The structure of boundary parameter property satisfying sets

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Abstract

Precise definitions of singularities in General Relativity rely on a set of curves. Many boundary constructions force a particular set of curves by virtue of the construction. The abstract boundary, however, allows the set of curves to be chosen. This set, therefore, plays a very important role in the use of the abstract boundary as the definition of a singularity or point at infinity depends on it. The sets of curves used in the abstract boundary must satisfy the boundary parameter property. This property obfuscates the construction of and relationships between these sets of curves. In this paper we lay the ground work for an analysis of these sets of curves by showing that they are in one-to-one correspondence with certain sets of inextendible curves. As an application of this result we show how the usual set operations can be extended to boundary parameter property satisfying sets of curves, allowing for their comparison. These results provide an interpretation of what information boundary parameter property satisfying sets give us, provide tools to analyse their use and allow for easier physical interpretation of the abstract boundary classification.

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1 Introduction

Boundary constructions in General Relativity provide rigorous definitions of singularities, points at infinity, regular boundary points and so on. To do this most boundary constructions use, implicitly or explicitly, certain sets of curves. For example the g-boundary, [1], relies on incomplete geodesics with affine parameter, the b-boundary, [2], on incomplete curves with generalised affine parameter and the c-boundary, [3], on endless causal curves.

The abstract boundary, [4], does not use sets of curves in it's construction but does need such a set for it's classification. Which set of curves to use is thus left to choice. With a set of curves it is possible to provide a complete, detailed, physically relevant classification of abstract boundary points (see [4] for the details) that complements the classification in terms of polynomials of the Riemann tensor, see [5]. That one has to choose a set of curves begs the question of what happens when the set of curves changes.

The sets of curves used in the abstract boundary must satisfy what is called the boundary parameter property or b.p.p. This property ensures that the curves have, mutually compatible, physically relevant interpretations. That is it prevents bad choices of sets of curves. Unfortunately the b.p.p. obfuscates the structure of b.p.p. satisfying sets of curves and prevents easy analysis of them and their relationships. For example, it is not clear how to construct a b.p.p. satisfying set of curves that has particular properties.

Clearly these are important, physically relevant, questions. Another such question asks what is the "correct" set of curves to use? It is possible to argue that the topological problems of the g-, b- and c-boundaries relate to their corresponding sets of curves being too big, including to many curves. Papers such as [6] reiterate the point that careful consideration of the set of curves is needed to get a correct definition of singularity.

This paper is the first step to answering all these questions. It lays the ground work for analysis of these sets of curves by describing, in detail, their structure. In particular we show that sets of curves with the b.p.p. are in one-to-one correspondence with sets of inextendible curves that are not contained in a compact set and which satisfy a technical condition which ensures that the set is chosen sensibly. This technical condition is in some sense equivalent to the b.p.p. , but is much easier to work this. Note we use various equivalence relations to build the one-to-one correspondence. Once the correspondence is demonstrated we show how to use it to define set-like operations, $\subset$, $\cup$, $\cap$, and $\setminus$ (or $-$), that respect the b.p.p. This immediately allows certain relationships, between b.p.p. satisfying sets of curves, to be analysed.

For the reader, we give a few of the results from [4] that are needed for this paper.

Definition 1 (Manifold)**.** *We shall only consider manifolds, $\mathcal{M}$, that are paracompact, Hausdorff, connected, C^∞-manifolds.*

Definition 2. *An embedding, $\phi : \mathcal{M} \to \mathcal{M}_\phi$ of $\mathcal{M}$ is an envelopment if $\mathcal{M}_\phi$ has the same dimension as $\mathcal{M}$. Let $\Phi(\mathcal{M})$ be the set of all envelopments of $\mathcal{M}$.*

Following the lead of [4] we work only with C^1 piecewise curves. Note, however, that since it is only accumulation points of images of curves that are used in [4], it is possible to define the a-boundary using C^0 curves. Since the same is true of this paper, all results below will hold for C^0 curves.

Definition 3 (Curves)**.** *A parametrised curve (or just curve) γ in the manifold $\mathcal{M}$ is a C^1 map $\gamma : [a, b) \to \mathcal{M}$ where $[a, b) \subset \mathbb{R}$, $a < b \leq \infty$ and such that its tangent vector $\gamma' : [a, b) \to T_\gamma\mathcal{M}$ vanishes nowhere on the interval $[a, b)$. We shall say that γ starts at $\gamma(a)$ and is bounded if $b < \infty$ otherwise γ is unbounded.*

A curve $\delta : [a', b') \to \mathcal{M}$ is a subcurve of γ if $a \leq a' < b' \leq b$ and $\delta = \gamma|_{[a',b')}$. That is a curve δ is a subcurve of γ if δ is the restriction of γ to some right-half open interval of $[a, b)$. We shall denote this by $\delta < \gamma$. If $a' = a$ and $b' < b$ we shall say that γ is an extension of δ.

A change of parameter is a monotone increasing surjective C^1 function, $s : [a, b) \to [a', b')$. A curve δ is obtained from the curve γ if $\delta = \gamma \circ s$.

Note that our definition of $<$ considers the parameterisation chosen for γ. That is if $\delta(t) < \gamma(t)$ then we know that $\delta(t) \not< \gamma(2t)$. Another way to say this is that we draw a distinction between reparameterisations of the same image of 'a curve'. Hence the relation $<$ is very fine.

Definition 4 (Bounded parameter property)**.** *A set $\mathcal{C}$ of parameterised curves is said to have the bounded parameter property (or b.p.p.) if the following properties are satisfied;*

1. *For all $p \in \mathcal{M}$ there exits $\gamma : [a, b) \to \mathcal{M} \in \mathcal{C}$ so that $p \in \gamma([a, b))$.*
2. *If $\gamma \in \mathcal{C}$ and $\delta < \gamma$ then $\delta \in \mathcal{C}$.*
3. *For all $\gamma, \delta \in \mathcal{C}$, if δ is obtained from γ by a change of parameter then either both curves are bounded or both are unbounded.*

Only the relative bounded or unboundedness of curves is important in definition 4. Indeed from definitions 28, 31 and 37 of [4] it is clear that we need not pay close attention to the domain of any curve. We only need know the curve's image and whether it is bounded or unbounded. Thus the relation $<$ is, in this sense, too fine for the a-boundary. In most uses of the a-boundary classification the b.p.p. satisfying set is fixed and nuanced points about distinctions between the domains of curves and their images is unimportant. It is, however, an important point when considering relationships between b.p.p. satisfying sets, as we shall see below.

