Expanding perfect fluid generalizations of the C-metric

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We reexamine Petrov type D gravitational fields generated by a perfect fluid with spatially homogeneous energy density and in which the flow lines form a timelike non-shearing and non-rotating congruence. It is shown that the anisotropic such spacetimes, which comprise the vacuum C-metric as a limit case, can have *non-zero* expansion, contrary to the conclusion in the original investigation by Barnes [1]. This class consists of cosmological models with generically one and at most two Killing vectors. We construct their line element and discuss some important properties. The methods used in this investigation incite to deduce testable criteria regarding shearfree normality and staticity op Petrov type D spacetimes in general, which we add in an appendix.

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1. INTRODUCTION

The C-metric is a well-known exact solution of Einstein's vacuum equation with zero cosmological constant. The static region of the corresponding spacetime was first described by Weyl [2]. At about the same time Levi-Civita [3] constructed its line element in closed form, arriving at essentially one cubic polynomial with two parameters as the metric structure function. The C-metric is a Petrov type D solution for which at each spacetime point both Weyl principal null directions (PNDs) are geodesic, non-shearing, non-rotating but diverging; it thus belongs to the Robinson-Trautman class of solutions and was rediscovered as such [4]. The label 'C' derives from the invariant classification of static degenerate Petrov type D vacuum spacetimes by Ehlers and Kundt [5]. The importance of this solution as summarized by Kinnersley and Walker [6], is threefold First, the C-metric describes a spacetime with only two Killing vectors which can be fully analyzed. Next, it is an 'example of almost everything', most notably it describes a radiative, locally asymptotically flat spacetime, whilst containing a static region. The C-metric is contained in the class of boost-rotation-symmetric spacetimes [7, 8], which are the only axially symmetric, radiative and asymptotically flat spacetimes with two Killing vectors. Finally, the solution has a clear physical interpretation as the anisotropic gravitational field of two Schwarzschild black holes being uniformly accelerated in opposite directions by a cosmic string or strut, provided that $m\alpha < 1/\sqrt{27}$, where the mass m and acceleration α are equivalents of the two essential parameters of Levi-Civita [6, 9].

Generalizations of the C-metric have been widely considered. Adding a cosmological constant Λ is straightforward, and we will henceforth refer with 'C-metric' to such Einstein spaces. Incorporating electromagnetic charge $q^2 = e^2 + g^2$ is equally natural and leads to quartic structure functions [6]. Recently, the question how to include rotation for the holes received a new answer [10, 11], avoiding the NUT-like behavior of the previously considered 'spinning C-metric' [12, 13]. All these generalizations fit in the well-established class $\mathcal{D}$ of Petrov type D Einstein-Maxwell solutions with a non-null electromagnetic field possessing geodesic and non-shearing null directions aligned with the PNDs [14, 15], which reduces for zero electromagnetic field to the subclass $\mathcal{D}_0$ of Petrov type D Einstein spaces and which contains all well-known 4D black hole metrics. In fact, all $\mathcal{D}$-metrics can be derived by performing 'limiting contractions' [16] from the most general member, the Plebianski-Demianski line element [17], which exhibits two quartic structure functions with six essential parameters m, α, q^2, Λ, NUT parameter l [18] and angular momentum a. A physically comprehensive and simplified treatment can be found in [19], also surveying recent work in this direction.

In this paper we present a new family of Petrov type D, expanding and anisotropic perfect fluid (PF) generalizations of the C-metric. The direct motivation and background for this work is the following.

According to the Goldberg-Sachs theorem [20] the two PNDs of any member of $\mathcal{D}_0$ are precisely those null directions which are geodesic and non-shearing. Such a member is purely electric (i.e. its Weyl scalar Ψ_2 is real) precisely when both PNDs are moreover non-twisting (i.e. hypersurface-orthogonal (HO)). This is in particular the case for the C-metric. As we will show, it implies the existence of an *umbilical synchronization* (US), i.e., a non-shearing and non-rotating unit *timelike* vector field (tangent to a congruence of observers). The importance of USs in cosmology was stressed in [21]. If a congruence of observers measuring isotropic radiation admits orthogonal hypersurfaces, an US exists. Only small deviations from isotropy are seen in the cosmic microwave background, and scalar perturbations of a Friedmann-Lemaître-Robertson-Walker universe preserve the existence of an US [22]. In general, spacetimes admitting an US have zero magnetic part of the Weyl tensor wrt it [23]

