $\epsilon > 0$ there is $B > 0$ such that

$$P(Z > u + a) \leq e^{-\lambda} \left[\lambda P(S_1 > u + a) + \frac{\lambda^2 P(S_2 > u + a)}{2!} \right] + \epsilon e^{-\lambda} \sum_{k=3}^{\infty} \frac{\lambda^k P(S_k > u)}{k!} + \frac{e^{-\lambda}}{P(X_1 > B)} \sum_{k=3}^{\infty} \frac{\lambda^k P(S_{k-1} > u)}{k!}\,. \tag{6.3}$$

We show first the the last sum is $o(P(Z > u))$ as $u \to \infty$. To this end fix an index $m > 3$. Then

$$\psi(u) := \sum_{k=3}^{\infty} \frac{\lambda^k P(S_{k-1} > u)}{k!} \leq \sum_{k=3}^{m} \frac{\lambda^k P(S_{k-1} > u)}{k!} + \frac{1}{m+1} \sum_{k=m+1}^{\infty} \frac{m+1}{k} \frac{\lambda^k P(S_{k-1} > u)}{(k-1)!}\,.$$

Taking into account (3.4) we see that

$$\limsup_{u \to \infty} \frac{\psi(u)}{P(Z > u)} \leq \frac{\lambda e^{\lambda}}{m+1}\,,$$

and letting $m \to \infty$ we come to the needed conclusion.

Now, (6.3) and (3.4) yield that

$$\limsup_{u\to\infty} \frac{P(Z > u + a)}{P(Z > u)} \le \epsilon$$

for each $\epsilon > 0$. So, the proposition follows. □

Proof of Theorem 2.3. The proof is a word for word repetition of the proof of Theorem 2 from [4]. To obtain formula (32) from this paper one should use Proposition 6.1.

7. Proof of Theorem 2.5

Let $\{X_k\}_{k=1}^{\infty}$ be iid random variables taking values $n!\,, n = 1, 2, \dots$, such that

$$P(X_1 = n!) = \frac{1}{(e-1)n!}\,. \tag{7.1}$$

Denote by $Z(t)$ a compound Poisson process with parameter $\lambda = 1$ and jumps X_k. It will be shown below that (2.14) holds for the sequence $u_n = n!$ and (2.15) holds for the sequence $u_n = n \cdot n!$.

7.1. **Estimates for sums.** Here we obtain asymptotics for probabilities $P(S_k + B(1) > n!)$ and $P(S_k + B(1) > n \cdot n!)$ as $n \to \infty$.

Lemma 7.1. *For* $2 \le k \le n$

$$P(S_k + B(1) > n!) = kP(X_1 = n!) \int_{-\infty}^{\infty} P(S_{k-1} > -t)\phi(t)dt + \frac{\alpha(k,n)}{(n+1)!}\,,$$

where ϕ *is* $(0,1)$*-normal density function and* $\sup_{2\le k\le n} |\alpha(k,n)| < \infty$.

Proof. We represent the considered probability as

$$P(S_k + B(1) > n!) = \left(\int_{-\infty}^{0} + \int_{0}^{n} + \int_{n}^{\infty}\right) P(S_k > n! - t)\phi(t)dt := I_1 + I_2 + I_3\,, \tag{7.2}$$

and start with the integral I_1. Assume first that $k = n$. The condition

$$\max\{X_1, \dots, X_n\} \le (n-1)!$$

implies $S_n \le n! - t$ for $t \le 0$. Hence,

$$\begin{aligned} P(S_n > n! - t) &= P\left(S_n > n! - t\,, \max\{X_1, \dots, X_n\} \ge n!\right) \\ &= P\left(S_n > n! - t\,, \text{exactly one of } X_1, \dots, X_n \text{ is non-less than } n!\right) \\ &\quad + P\left(S_n > n! - t\,, \text{at least two of } X_1, \dots, X_n \text{ are non-less than } n!\right) := p + q. \end{aligned}$$