The following definitions are inspired from [4]. They simplify the results that follow.

Definition 5. *Let $\phi \in \Phi$ and $\mathcal{C}$ be a set of curves with the b.p.p. Define* $\mathrm{App}(\phi, \mathcal{C})$ *to be the the set of points $p \in \partial\phi(\mathcal{M}) = \overline{\phi(\mathcal{M})} - \phi(\mathcal{M})$ so that there exists $\gamma : [a, b) \to \mathcal{M} \in \mathcal{C}$ such that p is an accumulation point of $\phi(\gamma[a, b))$. The set* $\mathrm{App}(\phi, \mathcal{C})$ *is the set of points that are approached by at least one curve in $\mathcal{C}$. Define* $\mathrm{Nonapp}(\phi, \mathcal{C})$ *by* $\mathrm{Nonapp}(\phi, \mathcal{C}) = \partial\phi(\mathcal{M}) - \mathrm{App}(\phi, \mathcal{C})$. *This is the set of points that are not the accumulation point of any curve in $\mathcal{C}$.*

Definition 6. *Let $\phi \in \Phi$ and $\mathcal{C}$ be a set of curves with the b.p.p. Define* $\mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{C})$ *to be the set of points $p \in \partial\phi(\mathcal{M})$ so that $p \in \mathrm{App}(\phi, \mathcal{C})$ and there exists $\gamma : [a, b) \to \mathcal{M} \in \mathcal{C}$, $b \in \mathbb{R}$ so that p is an accumulation point of $\phi(\gamma([a, b)))$. That is $p \in \mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{C})$ if and only if p is approached by a bounded curve. We use the symbol* $\mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{C})$ *as elements of* $\mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{C})$ *can be either regular or singular boundary points, see definition 37 of [4]. Let* $\mathrm{App}_{\mathrm{Inf}}(\phi, \mathcal{C}) = \mathrm{App}(\phi, \mathcal{C}) - \mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{C})$. *Elements of* $\mathrm{App}_{\mathrm{Inf}}(\phi, \mathcal{C})$ *are either regular boundary points or points at infinity, see definition 31 of [4].*

It is the sets $\mathrm{App}(\phi, \mathcal{C})$, $\mathrm{Nonapp}(\phi, \mathcal{C})$, $\mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{C})$ and $\mathrm{App}_{\mathrm{Inf}}(\phi, \mathcal{C})$ that are used to give the classification. See definitions 28, 31 and 37 of [4] for more detail on this. Hence when working with b.p.p. satisfying sets we really only care about the boundedness of curves and their limit points.

2 Inextendible curves and boundary parameter property satisfying sets

As mentioned, we shall show that every b.p.p. satisfying set of curves corresponds to a set of inextendible curves that are not contained in a compact set and satisfy a technical condition equivalent to condition 3 of definition 4. This demonstrates that the structure of b.p.p. satisfying sets of curves is, in fact, very simple. Just pick your favorite collection of inextendible curves that travel to infinity and, with a little work, you have a b.p.p. satisfying set of curves. When choosing particular inextendible curves it is easy to choose them so that the necessary technical condition is true. That is, you have a set of curves that can be used to classify the a-boundary of a spacetime.

The basic idea is that once you have some collection of inextendible curves that are not contained in a compact region, it is possible to fill in the "gaps" so as to ensure that the set satisfies the b.p.p. Since the way in which the "gaps" are filled makes no difference to the a-boundary classification this method ensures that all b.p.p. satisfying sets can be constructed in this way. At least, all b.p.p. satisfying sets can be constructed up to an equivalence relation that removes the dependence on how the "gaps" are filled.

We start by defining how to construct a particular b.p.p. satisfying set of curves. Given any $p \in \mathcal{M}$ there exists a normal neighbourhood V_p so that $p \in V_p$ and $\overline{V_p}$ is compact. Choose any $v \in T_p\mathcal{M}$ and let $\gamma_p : [-\epsilon, \epsilon) \to \mathcal{M}$ be the unique geodesic lying in V_p so that $\gamma_p(0) = p$ and $\gamma_p'(0) = v$. Let $\mathcal{C}_{\text{Norm}} = \{\delta \text{ a curve} : \exists \gamma_p \text{ so that } \delta < \gamma_p\}$.

For future reference let $\mathcal{C}_{\text{Norm}}(U) = \{\delta \text{ a curve} : \exists p \in U\ \exists \gamma_p \text{ so that } \delta < \gamma_p\}$. That is $\mathcal{C}_{\text{Norm}}(U)$ is the restriction of our construction to include only curves γ_p for the points in some $U \subset \mathcal{M}$.

Definition 7. *Let $\mathcal{C}_{\text{Norm}}$ and $\mathcal{C}_{\text{Norm}}(U)$ be defined as above.*

Proposition 8. *The set $\mathcal{C}_{\text{Norm}}$ satisfies the b.p.p. In addition all curves $\delta \in \mathcal{C}_{\text{Norm}}$ have bounded parameter.*

Proof.

1. For all $p \in \mathcal{M}$ the curve $\gamma_p \in \mathcal{C}_{\text{Norm}}$ is such that $p \in \gamma_p$.
2. Let $\delta \in \mathcal{C}_{\text{Norm}}$ and let $\beta < \delta$. Since $\delta \in \mathcal{C}_{\text{Norm}}$ there exists $\gamma_p \in \mathcal{C}$ so that $\delta < \gamma_p$. We know that $\beta < \delta < \gamma_p$ and therefore, by construction, $\beta \in \mathcal{C}_{\text{Norm}}$.
3. Let $\delta : [a, b) \to \mathcal{M} \in \mathcal{C}_{\text{Norm}}$ then there must exist $\gamma_p : [-\epsilon, \epsilon) \to \mathcal{M} \in \mathcal{C}_{\text{Norm}}$ so that $\delta < \gamma_p$. By definition, this implies that $[a, b) \subset [-\epsilon, \epsilon)$ and therefore that δ has bounded parameter. Suppose that there exists $\mu : [p, q) \to \mathcal{M} \in \mathcal{C}_{\text{Norm}}$ so that δ and μ are obtained from each other by a change of parameter. Since $\delta, \mu \in \mathcal{C}_{\text{Norm}}$ we know that both must be bounded.