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and thus are either of Petrov type O, or purely electric (PE) and of type D or I [16]. Conformally flat spacetimes always admit USs (see e.g. (6.15) in [16]). Trümper showed that algebraically general vacua with a US are static [24]. Motivated by this result and by his own work [25] on static PFs, Barnes [1] studied PF spacetimes with a US tangent to their flow lines. He was able to generalize Trümper's result to such PFs and recovered Stephani's results on conformally flat PF solutions which are either of generalized Schwarzschild type or of generalized Friedmann type (so called Stephani universes) [26]. The remaining type D solutions were integrated and invariantly partitioned, based on the direction of the gradient of the energy density w relative to the PNDs and the flow vector at each point. Class I, with w constant on the hypersurfaces orthogonal to the flow lines and thus the only class containing Einstein spaces as limit cases, was further subdivided using the gradient of Ψ_2 (cf. section § 2.2 for details). By solving the field equations, Barnes concludes that class ID, consisting of the anisotropic class I models, has solely non-expanding solutions. Hence, these PF solutions would not be viable as a cosmological model. However, based on an integrability analysis of class I in the Geroch-Held-Penrose (GHP) formalism [27], we found that this conclusion cannot be valid and this led to a detailed reinvestigation.

In this article the general line element of the full ID class, as originally defined in [1], will be constructed, and some elementary properties will be discussed. It contains both the known non-expanding perfect fluid models and the new expanding ones. We want to stress the following point. The full class represents a PF generalization of the C-metric in the naive sense that the C-metric is contained as the Einstein space limit. The physical interpretation of this fact is however not established. This would require to exhibit this solution for small masses as a perturbation of a known PF solution, just as the C-metric interpretation of small accelerating black holes has been established in a flat or (anti-)de Sitter background [6, 28–31].

However, the mathematical relation with the C-metric is useful. As already deduced in [1], the PF solution is, just as the C-metric, conformally related to the direct sum of two 2D metrics. The fact that one part is equal for the PF solution and the C-metric is helpful in the analysis, e.g. we will show that (a part of) the axis of symmetry can readily be identified as a cosmic string, analogous to the cosmic string present in the C-metric. The non-static spacetimes presented form exact perfect fluid solutions with only this symmetry, and the analysis appears to be within reach. For the expanding ID PF models both the matter density $w(t)$ and the expansion scalar $\theta(t)$ can be arbitrary functions. This freedom is displayed explicitly in the metric form, and makes the solutions more attractive as a cosmological model.

The paper is organized as follows. In section 2 we present the GHP approach to class I. We derive a closed set of equations, construct suitable scalar invariants, interpret the invariant subclassification of [1] and perform a partial integration. At the end we provide alternative characterizations for the Einstein space members. In section 3 we first finish the construction of the general ID line element in an elegant and transparent way, and point out the calculative error of [1] in the original approach. Finally the basic properties of the ID perfect fluid models are summarized. The work greatly benefited from the use of the GHP formalism, which at the same time elucidates the deviation of the C-metric. In appendix A we provide a concise introduction to this formalism for the non-expert reader. In appendix B, finally, we present criteria for deciding when a Petrov type D spacetime admits a (rigid) US or is static.

Notation. For spacetimes (M, g_{ab}) we take $(+ + + -)$ as the metric signature and use geometrized units $8\pi G = c = 1$, where G is the gravitational coupling constant and c the speed of light. Λ denotes the cosmological constant. We make consistent use of the abstract Latin index notation for tensor fields, as advocated in [32]. Round (square) brackets denote (anti-)symmetrization, η_{abcd} is the spacetime alternating pseudo-tensor and $\nabla_c T_{ab\ldots}$ designates the Levi-Civita covariant derivative of the tensor field $T_{ab\ldots}$. One has

$$\mathrm{d}_a f = \nabla_a f, \quad \mathrm{d}_b Y_a = \nabla_{[b} Y_{a]}$$

for the exterior derivative of a scalar field f, resp. one-form field Y_a, and we write

$$\mathbf{X}(f) \equiv X^a \mathrm{d}_a f, \quad f_{,x^i} \equiv \partial_{x^i}{}^a \mathrm{d}_a f$$

for the Leibniz action of a tetrad vector field X^a and x^i-coordinate vector field $\partial_{x^i}{}^a$. However, we use index-free notation in line elements $\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$. The specific GHP notation is introduced in appendix A.