Since X_k are iid random variables,

$$\begin{aligned} p &= nP\left(S_n > n! - t\,, X_1 \ge n!\,, \max\{X_2, \dots, X_n\} \le (n-1)!\right) \\ &= nP\left(S_n > n! - t\,, X_1 = n!\right) - nP\left(S_n > n! - t\,, X_1 = n!\,, \max\{X_2, \dots, X_n\} \ge n!\right) \\ &\quad + nP\left(S_n > n! - t\,, X_1 \ge (n+1)!\,, \max\{X_2, \dots, X_n\} \le (n-1)!\right) := n(p_1 - p_2 + p_3)\,. \end{aligned}$$

Obviously, $p_1 = P(X_1 = n!)P(S_{n-1} > -t)$, and according to (7.1),

$$p_2 \le P(X_1 = n!)P\left(\max\{X_2, \dots, X_n\} \ge n!\right) \le C \frac{1}{n!}\frac{n-1}{n!}$$

and

$$p_3 \leq P(X_1 \geq (n+1)!) \leq \frac{C_1}{(n+1)!},$$

It is clear that

$$q \leq n^2 \left[P(X_1 \geq n!)\right]^2 \leq \frac{C_2}{[(n-1)!]^2}.$$

From here

$$I_1 = nP(X_1 = n!) \int_{-\infty}^{0} P(S_{n-1} > -t)\phi(t)dt + O\left(\frac{1}{(n+1)!}\right). \tag{7.3}$$

Turn now to the integral I_2. For $0 < t < n$

$$P(S_n > n! - t) = P\left(S_n > n! - t\,,\, \max\{X_1, \ldots, X_n\} \geq n!\right)$$
$$+P\left(S_n > n! - t\,,\, \max\{X_1, \ldots, X_n\} \leq (n-1)!\right) := \widetilde{p} + \widetilde{q}\,,$$

and by the same reasons as above $\widetilde{p} = nP(X_1 = n!)P(S_{n-1} > -t) + O\left(1/(n+1)!\right)$. Further,

$$\widetilde{q} = P(S_n > n! - t\,, X_1 = \cdots = X_n = (n-1)!)$$
$$+P\left(S_n > n! - t\,, \max\{X_1, \ldots, X_n\} = (n-1)!\,,\, \min\{X_1, \ldots, X_n\} \leq (n-2)!\right)$$
$$+P\left(S_n > n! - t\,, \max\{X_1, \ldots, X_n\} \leq (n-2)!\right) := \widetilde{q}_1 + \widetilde{q}_2 + \widetilde{q}_3\,.$$

If $\max\{X_1, \ldots, X_n\} = (n-1)!$ and $\min\{X_1, \ldots, X_n\} \leq (n-2)!$, then

$$S_n \leq (n-1)\cdot(n-1)! + (n-2)! = n! - (n-2)\dot{(}n-2)! < n! - t$$

for $t < n$. So, $\widetilde{q}_2 = 0$. By similar reasons $\widetilde{q}_3 = 0$ and

$$\widetilde{q}_1 \leq \left[P(X_1 = (n-1)!)\right]^n .$$

Hence

$$I_2 = nP(X_1 = n!) \int_{0}^{n} P(S_{n-1} > -t)\phi(t)dt + O\left(\frac{1}{(n+1)!}\right).$$

Since

$$I_3 = O\left(\exp\left(-\frac{n^2}{2}\right)\right) = o\left(\frac{1}{(n+2)!}\right) \tag{7.4}$$

the last relations and (7.2) yield the lemma for $k = n$.

The case $2 \leq k < n$ is treated by the similar way. □

Remark 7.2. The same reasons give us

$$P(S_1 + B(1) > n!) = O\left(\frac{1}{(n+1)!}\right). \tag{7.5}$$

Lemma 7.3. *For* $2 \leq k \leq n$

$$P(S_k + B(1) > n \cdot n!) = kP(X_1 = (n+1)!) + \frac{\beta(k,n)}{(n+2)!},$$

where $\sup_{2 \leq k \leq n} |\beta(k,n)| < \infty$.