Therefore $\mathcal{C}_{\text{Norm}}$ satisfies the b.p.p. and all curves in $\mathcal{C}_{\text{Norm}}$ are bounded. □

Sets of curves with the b.p.p. such as above are not useful for the abstract boundary classification as they make no distinction between approachable and non-approachable boundary points for envelopments of $\mathcal{M}$.

Proposition 9. *The set* $\mathrm{Nonapp}(\phi, \mathcal{C}_{\mathrm{Norm}})$ *is equal to* $\partial\phi(\mathcal{M})$ *for all* $\phi \in \Phi(\mathcal{M})$*.*

Proof. By definition we know that $\mathrm{Nonapp}(\phi, \mathcal{C}_{\mathrm{Norm}}) \subset \partial\phi(\mathcal{M})$ so we need only show that $\partial\phi(\mathcal{M}) \subset \mathrm{Nonapp}(\phi, \mathcal{C}_{\mathrm{Norm}})$.

Let $p \in \partial\phi(\mathcal{M})$ if $p \notin \mathrm{Nonapp}(\phi, \mathcal{C}_{\mathrm{Norm}})$ then $p \in \mathrm{App}(\phi, \mathcal{C}_{\mathrm{Norm}})$. Thus there exists $\delta : [a, b) \to \mathcal{M} \in \mathcal{C}_{\mathrm{Norm}}$ so that p is a limit point of the curve $\phi \circ \delta$. We may choose $\{t_i\} \subset [a, b)$ so that $\phi \circ \delta(t_i) \to p$. By construction the sequence $\{\delta(t_i)\}$ cannot have any limit points. Since $\delta \in \mathcal{C}_{\mathrm{Norm}}$ there exists $\gamma_q : [-\epsilon, \epsilon) \to \mathcal{M}$, for some q so that $\gamma_q([-\epsilon, \epsilon)) \subset V_q$ and $\delta < \gamma_q$. We can conclude that $\{\delta(t_i)\} \subset \gamma_q([-\epsilon, \epsilon)) \subset \overline{V_q}$ which is compact. Hence $\{\delta(t_i)\}$ must have a limit point. Since this is a contradiction we know that $p \in \mathrm{Nonapp}(\phi, \mathcal{C}_{\mathrm{Norm}})$ as required. $\square$

Thus from the point of view of the a-boundary $\mathcal{C}_{\mathrm{Norm}}$ provides no information. Hence given a set of inextendible curves S that are not contained in a compact set we can add curves from $\mathcal{C}_{\mathrm{Norm}}(U)$, for some suitable U, to S to get a set which satisfies conditions 1 and 2 of definition 4 and which gives the same a-boundary information as S. We will use this latter, for the moment we prove a more useful result than proposition 9.

Proposition 10. *Let* γ *be a curve in* $\mathcal{M}$ *then* $\overline{\gamma}$ *is compact if and only if for all* $\phi \in \Phi$*,* $\partial\phi(\mathcal{M}) \cap \overline{\phi \circ \gamma} = \emptyset$*.*

Proof. Let $\gamma : [a, b) \to \mathcal{M}$ be a curve. Suppose that $\overline{\gamma}$ is compact and let $p \in \partial\phi(\mathcal{M}) \cap \overline{\phi \circ \gamma}$, from some $\phi \in \Phi$. The same argument used in the proof of proposition 9 can be used to derive a contradiction. Hence $\partial\phi(\mathcal{M}) \cap \overline{\phi \circ \gamma} = \emptyset$.

Suppose that for all $\phi \in \Phi$, $\partial\phi(\mathcal{M}) \cap \overline{\phi(\gamma)} = \emptyset$ and that $\overline{\gamma([a, b))}$ is not compact. Then there exists a sequence $\{x_i\} \subset \overline{\gamma([a, b))}$ with no limit points in $\mathcal{M}$. By the Endpoint Theorem (see [7]) there exists $\psi \in \Phi(\mathcal{M})$ and $x \in \partial\psi(\mathcal{M})$ so that $\{\psi(x_i)\} \to x$. Then $x \in \partial\psi(\mathcal{M}) \cap \overline{\psi \circ \gamma}$ which is a contradiction. Thus $\overline{\gamma([a, b))}$ is compact. $\square$

Corollary 11. *Let* $\mathcal{C}$ *be a set of b.p.p. curves so that for all* $\gamma \in \mathcal{C}$*,* γ *is extendible, then* $\mathrm{Nonapp}(\phi, \mathcal{C}) = \partial\phi(\mathcal{M})$ *for all* $\phi \in \Phi(\mathcal{M})$*.*

Proof. If $\gamma : [a, b) \to \mathcal{M}$ is extendible then there exists $\delta : [a, b') \to \mathcal{M}$, $b < b'$, so that $\gamma = \delta|_{[a,b)}$. Then $\overline{\gamma} = \delta([a, b])$ and thus must be compact. Therefore for all $\gamma \in \mathcal{C}$ we know that γ is compact. This implies that for all $\gamma \in \mathcal{C}$ and for all $\phi \in \Phi$ we know that $\partial\phi(\mathcal{M}) \cap \overline{\phi \circ \gamma} = \emptyset$, by proposition 10. By definition this implies that $\text{App}(\phi, \mathcal{C}) = \emptyset$. Hence $\text{Nonapp}(\phi, \mathcal{C}) = \partial\phi(\mathcal{M})$. □

Proposition 10 and corollary 11 prove our claim that the a-boundary is only interested in b.p.p. sets that contain inextendible curves, γ, that are not contained in a compact set. This is an important point as we use this to show that b.p.p. satisfying sets are in some sense the same as sets of inextendible curves that are not contained in a compact set.

Definition 12. *Let $BPP(\mathcal{M})$ be the set of all sets of curves with the b.p.p. That is $BPP(\mathcal{M}) = \{\mathcal{C} : \mathcal{C}$ is a set of curves with the b.p.p.$\}$.*

Remember that we draw a distinction between curves that may have the same image, yet different domains. Therefore the set $BPP(\mathcal{M})$ will contain multiple sets with curves that have the same images but different domains. For example given $\mathcal{C} \in BPP(\mathcal{M})$. We can define a new b.p.p. satisfying set of curves by $\mathcal{C}' = \{\gamma(t-1) : \gamma \in \mathcal{C}\}$. Since we have not changed the "boundedness" of any curve in $\mathcal{C}$, from the point of view of the a-boundary the sets $\mathcal{C}$ and $\mathcal{C}'$ are considered to be the same. That is $\text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{C}')$ and $\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}_{\text{Sing}}(\phi, \mathcal{C}')$ for all $\phi \in \Phi$. Thus $BPP(\mathcal{M})$ will contain multiple copies of each b.p.p. satisfying set that from the view point of the a-boundary are redundant. We deal with this issue below.