2. GHP APPROACH TO CLASS I

2.1. Definition and integrability

We consider Barnes' class I [1], consisting of spacetimes (M, g_{ab}) with the following properties:

(i) the spacetime admits a unit timelike vector field u^a ($u^a u_a = -1$) which is non-shearing and non-rotating, i.e., its covariant derivative is of the form

$$\nabla_b u_a = \theta h_{ab} - \dot{u}_a u_b, \quad h_{ab} \equiv g_{ab} + u_a u_b, \tag{1}$$

where the acceleration $\dot{u}_a = u^b \nabla_b u_a$ and expansion rate $\theta = \nabla_a u^a$ are the remaining kinematic quantities of u^a;

(ii) the Weyl-Petrov type is D throughout;

(iii) the Einstein tensor has the structure

$$G_{ab} = S u_a u_b + p g_{ab} = w u_a u_b + p h_{ab}, \tag{2}$$

$$D_a w \equiv h_a{}^b \nabla_b w = 0, \tag{3}$$

i.e., the spacetime represents the gravitational field of either a perfect fluid with shearfree normal four-velocity u^a, pressure $p+\Lambda$ and spatially homogeneous energy density $w-\Lambda$ (case $S\equiv w+p\neq 0$) or a vacuum (*Einstein space* case $S=0$, where $w=-p$ may be identified with Λ).

By virtue of condition (i) the Weyl tensor C_{abcd} of the spacetime is purely electric wrt u^a [23]:

$$\begin{aligned} E_{ab} &\equiv C_{acbd}u^c u^d \neq 0, \\ H_{ab} &\equiv \frac{1}{2}\eta_{acmn}C^{mn}{}_{bd}u^c u^d = 0. \end{aligned} \tag{4}$$

The vector field u^a is Weyl principal [16] which, in conjunction with condition (ii), lies in the plane Σ of Weyl PND's at each point, i.e., Weyl principal null vectors k^a and l^a (subject to the normalization condition $k^a l_a=-1$) and $q=Q^2>0$ exist such that

$$u^a = \frac{1}{\sqrt{2q}}\left(qk^a + l^a\right). \tag{5}$$

Within the GHP formalism (cf. appendix A) based on a Weyl principal null tetrad (WPNT) $(k^a, l^a, m^a, \overline{m}^a)$, q is (-2,-2)-weighted and the conditions (i)-(iii) translate into

$$\begin{aligned}
\text{(i)}: \quad & \lambda = q\overline{\sigma}, && (6)\\
& \mu - \overline{\mu} + q(\rho-\overline{\rho}) = 0, && (7)\\
& \pi + \overline{\tau} = q\overline{\kappa} + q^{-1}\nu, && (8)\\
& Þ' q - qÞ q = -2q(\mu - q\overline{\rho}), && (9)\\
& ð q = ð' q = 0, && (10)\\
\text{(ii)}: \quad & \Psi_0=\Psi_1=\Psi_3=\Psi_4=0\neq\Psi_2; && (11)\\
\text{(iii)}: \quad & \Phi_{01}=\Phi_{12}=\Phi_{02}=0, && (12)\\
& 8\Phi_{11} = 4q\Phi_{00} = 4q^{-1}\Phi_{22} = S, && (13)\\
& ð w = ð' w = 0, \quad Þ' w - qÞ w = 0. && (14)
\end{aligned}$$

The $[ð, ð'](q)$ commutator relation yields

$$\overline{\Psi}_2 = \Psi_2, \tag{15}$$

which expresses (4), and we denote

$$\Psi \equiv -E_{ab}v^a v^b = 2\Psi_2, \quad v^a \equiv \frac{1}{\sqrt{2q}}\left(qk^a - l^a\right). \tag{16}$$

Combining (6)-(16) with the GHP Bianchi equations results in

$$\begin{aligned}
& \kappa=\nu=0, \quad \sigma=\lambda=0, && (17)\\
& \overline{\rho}=\rho, \quad \overline{\mu}=\mu, \quad \pi=-\overline{\tau}, && (18)\\
& Þ\Psi = 3\rho\Psi, \quad Þ'\Psi = -3\mu\Psi, && (19)\\
& ð\Psi = 3\tau\Psi, \quad ð'\Psi = -3\pi\Psi, && (20)\\
& Þ' S - qÞ S = S(Þ q - \mu + q\rho), && (21)\\
& ð S = \tau S, \quad ð' S = \overline{\tau} S, && (22)\\
& Þ' w = qÞ w = -\frac{3S(\mu - q\rho)}{2}. && (23)
\end{aligned}$$