Proof. Since the conditions $\max\{X_1, \ldots, X_n\} = n!$ and $\min\{X_1, \ldots, X_n\} \le (n-1)!$ imply $S_n \le (n-1)n! + (n-1)! = n \cdot n! - (n-1)(n-1)! < n \cdot n! - n$, then for $t < n$

$$P(S_n > n \cdot n! - t) = P\left(S_n > n \cdot n! - t\,,\ X_1 = \cdots = X_n = n!\right)$$
$$+P\left(S_n > n \cdot n! - t\,,\ \max\{X_1, \ldots, X_n\} \ge (n+1)!\right)\,.$$

From here, as in the proof of the previous lemma,

$$P(S_n > n \cdot n! - t) = nP(X_1 = (n+1)!)P(S_{n-1} > -n! - t) + O\left(\frac{1}{(n+2)!}\right)\,,$$

because $n \cdot n! - (n+1)! = -n!$. In the case $2 \le k < n$ we get similarly for $t < n$

$$P(S_k > n \cdot n! - t) = kP(X_1 = (n+1)!)P(S_{k-1} > -n! - t) + \frac{\mu(k,n,t)}{(n+2)!}\,, \tag{7.6}$$

where $\sup_{2\le k<n\,;t<n} |\mu(k,n,t)| < \infty$. These equalities and (7.4) give us

$$P(S_k + B(1) > n \cdot n!) = kP(X_1 = (n+1)!)\int_{-\infty}^{\infty} P(S_{k-1} > -n! - t)\phi(t)dt + \frac{\widetilde{\mu}(k,n)}{(n+2)!} \tag{7.7}$$

and $\sup_{2\le k\le n} |\widetilde{\mu}(k,n)| < \infty$. Since X_k are positive, we have for $t > -n!$

$$P(S_{k-1} > -n! - t) = 1\,. \tag{7.8}$$

Hence, the last integral is

$$\int_{-\infty}^{-n!} P(S_{k-1} > -n! - t)\phi(t)dt + P(B(1) > -n!) = 1 + o\left(\frac{1}{(n+3)!}\right)\,.$$

From here and the previous relations the lemma follows. □

Remark 7.4. By the same way we obtain

$$P(S_1 + B(1) > n \cdot n!) = P(X_1 = (n+1)!)P(B(1) > -n!) + O\left(\frac{1}{(n+2)!}\right) \tag{7.9}$$
$$= P(X_1 = (n+1)!) + O\left(\frac{1}{(n+2)!}\right)$$

7.2. **Estimates for $X(1)$.** Here we find asymptotics for the probabilities $P(X(1) > n!)$ and $P(X(1) > n \cdot n!)$.

Lemma 7.5. *The following hold:*

$$P(X(1) > n!) = P(X_1 = n!)\sum_{k=2}^{n} I_k + O\left(\frac{1}{(n+1)!}\right)\,, \tag{7.10}$$

where

$$I_k = \frac{1}{e(k-1)!}\int_{-\infty}^{\infty} P(S_{k-1} > -t)\phi(t)dt\,. \tag{7.11}$$

Proof. We can write the considered probability as a sum of three sums:

$$P(X(1) > n!) = e^{-1}\left[P(B(1) > n!) + P(S_1 + B(1)) > n!\right]$$

$$+e^{-1}\sum_{k=2}^{n}\frac{P(S_k + B(1) > n!)}{k!} + e^{-1}\sum_{n+1}^{\infty}\frac{P(S_k + B(1) > n!)}{k!},$$

and (7.5) implies that the first sum is $O\left(1/(n+1)!\right)$. The same is true for the third sum. As for the second one, Lemma 7.1 yields that it is

$$P(X_1 = n!)\sum_{k=2}^{n}\frac{1}{e(k-1)!}\int_{-\infty}^{\infty}P(S_{k-1} > -t)\phi(t)dt + O\left(\frac{1}{(n+1)!}\right)$$

and (7.10) follows. □

Lemma 7.6. *The following relation holds:*

$$P(X(1) > n\cdot n!) = P(X_1 = (n+1)!) + O\left(\frac{1}{(n+2)!}\right). \tag{7.12}$$