Definition 13. *Let $X(\mathcal{M})$ be the set of inextendible curves that are not contained in a compact set. Let $X_{\text{b.p.p.}}(\mathcal{M})$ be the set of subsets of $X(\mathcal{M})$ such that for all $S \in X_{\text{b.p.p.}}(\mathcal{M})$ and for all $\gamma : [a, b) \to \mathcal{M}, \delta : [p, q) \to \mathcal{M} \in S$, if there exists $c, r \in \mathbb{R}$ and a change of parameter $s : [c, b) \to [r, q)$ so that $\gamma|_{[r,q)} \circ s = \delta|_{[c,b)}$ then either both γ and δ are bounded or both are unbounded.*

The technical condition in the above definition is to prevent us from choosing subsets of $X(\mathcal{M})$ which are inappropriate from the point of view of the a-boundary. It is a translation of part 3 of definition 4 to subsets of $X(\mathcal{M})$. Hence the subscript b.p.p. Note that in order to accommodate our distinction between curves with the same image but different domains, given by the relation $<$, we need to include many more sets of curves in $X(\mathcal{M})$

than strictly necessary. In effect this is the same problem with $BPP(\mathcal{M})$, mentioned above.

Proposition 14. *There exists a function* $f : BPP(\mathcal{M}) \to X_{\text{b.p.p.}}(\mathcal{M})$*, given by* $f(\mathcal{C}) = X(\mathcal{M}) \cap \mathcal{C}$*.*

Proof. The function is certainly well defined and by definition of $\mathcal{C}$ we know that $X(\mathcal{M}) \cap \mathcal{C} \in X_{\text{b.p.p.}}(\mathcal{M})$. □

The function f takes a b.p.p. satisfying set and extracts the inextendible curves that are not contained in a compact set. That is it extracts the curves that are important with regards to the a-boundary classification using $\mathcal{C}$. We now show how to construct a b.p.p. satisfying set from an element of $X_{\text{b.p.p.}}(\mathcal{M})$.

Definition 15. *Given a set of curves* S *let* $S_p = \{p \in \mathcal{M} : \exists \gamma : [a,b) \to \mathcal{M} \in S$ *so that* $p \in \gamma([a,b))\}$*.*

Thus S_p is the set of all points in the manifold that are contained in the image of some curve in S.

Proposition 16. *Let* $S \in X_{\text{b.p.p.}}(\mathcal{M})$*, then we can define a function* $g : X_{\text{b.p.p.}}(\mathcal{M}) \to BPP(\mathcal{M})$ *by letting*

$$g(S) = \{\delta : \exists \gamma \in S \ \delta < \gamma\} \cup \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p).$$

Proof. As before, it is clear that g is well defined, so we must only check that $g(S)$ satisfies the b.p.p. We must check the three conditions of definition 4.

1. Let $p \in \mathcal{M}$. If $p \in S_p$ then there exists $\alpha : [a,b) \to \mathcal{M} \in S$ so that $p \in \alpha([a,b))$. Since $\alpha < \alpha$ we know that $\alpha \in g(S)$. If $p \notin S_p$ then $p \in \mathcal{M} - S_p$ then by definition of $\mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ we know that there must exist some $\gamma_p : [-\epsilon, \epsilon) \to \mathcal{M} \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ so that $p \in \gamma_p([-\epsilon, \epsilon))$, as required.

2. Let $\alpha \in g(S)$ and $\beta < \alpha$. We know that either $\alpha \in \{\delta : \exists \gamma \in S \ \delta < \gamma\}$ or $\alpha \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$. If $\alpha \in \{\delta : \exists \gamma \in S \ \delta < \gamma\}$, then by definition $\beta \in \{\delta : \exists \gamma \in S \ \delta < \gamma\}$. Likewise if $\alpha \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ then by the definition of $\mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ we know that $\beta \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ as required.

3. Let $\alpha : [a,b) \to \mathcal{M}, \beta : [p,q) \to \mathcal{M} \in g(S)$ be such that β and α are obtained from each other by a change of parameter $s : [a,b) \to [p,q)$, that is $\alpha \circ s = \beta$.

Suppose that $\beta, \alpha \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ then from corollary 11 we know that both must be bounded.

Suppose that $\beta \in \{\delta : \exists \gamma \in S\ \delta < \gamma\}$, $\alpha \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$. Since $\beta \in \{\delta : \exists \gamma \in S\ \delta < \gamma\}$ there exists $\gamma_x : [p', q') \to \mathcal{M} \in S$ so that $\beta < \gamma_x$, that is $[p, q) \subset [p', q')$ and $\beta = \gamma_x|_{[p,q)}$. Since $\gamma_x \in S$ it must be the case that $\overline{\gamma_x([p', q'))}$ is not compact. As $\alpha \in \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)$ we know that $\overline{\alpha([a, b))}$ is necessarily compact. Hence as $\overline{\alpha([a, b))} = \overline{\alpha \circ s([p, q))} = \overline{\beta([p, q))}$ we know that $\overline{\beta([p, q))}$ must be compact. Since $\overline{\beta([p, q))} \subset \overline{\gamma_x([p', q'))}$ and $\overline{\gamma_x([p', q'))}$ is not compact we can conclude that $[p, q)$ is a proper subset of $[p', q')$ and in particular that $q < q'$. Hence $q \in \mathbb{R}$ and therefore β must be bounded.

Now suppose $\beta, \alpha \in \{\delta : \exists \gamma \in S\ \delta < \gamma\}$. As $\alpha \in \{\delta : \exists \gamma \in S\ \delta < \gamma\}$ there must exist $\gamma_y : [a', b') \to \mathcal{M} \in S$ so that $\overline{\gamma_y([a', b'))}$ is not compact, $[a, b) \subset [a', b')$ and $\alpha = \gamma_y|_{[a,b)}$. Let $\gamma_x : [p', q') \to \mathcal{M}$ be as above.