Herewith the Ricci equations reduce to

$$\begin{aligned}
Þ\mu &= -Þ'\rho && (24)\\
&= -ð'\tau + \mu\overline{\rho} + \tau\overline{\tau} + \frac{\Psi}{2} + \frac{w}{3} - \frac{S}{4}, && (25)\\
Þ'\mu &= -\mu^2 - \frac{qS}{4}, \quad ð\mu = ð'\mu = 0, && (26)\\
Þ\rho &= \rho^2 + \frac{S}{4q}, \quad ð\rho = ð'\rho = 0, && (27)\\
Þ\tau &= Þ'\tau = 0, \quad ð\tau = \tau^2, && (28)\\
-ð\pi &= ð'\tau = ð\overline{\tau} \equiv \frac{H}{2} && (29)
\end{aligned}$$

and the complex conjugates of (28), while the commutator relations applied to a (w_p, w_q)-weighted scalar η become

$$\begin{aligned}
[Þ, Þ']\eta &= (\mathrm{w}_p + \mathrm{w}_q)\left(-\tau\overline{\tau} + \frac{\Psi}{2} - \frac{w}{6} + \frac{S}{4}\right)\eta, && (30)\\
[ð, ð']\eta &= (\mathrm{w}_p - \mathrm{w}_q)\left(-\mu\rho + \frac{\Psi}{2} - \frac{w}{6}\right)\eta, && (31)\\
[Þ, ð]\eta &= (-\tauÞ + \rhoð + \mathrm{w}_q\,\rho\tau)\eta, && (32)\\
[Þ, ð']\eta &= (-\overline{\tau}Þ + \rhoð' + \mathrm{w}_p\,\rho\overline{\tau})\eta, && (33)\\
[Þ', ð]\eta &= (-\tauÞ' - \muð + \mathrm{w}_p\,\mu\tau)\eta, && (34)\\
[Þ', ð']\eta &= (-\overline{\tau}Þ' - \muð' + \mathrm{w}_q\,\mu\overline{\tau})\eta. && (35)
\end{aligned}$$

Then the $[ð, ð'](\tau)$, $[ð, ð'](\overline{\tau})$, $[Þ, ð'](\tau)$ and $[Þ', ð'](\tau)$ commutator relations imply

$$\begin{aligned} & ð H = 2\tau(H+\Psi-G), \quad ð' H = 2\overline{\tau}(H+\Psi-G), \\ & Þ H = \rho(H+F), \quad Þ H = -\mu(H+F), \end{aligned} \tag{36}$$

where

$$F \equiv 2\tau\overline{\tau}, \quad G \equiv 2\mu\rho + \frac{w}{3}. \tag{37}$$

One checks that the integrability conditions for the formal system of PDE's (9)-(35) are now identically satisfied, indicating that corresponding solutions exist. Those for which u^a is non-expanding additionally satisfy

$$\theta \sim \mu - q\overline{\rho} = 0 \tag{38}$$

(cf. (92) below). However, (38) does not follow as a consequence of the ansätze; this implies the existence of expanding anisotropic perfect fluid models in class I (§ 3). Neither do the equations tell us about the sign of the scalar invariant $\mu\rho$; consistency with (38) requires that $\mu\rho \geq 0$, which prevents the class I Einstein spaces to be static in general (§ 2.3).

2.2. Metric structure and subclassification

The first, second and last parts of (17)-(18) precisely account for the hypersurface-orthogonality of k^a, l^a and m^a, respectively. Thus real scalar fields u, v, (zero-weighted) and U, V ($(-1,-1)$- resp. $(1,1)$-weighted), and complex scalar fields ζ (zero-weighted) and Z ((1,-1)-weighted) exist such that

$$d_a u = \frac{\Psi^{1/3}}{U} k_a, \quad d_a v = \frac{\Psi^{1/3}}{V} l_a, \quad d_a \zeta = \frac{\Psi^{1/3}}{Z} m_a. \quad (39)$$

By (A10) this is equivalent to

$$Þ' u = -\Psi^{1/3}/U, \quad Þ u = ð u = ð' u = 0, \quad (40)$$
$$Þ v = -\Psi^{1/3}/V, \quad Þ' v = ð v = ð' v = 0, \quad (41)$$
$$ð' \zeta = \Psi^{1/3}/Z, \quad Þ \zeta = Þ' \zeta = ð \zeta = 0, \quad (42)$$
$$ð \overline{\zeta} = \Psi^{1/3}/\overline{Z}, \quad Þ \overline{\zeta} = Þ' \overline{\zeta} = ð' \overline{\zeta} = 0. \quad (43)$$