Proof. We can write, using (7.9) and Lemma 7.3,

$$P(X(1) > n\cdot n!) = e^{-1}P(B(1) > n\cdot n!) + P(X_1 = (n+1)!)e^{-1}\sum_{k=1}^{n}\frac{1}{(k-1)!}$$

$$+e^{-1}\frac{P(S_{n+1} > n\cdot n!)}{(n+1)!} + O\left(\frac{1}{(n+2)!}\right).$$

Since the condition $\max\{X_1,\dots,X_{n+1}\} \le (n-2)!$ implies $S_{n+1} \le (n+1)(n-2)! < n\cdot n!$, we see that

$$P(S_{n+1} > n\cdot n!) = P\left(S_{n+1} > n\cdot n!\,,\ \max\{X_1,\dots,X_{n+1}\} \ge (n-1)!\right) \le \frac{C(n+1)}{(n-1)!}. \tag{7.13}$$

So, the needed relation follows. □

7.3. **Proof of (2.14).** We have

$$P\left(\sup_{0\le t\le 1} X(t) > n!\right) \ge P(X(1) > n!) + P(X(1) \le n!\,,\ X(\Gamma_{\tau-1}) > n!)\,, \tag{7.14}$$

where τ is given by (3.7). Further,

$$p_k(n!) := P(\tau = k\,,\ X(1) \le n!\,,\ X(\Gamma_{k-1}) > n!)$$

$$= P\left(\tau = k\,,\ S_k + B(1) \le n!\,,\ S_{k-1} + B(\Gamma_{k-1}) > n!\right)$$

Because $B(t)$ is symmetric and independent of $Z(t)$,

$$p_k(n!) \ge \frac{1}{2}P\left(\tau = k\,,\ S_k + B(\Gamma_{k-1}) \le n!\,,\ S_{k-1} + B(\Gamma_{k-1}) > n!\right) := \frac{1}{2}q_k(n!)\,. \tag{7.15}$$

Elementary calculations give us

$$q_k(n!) = \frac{1}{e(k-2)!}\int_0^1 P\left(S_k + B(y) \le n!\,,\ S_{k-1} + B(y) > n!\right)y^{k-1}(1-y)dy\,.$$

Assume now that $3 \le k \le n$. The same reasons as above and the well known formula for the density of $B(y)$ imply

$$q_k(n!) = (k-1)P(X_1 = n!) \tag{7.16}$$

$$\times \frac{1}{e(k-2)!}\int_0^1 \left[\int_{-\infty}^{\infty} P(S_{k-1} \le -t\,,\, S_{k-2} > -t)\frac{1}{\sqrt{2\pi y}}e^{-\frac{t^2}{2y}}dt\right] y^{k-1}(1-y)dy$$

$$+\frac{\nu(k,n)}{(n+1)!} := (k-1)P(X_1 = n!)J_k + \frac{\nu(k,n)}{(n+1)!}\,,$$

where $\sup_{3\le k\le n}|\nu(k,n)| < \infty$. Because the jumps X_k are positive, the inner integral coinsides with the integral over $(-\infty,-1)$, and it is positive. So,

$$J_k > 0 \quad \text{for all } k \ge 3 \tag{7.17}$$

and (7.14) and (7.15) imply

$$P\left(\sup_{0\le t\le 1} X(t) > n!\right) \ge P(X(1) > n!) + \frac{1}{2}P(X_1 = n!)\sum_{k=3}^{n}(k-1)J_k + O\left(\frac{1}{(n+1)!}\right). \tag{7.18}$$

According to (7.11) $\sum_{k=2}^{\infty} I_k \le 1$. From here, (7.18), (7.1), (7.10) and (7.17)

$$\liminf_{n\to\infty}\frac{P\left(\sup_{0\le t\le 1} X(t) > n!\right)}{P(X(1) > n!)} \ge 1 + \frac{\frac{1}{2}\sum_{k=3}^{\infty}(k-1)J_k}{\sum_{k=2}^{\infty} I_k} > 1\,,$$

and (2.14) follows. □

7.4. **Proof of (2.15).** Using (4.3) and the positivity of $Z(t)$ we may write

$$P\left(\sup_{0\le t\le 1} X(t) > n\cdot n!\right) \le P\left(Z(1) + |B(1)| > n\cdot n!\right). \tag{7.19}$$