If $q < q'$ then $\overline{\beta([p, q))}$ is compact and $\overline{\beta([p, q))} = \overline{\alpha \circ s([p, q))} = \overline{\alpha([a, b))}$ must also be compact. Therefore $b < b'$ and α is bounded. Applying the same argument for α shows that $q < q'$ if and only if $b < b'$. Thus, in this case, both β and α are bounded.

Thus we need only now consider the case when $q = q'$ and $b = b'$. We see that $\gamma_x|_{[p,q')} = \beta = \alpha \circ s = \gamma_y|_{[a,b')} \circ s$. Since $\gamma_x, \gamma_y \in S$ from the technical condition in definition 13 we know that γ_x and γ_y are either both bounded or both unbounded. Since $q = q'$ and $b = b'$ the same applies to α and β. That is either both are bounded or both are unbounded, as required.

□

We have the following lemma which implies that $f \circ g(S) = S$.

Lemma 17. *Let $S \subset X(\mathcal{M})$ then*

$$\Big(\{\delta : \exists \gamma \in S\ \delta < \gamma\} \cup \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)\Big) \cap X(\mathcal{M}) = S$$

Proof. The intersection $\mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p) \cap X(\mathcal{M})$ must be empty by corollary 11. Let $\delta \in \{\delta : \exists \gamma \in S\ \delta < \gamma\}$ then there exists $\gamma \in S$ so that $\delta < \gamma$. If $\delta \neq \gamma$ then δ is extendible and δ is contained in a compact set. Thus we know that $\delta \notin X(\mathcal{M})$. If, however, $\delta = \gamma$ then $\gamma \in S$ and as $S \subset X(\mathcal{M})$ by definition, we can conclude that $S = S \cap X(\mathcal{M})$. Therefore

$$\Big(\{\delta : \exists \gamma \in S\ \delta < \gamma\} \cup \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)\Big) \cap X(\mathcal{M}) = S,$$

as required. ∎

Corollary 18. *Let $S \in X_{\text{b.p.p.}}(\mathcal{M})$ then $f \circ g(S) = S$.*

Proof. We can calculate that,

$$\begin{aligned} f \circ g(S) &= f\left(\{\delta : \exists \gamma \in S\ \delta < \gamma\} \cup \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)\right) \\ &= \Big(\{\delta : \exists \gamma \in S\ \delta < \gamma\} \cup \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)\Big) \cap X(\mathcal{M}) \\ &= S, \end{aligned}$$

by lemma 17 as required. ∎

We have demonstrated that only the curves in an b.p.p. satisfying set that are also in $X(\mathcal{M})$ provide information regarding the classification of a-boundary points. We have also shown how sets of curves in $X_{\text{b.p.p.}}(\mathcal{M})$ correspond to b.p.p. satisfying sets of curves. We have not shown that there is a one-to-one correspondence between the sets $BPP(\mathcal{M})$ and $X_{\text{b.p.p.}}(\mathcal{M})$, however. There are two problems preventing this. The first comes from the fineness of $<$. The second because given an element of $X_{\text{b.p.p.}}(\mathcal{M})$ there are multiple ways to add additional extendible curves so that definition 4 is satisfied. From above, however, we know that the choices related to these problems are immaterial, therefore it makes sense to define an equivalence relations on $BPP(\mathcal{M})$ and $X_{\text{b.p.p.}}(\mathcal{M})$, which remove these choices. The result will be the desired one-to-one correspondence.

3 Formalising the one-to-one correspondence

3.1 The obvious equivalence relation

The obvious choice, from the point of view of the a-boundary, is to say that two b.p.p. satisfying sets of curves, $\mathcal{C}$ and $\mathcal{D}$, are equivalent if $\text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{D})$ and $\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}_{\text{Sing}}(\phi, \mathcal{D})$ for all $\phi \in \Phi$. This ensures that $\mathcal{C}$ and $\mathcal{D}$ give the same classification of boundary points.

Unfortunately under this equivalence relation g is not guaranteed to remain injective. The necessary example is, predictably, given by the Misner spacetime, [8]. The reason why is very interesting and highlights an important area of research for the Abstract Boundary. We don't go into the details here. We do, however, have the following result, which provides some guidance.

Lemma 19. *Let $\mathcal{C} \in BPP(\mathcal{M})$ and let $\mathcal{D} = g \circ f(\mathcal{C})$. Then* $\mathrm{App}(\phi, \mathcal{C}) = \mathrm{App}(\phi, \mathcal{D})$ *and* $\mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{C}) = \mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{D})$ *for all* $\phi \in \Phi$.

Proof. Let $p \in \mathrm{App}(\phi, \mathcal{C})$ then there exists $\gamma \in \mathcal{C}$ so that p is an accumulation point of $\phi \circ \gamma$. Since $p \in \partial\phi(\mathcal{M})$ we know that γ must be inextendible and not contained in a compact region. That is $\gamma \in f(\mathcal{C})$. By definition we then know that $\gamma \in g \circ f(\mathcal{C})$. That is $\gamma \in \mathcal{D}$ so that $p \in \mathrm{App}(\phi, \mathcal{D})$. We note that if $p \in \mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{C})$ then there exists $\gamma \in \mathcal{C}$ so that γ is bounded and p is an accumulation point of $\phi \circ \gamma$. By the same argument $\gamma \in \mathcal{D}$ and therefore $p \in \mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{D})$ as required. □

This suggests that to solve the failure of g to remain injective we should define an equivalence relation on $X(\mathcal{M})$ as well. Hence we make the following definitions.

Definition 20. *Let $\approx_1$ be the equivalence relation on $BPP(\mathcal{M})$ given by $\mathcal{C} \approx_1 \mathcal{D}$ if and only if for all $\phi \in \Phi$,* $\mathrm{App}(\phi, \mathcal{C}) = \mathrm{App}(\phi, \mathcal{D})$ *and* $\mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{C}) = \mathrm{App}_{\mathrm{Sing}}(\phi, \mathcal{D})$. *We denote the equivalence class of $\mathcal{C}$ by $[\mathcal{C}]_1$.*

Definition 21. *Let $\equiv_1$ be the equivalence relation on $X(\mathcal{M})$ given by $S \equiv_1 P$ if and only if $g(S) \approx_1 g(P)$. We denote the equivalence class of S by $[S]_1$.*

Both equivalence relations are clearly well defined, so we can now show that the induced functions are bijective and mutually inverse.