The commutator relations (32)-(35) applied to u, v, ζ and $\overline{\zeta}$ then yield

$$ð U = ð' U = ð V = ð' V = 0, \quad (44)$$
$$Þ Z = Þ' Z = Þ \overline{Z} = Þ' \overline{Z} = 0. \quad (45)$$

Hence, when we take these fields as coordinates, (39)-(45) imply that the zero-weighted fields UV and $Z\overline{Z}$ only depend on (u, v), resp. $(\zeta, \overline{\zeta})$, such that all class I metrics are conformally related to direct sums of metrics on two-spaces:

$$g_{ab} = \Psi^{-2/3}(g^{\perp}_{ab} \oplus g^{\Sigma}_{ab}), \quad (46)$$
$$g^{\perp}_{ab} \equiv 2\Psi^{2/3} m_{(a}\overline{m}_{b)} = 2Z\overline{Z}(\zeta, \overline{\zeta})\, d_{(a}\zeta\, d_{b)}\overline{\zeta}, \quad (47)$$
$$g^{\Sigma}_{ab} \equiv -2\Psi^{2/3} k_{(a} l_{b)} = -2UV(u, v)\, d_{(a} u\, d_{b)} v. \quad (48)$$

The line elements of $g^{\perp}_{ab}$ and g^{Σ}_{ab} will be denoted by $ds^2_{\perp}$, resp. ds^2_{Σ}.

In the case where such a two-space is not of constant curvature, however, we will construct more suitable coordinates in the sequel. Inspired by the GHP manipulations (48)-(53) of [33] for type D vacua [47], we start this construction by deducing suitable combinations of the scalar invariants F, G, H and Ψ. From (A10), (14), (23) and (28)-(37) it is found that

$$d_a F = 3\Psi^{1/3} \varphi\, \alpha_a, \quad d_a G = 3\Psi^{1/3} \gamma\, \beta_a, \quad (49)$$
$$d_a \varphi = 2\Psi^{1/3} x\, \alpha_a, \quad d_a \gamma = 2\Psi^{1/3} y\, \beta_a, \quad (50)$$
$$d_a x = \Psi^{1/3}\, \alpha_a, \quad d_a y = \Psi^{1/3}\, \beta_a, \quad (51)$$

where

$$\alpha_a \equiv \overline{\tau} m_a + \tau \overline{m}_a, \quad \beta_a \equiv \mu k_a - \rho l_a \quad (52)$$

are invariantly defined one-forms and

$$\varphi \equiv \frac{H + F}{3\Psi^{1/3}}, \quad \gamma \equiv \frac{-H + \Psi + F + 2G}{3\Psi^{1/3}}, \quad (53)$$
$$x \equiv \frac{H + \Psi - G}{3\Psi^{2/3}}, \quad y \equiv \frac{-H + 2\Psi + G}{3\Psi^{2/3}}. \quad (54)$$

Consequently, the scalar invariants

$$C \equiv 3(\varphi - x^2) = 3(\gamma - y^2), \quad (55)$$
$$D \equiv -x^3 - Cx + F = y^3 + Cy - G. \quad (56)$$

are *constant* ($d_a C = d_a D = 0$). From (54) and (56) we conclude that F, G, H and Ψ are biunivocally related to x, y, C and D, where

$$2\tau\overline{\tau} \equiv F = x^3 + Cx + D, \quad (57)$$
$$2\mu\rho \equiv G - \frac{w}{3} = y^3 + Cy - D - \frac{w}{3}, \quad (58)$$
$$2ð'\tau \equiv H = 2x^3 + 3x^2 y + Cy - D, \quad (59)$$
$$\Psi = (x + y)^3 \neq 0. \quad (60)$$

Barnes [1] partitioned class I according to the position of the gradient $\nabla^a \Psi$ relative to Σ and $\Sigma^{\perp}$. This relates to the vanishing of the invariants $\tau\overline{\tau} = -\pi\tau$ or $\mu\rho$, maximal symmetry of $g^{\perp}_{ab}$ or g^{Σ}_{ab} and spatial rotation or boost isotropy of g_{ab}, as follows.