Applying (7.6) and (7.8) we see that for $2 \le k \le n$ and $0 < t < n$

$$P\left(S_k > n\cdot n! - t\right) = kP(X_1 = (n+1)!) + \frac{\widetilde{\alpha}(k,n,t)}{(n+2)!}$$

and $\sup_{2\le k\le n\,;\,0<t<n}|\widetilde{\alpha}(k,n,t)| < \infty$. Integrating with respect to the distribution of $|B(1)|$ and using (7.4) imply for $1 \le k \le n$:

$$P\left(S_k + |B(1)| > n\cdot n!\right) = kP(X_1 = (n+1)!) + O\left(\frac{1}{(n+2)!}\right). \tag{7.20}$$

The same reasons as in the proof of (7.13) yield $P\left(S_{n+1} > n\cdot n! - t\right) \le C/(n-2)!$ for $0 < t < n$. Applying (7.4) we conclude that

$$P\left(S_{n+1} + |B(1)| > n\cdot n!\right) = O\left(\frac{1}{(n-2)!}\right).$$

From here and (7.19)

$$P\left(\sup_{0\le t\le 1} X(t) > n\cdot n!\right) \le P(X_1 = (n+1)!)\sum_{k=1}^{n}\frac{1}{e(k-1)!} + O\left(\frac{1}{(n+2)!}\right),$$

and (7.12) and (7.1) imply that

$$\limsup_{n\to\infty} \frac{P\left(\sup_{0\le t\le 1} X(t) > n\cdot n!\right)}{P\left(X(1) > n\cdot n!\right)} \le 1\,.$$

So, (2.15) follows. □

Remark 7.7. According to (7.1), the jumps X_k of compound Poisson process Z have not moments of positive order. But one can consider jumps with the distribution

$$P(X_1 = n!) = \frac{C(v)}{(n!)^v}\,,\; n = 1, 2, \ldots,$$

where v is a positive constant and $C(v)$ is the corresponding norming constant. Now jumps have finite moments of order less than v, and almost the same proof gives Theorem 2.5.

8. Some comments

8.1. **About the proof of Theorem 2.1.** Looking on (3.3) one may assume that the relation

$$\lim_{u\to\infty} \frac{P\left(\sup_{0\le t\le 1} B(t) + \sup_{0\le t\le 1} Z(t) > u\right)}{P(B(1) + Z(1) > u)} = 1$$

also holds, which might shorten the proof of (4.1). It is true if the tail of X_k is subexponential (see, for example, Proposition 2.1 from [11] and references therein). Here we show that it is not true for light tails.

Proposition 8.1. *If (2.7) holds for X_k, then*

$$\lim_{u\to\infty} \frac{P\left(\sup_{0\le t\le 1} B(t) + \sup_{0\le t\le 1} Z(t) > u\right)}{P(B(1) + Z(1) > u)} = 2\,.$$

Proof. Clearly that

$$P(B(1) + Z(1) > u) = P(B(1) + Z(1) > u\,,\, B(1) > 0) + P(B(1) + Z(1) > u\,,\, B(1) < 0)\,,$$

and the last probability does not exceed $P(Z(1) > u)$. On the other hand.

$$P(B(1) + Z(1) > u) \ge P(Z(1) > u - a)P(B(1) > a)$$

for $a > 0$. From here and Proposition 6.1

$$\lim_{u\to\infty} \frac{P(B(1) + Z(1) > u\,,\, B(1) < 0)}{P(B(1) + Z(1) > u)} = 0\,,$$

and , therefore,

$$\lim_{u\to\infty} \frac{P(B(1) + Z(1) > u\,,\, B(1) > 0)}{P(B(1) + Z(1) > u)} = 1\,.$$