Theorem 22. *The induced functions $f_1 : \frac{BPP(\mathcal{M})}{\approx_1} \to \frac{X(\mathcal{M})}{\equiv_1}$ and $g_1 : \frac{X(\mathcal{M})}{\equiv_1} \to \frac{BPP(\mathcal{M})}{\approx_1}$ are bijective and mutually inverse.*

Proof. We need to show that f_1 and g_1 are well-defined. Suppose that $\mathcal{C} \approx_1 \mathcal{D}$ we need to show that $f(\mathcal{C}) \equiv_1 f(\mathcal{D})$. That is we need to show that $g \circ f(\mathcal{C}) \approx_1 g \circ f(\mathcal{D})$. From lemma 19, however, we know that $g \circ f(\mathcal{C}) \approx_1 \mathcal{C} \approx_1 \mathcal{D} \approx_1 g \circ f(\mathcal{D})$ as required. Likewise suppose that $S \equiv_1 P$ we need to show that $g(S) \approx_1 g(P)$, but this follows directly from definition 21. Hence f_1 and g_1 are well defined.

We now show that $f_1 \circ g_1([S]_1) = [S]_1$ and $g_1 \circ f_1([\mathcal{C}]_1) = [\mathcal{C}]_1$. Let $S \in X_{\mathrm{b.p.p.}}(\mathcal{M})$, then we can calculate that

$$\begin{aligned} f_1 \circ g_1([S]_1) &= [f \circ g(S)]_1 \\ &= \Big[\Big(\{\delta : \exists \gamma \in S \ \delta < \gamma\} \cup \mathcal{C}_{\mathrm{Norm}}(\mathcal{M} - S_p)\Big) \cap X(\mathcal{M})\Big]_1 \\ &= [S]_1, \end{aligned}$$

by lemma 17 as required. Let $[\mathcal{C}]_1 \in \frac{BPP(\mathcal{M})}{\approx_1}$, from lemma 19 we know that $g \circ f(\mathcal{C}) \approx_1 \mathcal{C}$ so that

$$\begin{aligned} g_1 \circ f_1([\mathcal{C}]_1) &= [g \circ f(\mathcal{C})]_1 \\ &= [\mathcal{C}]_1, \end{aligned}$$

as required. Therefore f_1 and g_1 are both bijective are mutually inverse. □

One of the reasons for investigating the link between $BPP(\mathcal{M})$ and $X_{\text{b.p.p.}}(\mathcal{M})$ is to derive a simple description of b.p.p. satisfying sets. It is therefore unfortunate that we have been forced to use $\equiv_1$. The definition of $\equiv_1$, definition 21, prevents easy computation of the equivalence classes of $X_{\text{b.p.p.}}(\mathcal{M})$. This almost defeats the purpose of using elements of $X_{\text{b.p.p.}}(\mathcal{M})$.

We have been put into this situation by defining $\approx_1$ as we have. While this is the most sensible equivalence relation it may not be the best. At it's most base level the reason for this difficulty boils down to having no easy way to distinguish classes of curves based on their limit points in any envelopment. Fortunately there is another alternative which gives us better tools for when using elements of $X_{\text{b.p.p.}}(\mathcal{M})$, but is not quite as mathematically nice as $\approx_1$.

3.2 The computationally nice equivalence relation

If we are willing to forgo the insistence that b.p.p. satisfying sets $\mathcal{C}$ and $\mathcal{D}$ must be identified if

$$\text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{D})$$

and

$$\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}_{\text{Sing}}(\phi, \mathcal{D}).$$

Then another pair of equivalence relations can be investigated.

Rather than identifying b.p.p. satisfying sets based on limit points we can identify them based on images of curves. The result is clearer and allows us to work with elements of $BPP(\mathcal{M})$ via elements of $X_{\text{b.p.p.}}(\mathcal{M})$. This will still remove the unnecessary fineness of $<$. Since we would also like to remove the distinction between different ways of adding extendible curves to elements of $X_{\text{b.p.p.}}(\mathcal{M})$, it makes sense to start with the following definition.

Definition 23. *Define an equivalence relation $\equiv_2$ on the set $X_{\text{b.p.p.}}(\mathcal{M})$ by $S \equiv_2 P$ if and only if*

$$\forall \gamma : [a,b) \to \mathcal{M} \in S \ \exists \delta : [p,q) \to \mathcal{M} \in P, c, r \in \mathbb{R}$$

so that $\gamma([c,b)) = \delta([r,q))$ *and either both* γ, δ *are bounded or unbounded, and,*

$$\forall \delta : [p,q) \to \mathcal{M} \in P \; \exists \gamma : [a,b) \to \mathcal{M} \in S, c, r \in \mathbb{R}$$

so that $\gamma([c,b)) = \delta([r,q))$ *and either both* γ, δ *are bounded or unbounded.*

That is $S \equiv_2 P$ *if and only if the images of all curves in* S *is equal to the images of all curves in* P *and that the boundedness and unboundedness of curves that have the same image, excluding some finite length portion of the domains of each curves, are the same.*

We shall write the equivalence class of S *by* $[S]_2$*.*

Lemma 24. *The equivalence relation* $\equiv_2$ *on* $X_{\text{b.p.p.}}(\mathcal{M})$ *is well defined.*

Proof. Let $S \in X_{\text{b.p.p.}}(\mathcal{M})$ then as $\gamma < \gamma$ for all $\gamma \in S$ we can see that $S \equiv_2 S$.

It is clear that the symmetry of $\equiv_2$ is satisfied by definition.

Suppose that $S, P, Q \in X_{\text{b.p.p.}}(\mathcal{M})$ are such that $S \equiv_2 P$ and $P \equiv_2 Q$. Let $\gamma : [a,b) \to \mathcal{M} \in S$ then there exists $\delta : [p,q) \to \mathcal{M} \in P$ and $c, r \in \mathbb{R}$ so that $\gamma([c,b)) = \delta([r,q))$ and either both γ and δ are bounded or unbounded. Likewise as $\delta \in P$ there exists $\mu : [u,v) \to \mathcal{M} \in Q$ and $s, w \in \mathbb{R}$ so that $\delta([s,q)) = \mu([w,v))$ and either both are bounded or unbounded. Without loss of generality assume that $r < s$, then as $\gamma([c,b)) = \delta([r,q))$ there must exist $d \in \mathbb{R}$ so that $\gamma([d,b)) = \delta([s,q)) = \mu([w,v))$. If γ is bounded then δ must be bounded and therefore μ must also be bounded. Likewise, if γ is unbounded then μ must be unbounded. The reverse direction follows similarly.