Relation (4.3) implies

$$P(B(1) + Z(1) > u\,,\, B(1) > 0) = \frac{1}{2}P(|B(1)| + Z(1) > u) = \frac{1}{2}P\left(\sup_{0\le t\le 1} B(t) + Z(1) > u\right)\,.$$

But, according to Theorem 1 from [4]

$$P\left(\sup_{0\le t\le 1} Z(t) > u\right) \sim P\left(Z(1) > u\right) \tag{8.1}$$

as $u \to \infty$. So, Lemma 3.1 implies

$$\lim_{u\to\infty} \frac{P\left(|B(1)| + \sup_{0\le t\le 1} Z(t) > u\right)}{P(|B(1)| + Z(1) > u)} = 1\,.$$

The last equalities yield the proposition. □

The limit considered in Proposition 8.1 can belong to the interval $(1,2)$. To show this, assume that jumps X_k satisfy the condition

$$P(X_k > u) \sim e^{-\alpha u} u^{\gamma} \quad \text{as } u \to \infty\,,$$

where $\alpha > 0$ and $\gamma > -1$ are constants. Then (2.1) holds. For a random variable Y denote

$$m_Y^+(\alpha) = \int_0^\infty e^{\alpha t} F_Y(dt)\,,\; m_Y^-(\alpha) = \int_{-\infty}^0 e^{\alpha t} F_Y(dt)\,,$$

and $m_Y(\alpha) = m_Y^+(\alpha) + m_Y^-(\alpha)$. Lemma 4 from [5] gives us

$$\lim_{u\to\infty} \frac{P(|B(1)| + Z(1) > u)}{P(Z(1) > u)} = m_{|B(1)|}(\alpha) = 2m_{B(1)}^+(\alpha)$$

and

$$\lim_{u\to\infty} \frac{P(B(1) + Z(1) > u)}{P(Z(1) > u)} = m_{B(1)}(\alpha)\,.$$

As above, applying (8.1) and Lemma 3.1 we see that the limit under consideration is

$$l := \frac{2m_{B(1)}^+(\alpha)}{m_{B(1)}^+(\alpha) + m_{B(1)}^-(\alpha)}\,.$$

Because $m_{B(1)}^-(\alpha) < m_{B(1)}^+(\alpha)$, we conclude that $1 < l < 2$.

8.2. **A conjecture.** It is well known that if $X(t)$ is a symmetric Lévy process, then

$$P\left(\sup_{0\le t\le 1} X(t) > u\right) \le 2P(X(1) > u) \tag{8.2}$$

for all $u > 0$. Comparing it with (1.5) and Theorem 2.1 naturally yields the following

Conjecture. *Let $X(t)$ be a symmetric Lévy process such that*

$$\limsup_{u\to\infty} \frac{P\left(\sup_{0\le t\le 1} X(t) > u\right)}{P(X(1) > u)} = 2\,. \tag{8.3}$$

Then $X(t) = \sigma B(t)$ for a positive constant σ.

The following statement supports this conjecture.

Proposition 8.2. *Let $X(t)$ be a symmetric compound Poisson process. Then the strong inequality holds:*

$$\limsup_{u\to\infty} \frac{P(\sup_{0\le t\le 1} X(t) > u)}{P(X(1) > u)} < 2. \tag{8.4}$$

To show it we need the following

Lemma 8.3. *Let $X(t)$ be a symmetric compound Poisson process with jumps X_k and parameter λ. Then for all positive u:*

$$P\left(\sup_{0\le t\le 1} X(t) > u\right) \le 2P\left(X(1) > u\right) - D(u)\,,$$

where

$$D(u) = e^{-\lambda}\left[\lambda P(X_1 > u) + \sum_{n=2}^{\infty}\frac{\lambda^n}{n!}P\left(\max_{1\le k\le n-1} S_k \le u,\, S_n > u\right)\right]. \tag{8.5}$$