Therefore $\equiv_2$ is well defined. □

Definition 25. *Define* $\mathcal{C}, \mathcal{D} \in BPP(\mathcal{M})$ *to be equivalent, denoted* $\mathcal{C} \approx_2 \mathcal{D}$*, if and only if* $f(\mathcal{C}) \equiv_2 f(\mathcal{D})$*. It is clear from the definition that this provides a well-defined equivalence relation. Denote the equivalence class of* $\mathcal{C}$ *by* $[\mathcal{C}]_2$*.*

In effect we are saying that two b.p.p. satisfying sets of curves are equivalent if the subset of inextendible non-compact curves contained in each are equivalent. This makes sense as we know from proposition 10 that only the inextendible curves that are not contained in a compact set tell us anything about the a-boundary and the set of these curves is precisely the set $f(\mathcal{C})$. The following two results tell us how to interpret the equivalence relation $\approx_2$ and therefore also the equivalence relation $\equiv_2$.

Proposition 26. *Let* $\mathcal{C}, \mathcal{D} \in BPP(\mathcal{M})$ *then* $\mathcal{C} \approx_2 \mathcal{D}$ *implies that* $\mathcal{C} \approx_1 \mathcal{D}$*.*

Proof. Suppose that $\mathcal{C} \approx_2 \mathcal{D}$ and let $p \in \text{App}(\phi, \mathcal{C})$. Then there exists $\gamma \in \mathcal{C}$ so that p is an accumulation point of $\phi \circ \gamma$. Since $p \in \partial\phi(\mathcal{M})$ it must be the case that $\gamma \in f(\mathcal{C})$. Since $\mathcal{C} \approx_2 \mathcal{D}$ we know that $f(\mathcal{C}) \equiv_2 f(\mathcal{D})$. Hence there must exist $\delta \in f(\mathcal{D})$ so that the images of γ and δ agree, except on some compact portion. Thus p must be an accumulation point of $\phi\circ\delta$ and therefore $p \in \text{App}(\phi, \mathcal{D})$. Moreover, if γ is bounded by definition 23 δ must also be bounded. Therefore $p \in \text{App}_{\text{Sing}}(\phi, \mathcal{C})$ implies that $p \in \text{App}_{\text{Sing}}(\phi, \mathcal{D})$. Therefore $\mathcal{C} \approx_1 \mathcal{D}$ as required. ☐

Thus $\approx_2 \subset \approx_1$, so that if $\mathcal{C} \approx_2 \mathcal{D}$ then

$$\text{App}(\phi, \mathcal{C}) = \text{App}(\phi, \mathcal{D})$$

and

$$\text{App}_{\text{Sing}}(\phi, \mathcal{C}) = \text{App}_{\text{Sing}}(\phi, \mathcal{D})$$

for all $\phi \in \Phi$. The Misner spacetime gives an example showing that this inclusion is proper. We now show that the functions f_2 and g_2 induced by $\approx_2$ and $\equiv_2$ are bijective and mutually inverse.

Proposition 27. *The functions*

$$f_2 : \frac{BPP(\mathcal{M})}{\approx_2} \to \frac{X_{\text{b.p.p.}}(\mathcal{M})}{\equiv_2}$$

and

$$g_2 : \frac{X_{\text{b.p.p.}}(\mathcal{M})}{\equiv_2} \to \frac{BPP(\mathcal{M})}{\approx_2}$$

are bijective and mutually inverse.

Proof. We first show that f_2 and g_2 are well defined. Let $\mathcal{C} \approx_2 \mathcal{D}$, we must show that $f(\mathcal{C}) \equiv_2 f(\mathcal{D})$. This is true by definition and therefore f_2 is well-defined. Now let $S \equiv_2 P$, then we know that $f \circ g(S) = S$ from corollary 18. Hence $f(g(S)) = S \equiv_2 P = f(g(P))$ so by definition $g(S) \approx_2 g(P)$ as required. Thus the function g_2 is well-defined.

We now show that $f_2 \circ g_2([S]_2) = [S]_2$ and $g_2 \circ f_2([\mathcal{C}]_2) = [\mathcal{C}]_2$. Let $[\mathcal{C}]_2 \in \frac{BPP(\mathcal{M})}{\approx_2}$ and note that $f \circ g \circ f(\mathcal{C}) = f(\mathcal{C})$ by corollary 18. By definition 25 we know that $g \circ f(\mathcal{C}) \approx_2 \mathcal{C}$ or rather that $g_2 \circ f_2([\mathcal{C}]_2) = [\mathcal{C}]_2$. Let $S \in X_{\text{b.p.p.}}(\mathcal{M})$ then,

$$\begin{aligned} f_2 \circ g_2([S]_2) &= f_2\Big([\{\delta : \exists\gamma \in S \ \delta < \gamma\} \cup \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)]_2\Big) \\ &= \Big[\big(\{\delta : \exists\gamma \in S \ \delta < \gamma\} \cup \mathcal{C}_{\text{Norm}}(\mathcal{M} - S_p)\big) \cap X(\mathcal{M})\Big]_2 \\ &= [S]_2. \end{aligned}$$

Thus we see that $f_2 \circ g_2([S]_2) = [S]_2$ and $g_2 \circ f_2([\mathcal{C}]_2) = [\mathcal{C}]_2$. This is sufficient to prove that f_2 and g_2 are bijective and mutually inverse as required. □

Thus using $\approx_2$ and $\equiv_2$, rather than $\approx_1$ and $\equiv_1$, we still get a one-to-one correspondence. It is, however, easier to work with $\equiv_2$ and therefore easier to exploit the structure of elements of $BPP(\mathcal{M})$ outlined here.

The downside of $\approx_2$ is that there might be sets $\mathcal{C}, \mathcal{D} \in BPP(\mathcal{M})$ so that $\mathcal{C} \approx_1 \mathcal{D}$ but $\mathcal{C} \not\approx_2 \mathcal{D}$. The two classes of null geodesics and two maximal embeddings of the Misner space discussed in [9, page 171] give an explicit example of this. That is there might exist sets $\mathcal{C}$ and $\mathcal{D}$ so that the classification of a-boundary points is the same, but $\approx_2$ fails to encode this. The existence of $\mathcal{C}$ and $\mathcal{D}$ will depend on the manifold $\mathcal{M}$ and its metric. Because of how $\approx_2$ is defined this can only occur when the curves in $\mathcal{C}$ and $\mathcal{D}$ do not have the same images, but do have the same limit points.

4 Set-like operations that respect the b.p.p.

Using the structures given above it is now possible to define set-like operations on $BPP(\mathcal{M})$ so that the b.p.p. is preserved. For example we can now define $\mathcal{C} \cup \mathcal{D}$. Under the usual set operation the result of $\mathcal{C} \cup \mathcal{D}$ is not guaranteed to satisfy the b.p.p. Since the normal set operations are easier to use on $X_{\text{b.p.p.}}(\mathcal{M})$, the basic idea is to exploit this by defining the new operations on $BPP(\mathcal{M})$ by using f and g.

Let $\approx$ and $\equiv$ be either of the two pairs of equivalence relations $\approx_1$ and $\equiv_1$ or $\approx_2$ and $\equiv_2$. Let $[\mathcal{C}]$ be either $[\mathcal{C}]_1$ or $[\mathcal{C}]_2$ and let f_* and g_* be either f_1 and g_1 or f_2 and g_2, respectively.

An important point is that it does not make sense to define the set-like operations on $BPP(\mathcal{M})$ on all elements of $BPP(\mathcal{M})$. For example if $\gamma : [0, \infty) \to \mathcal{M} \in \mathcal{C}$ and $\gamma \circ \arctan(t) \in \mathcal{D}$ and $\gamma \in X(\mathcal{M})$ then it makes no sense to define $\mathcal{C} \cup \mathcal{D}$. Should γ or $\gamma \circ \arctan$ be in $\mathcal{C} \cup \mathcal{D}$? We can't have both as they are reparametrisations of each other and one is bounded while the other in unbounded. Thus we give the following definition, which allows for this situation.

Definition 28. *Let $\mathcal{C}, \mathcal{D} \in BPP(\mathcal{M})$ we make the following definitions.*

Subset *We say that $\mathcal{C}$ is a subset of $\mathcal{D}$, denoted $\mathcal{C} \subset_\approx \mathcal{D}$, if and only if $f_*([\mathcal{C}]) \subset f_*([\mathcal{D}])$.*

Union *A union of the sets $\mathcal{C}$ and $\mathcal{D}$, is defined as $g_*(S)$, where S is a maximal element of $X_{\text{b.p.p.}}(\mathcal{M})$ so that $[S] \subset f_*([\mathcal{C}]) \cup f_*([\mathcal{D}])$. Where* *there exists a maximum S we shall refer to $g_*(S)$ as the union of the sets $\mathcal{C}$ and $\mathcal{D}$ and denote it by $\mathcal{C} \cup_\approx \mathcal{D}$. In this case $\mathcal{C} \cup_\approx \mathcal{D} = g_*\left(f_*([\mathcal{C}]) \cup f_*([\mathcal{D}])\right)$.*

Intersection *An intersection of the sets $\mathcal{C}$ and $\mathcal{D}$, is defined as $g_*(S)$, where S is a maximal element of $X_{\text{b.p.p.}}(\mathcal{M})$ so that $S \subset f_*([\mathcal{C}]) \cap f_*([\mathcal{D}])$. Where there exists a maximum S we shall refer to $g_*(S)$ as the intersection of the sets $\mathcal{C}$ and $\mathcal{D}$ and denote it by $\mathcal{C} \cap_\approx \mathcal{D}$. In this case $\mathcal{C} \cap_\approx \mathcal{D} = g_*\left(f_*([\mathcal{C}]) \cap f_*([\mathcal{D}])\right)$.*

Relative Complement *A relative complement of the set $\mathcal{C}$ and $\mathcal{D}$, is defined as $g_*(S)$, where S is a maximal element of $X_{\text{b.p.p.}}(\mathcal{M})$ so that $S \subset f_*([\mathcal{C}]) - f_*([\mathcal{D}])$. Where there exists a maximum S we shall refer to $g_*(S)$ as relative complement of the sets $\mathcal{C}$ and $\mathcal{D}$ and denote it by $\mathcal{C} -_\approx \mathcal{D}$ or $\mathcal{C} \backslash_\approx \mathcal{D}$. In this case $\mathcal{C} -_\approx \mathcal{D} = g_*\left(f_*([\mathcal{C}]) - f_*([\mathcal{D}])\right)$.*

As mentioned above it maybe the case that, for example, $f^*([\mathcal{C}]) \cup f^*([\mathcal{D}])$ is not in $X_{\text{b.p.p.}}(\mathcal{M})$. Hence the definitions appeal to the existence of a largest subset, S, of $f^*([\mathcal{C}]) \cup f^*([\mathcal{D}])$ that is in $X_{\text{b.p.p.}}(\mathcal{M})$. Such a subset must exist by Zorn's lemma and the fact that the union of a chain of sets in $X_{\text{b.p.p.}}(\mathcal{M})$ is also in $X_{\text{b.p.p.}}(\mathcal{M})$. Note that this only implies the existence of maximal subsets and not necessarily a maximum subset.

In the case that $f^*([\mathcal{C}]) \cup f^*([\mathcal{D}])$ is not in $X_{\text{b.p.p.}}(\mathcal{M})$ it is natural that a choice needs to be made, since we must include some curves and exclude others, as mentioned before the definition. Thus each maximal subset gives us one of those possible choices. Where $f^*([\mathcal{C}]) \cup f^*([\mathcal{D}])$ is in $X_{\text{b.p.p.}}(\mathcal{M})$, however, there will be a maximum subset, namely $f^*([\mathcal{C}]) \cup f^*([\mathcal{D}])$, thus the definition above fits with our intuition regarding b.p.p. satisfying sets.

5 Conclusion and Future Work

We have shown that sets that satisfy the b.p.p. are the same as sets of inextendible curves that are not contained in a compact region, modulo a technical condition and a pair of equivalence relations. We have also shown how to define the usual set relations on the set $BPP(\mathcal{M})$. These relations and the equivalence provide new ways to explore the effect of different choices of b.p.p. satisfying sets on the a-boundary classification.

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