Proof. The proof is a modification of the proof of Levy inequality (see [10], p. 50). We have

$$P\left(\sup_{0\le t\le 1} X(t) > u\right) = P(X(1) > u) + P\left(\sup_{0\le t\le 1} X(t) > u,\, X(1) \le u\right),$$

and

$$P\left(\sup_{0\le t\le 1} X(t) > u,\, X(1)\le u\right) = e^{-\lambda}\left[\sum_{n=2}^{\infty}\frac{\lambda^n}{n!}P\left(\max_{1\le k\le n-1} S_k > u,\, S_n \le u\right)\right]. \tag{8.6}$$

For any $n > 2$

$$P\left(\max_{1\le k\le n-1} S_k > u,\, S_n \le u\right)$$
$$= P(S_1 > u,\, S_n \le u) + \cdots + P(S_1 \le u, \ldots, S_{n-2} \le u,\, S_{n-1} > u, S_n \le u)$$
$$\le P(S_1 > u,\, X_2 + \cdots + X_n \le 0) + \cdots + P(S_1 \le u, \ldots, S_{n-2} \le u, S_{n-1} > u,\, X_n \le 0).$$

Since random variables X_k are independent and symmetric, the last line can be written as

$$P(S_1 > u,\, X_2 + \cdots + X_n \ge 0) + \cdots + P(S_1 \le u, \ldots, S_{n-2} \le u,\, S_{n-1} > u, X_n \ge 0)$$
$$\le P(S_1 > u,\, S_n > u) + \cdots + P(S_1 \le u, \ldots, S_{n-2} \le u,\, S_{n-1} > u,\, S_n > u)$$
$$= P\left(\max_{1\le k\le n-1} S_k > u,\, S_n > u\right) = P(S_n > u) - P\left(\max_{1\le k\le n-1} S_k \le u,\, S_n > u\right).$$

The same inequality holds for $n = 2$. Therefore,

$$P\left(\sup_{0\le t\le 1} X(t) > u\right) \le P(X(1) > u) + e^{-\lambda}\sum_{n=2}^{\infty}\frac{\lambda^n}{n!}P(S_n > u)$$
$$-e^{-\lambda}\sum_{n=2}^{\infty}\frac{\lambda^n}{n!}P\left(\max_{1\le k\le n-1} S_k \le u,\, S_n > u\right) = 2P(X(1) > u) - D(u)\,.$$

□

Proof of Proposition 8.2. If the upper limit in (8.4) is equal to 2, there is a sequence $u_j \to \infty$ such that

$$\lim_{j\to\infty}\frac{P(\sup_{0\le t\le 1} X(t) > u_j)}{P(X(1) > u_j)} = 2\,. \tag{8.7}$$

Then Lemma 8.3 yields

$$\lim_{j\to\infty}\frac{D(u_j)}{P(X(1) > u_j)} = 0\,, \tag{8.8}$$

which implies, in particular,

$$\lim_{j\to\infty}\frac{P(X_1>u_j)}{P(X(1)>u_j)}=0\,.$$

Applying Lévy inequality we get for $n\geq 2$

$$P\left(\max_{1\leq k\leq n-1}S_k\leq u,\,S_n>u\right)\geq P(S_n>u)-P\left(\max_{1\leq k\leq n-1}S_k>u\right)$$
$$\geq P(S_n>u)-2P(S_{n-1}>u)\,.$$

Using (8.8), (8.5) and the induction one comes to the relation

$$\lim_{j\to\infty}\frac{P(S_n>u_j)}{P(X(1)>u_j)}=0 \tag{8.9}$$

for all $n\geq 2$.

Further, once again using Lévy inequality we get from (8.6)

$$P\left(\sup_{0\leq t\leq 1}X(t)>u_j,\,X(1)\leq u_j\right)\leq 2e^{-\lambda}\sum_{n=2}^{\infty}\frac{\lambda^n}{n!}P(S_{n-1}>u_j),$$

and the same reasons as in the proof of Proposition 6.1 show that this sum is $o(P(X(1))>u_j)$ as $j\to\infty$. Hence,

$$\lim_{j\to\infty}\frac{P(\sup_{0\leq t\leq 1}X(t)>u_j)}{P(X(1)>u_j)}=1\,,$$

which contradicts to (8.7). □

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