First assume $\tau = 0$. In this case it follows from (29) and the first parts of (37) and (53)-(56) that x is constant and

$$H = F = \varphi = 0, \quad C = -3x^2, \quad D = 2x^3,$$
$$\Psi - G = 3x\Psi^{2/3}. \quad (61)$$

In combination with (18), (20) and the first parts of (51)-(52) one gets

$$\tau\overline{\tau} = 0 \Leftrightarrow \pi = \tau = 0 \Leftrightarrow x = \mathit{const} \Leftrightarrow \nabla^a \Psi \in \Sigma. \quad (62)$$

The $[ð, ð']$ commutator relation applied to ζ, $\overline{\zeta}$ and Z imply $ð Z = ð' \overline{Z} = 0$ and $ð ð' Z = 3x\Psi^{2/3} Z$. Herewith the Gaussian curvature of the two-space with metric $g^{\perp}_{ab}$ becomes

$$K^{\perp} = -(Z\overline{Z})^{-1}(\ln(Z\overline{Z}))_{,\zeta\overline{\zeta}} = -\Psi^{-2/3} ð ð' (\ln Z\overline{Z})$$
$$= -\Psi^{-2/3} ð \left(\frac{ð' Z}{Z}\right) = -3x,$$

where the dual of (39) was used in the calculation. In conjunction with the results of Goode and Wainwright [46], we conclude that (62) yields the class I solutions which are locally rotationally symmetric (LRS) of label II in the Stewart-Ellis classification [34], characterized by $g^{\perp}_{ab}$ having constant curvature $K^{\perp} = -3x$. As well known (see e.g. the appendix of [35]) the coordinates ζ and $\overline{\zeta}$ may then be adapted such that $Z\overline{Z}(\zeta, \overline{\zeta}) = (1 + K^{\perp}\zeta\overline{\zeta}/2)^{-1}$ in (47), or an alternative form may be taken:

$$ds^2_{\perp} = \frac{2 d\zeta d\overline{\zeta}}{1 + \frac{K^{\perp}}{2}\zeta\overline{\zeta}} = Y_{\perp}^2 (dx_1^2 + \cos(\sqrt{k_{\perp}} x_1)^2 dx_2^2),$$
$$K^{\perp} = k_{\perp} Y_{\perp}^{-2}, \quad k_{\perp} \in -1, 0, 1. \quad (63)$$

Now assume $\mu\rho = 0$. It follows from (24)-(27) and the second parts of (37) and (50)-(56) that

$$S = 0, \quad G = \frac{w}{3} \equiv \frac{\Lambda}{3}, \quad -H + 2\Psi + \frac{\Lambda}{3} = 3y\Psi^{2/3},$$
$$\gamma = \mu = \rho = 0, \quad C = -3y^2, \quad D = -2y^3 - \frac{\Lambda}{3}. \quad (64)$$

In conjunction with (19) and the second parts of (51)-(52) this implies

$$\mu\rho = 0 \Leftrightarrow \mu = \rho = 0 \Leftrightarrow y\ =\ const \Leftrightarrow \nabla^a\Psi \in \Sigma^\perp. \quad (65)$$

By a similar reasoning as in the case $\tau = 0$ one concludes that (65) yields the locally boost isotropic Einstein spaces of Petrov type D, characterized by g^Σ_{ab} having constant curvature $K^\Sigma = -3y$, such that in this case one may take $UV(u, v) = (1 - K^\Sigma uv/2)^{-1}$ in (48) and we have

$$\mathrm{d}s^2_\Sigma = -\frac{2\mathrm{d}u\mathrm{d}v}{1-\frac{K^\Sigma}{2}uv} = Y_\Sigma^2(\mathrm{d}x_3^2 - \cos(\sqrt{k_\Sigma}x_3)^2\mathrm{d}x_4^2),$$
$$K^\Sigma = k_\Sigma Y_\Sigma^{-2}, \quad k_\Sigma \in -1, 0, 1. \quad (66)$$

With the two-surface element written in the second form, it is clear that

$$\partial_{x_4}{}^a = -\Psi^{-2/3}\cos(\sqrt{k_\Sigma}x_3)^2\mathrm{d}^a x_4 \quad (67)$$

is a HO timelike Killing vector field.

Four subclasses of class I thus arise, which were labeled by Barnes as follows:

$$\text{IA}: \ \tau = 0 = \mu\rho, \quad \text{IB}: \ \tau = 0 \neq \mu\rho,$$
$$\text{IC}: \ \tau \neq 0 = \mu\rho, \quad \text{ID}: \ \tau \neq 0 \neq \mu\rho.$$

We proceed with the respective integrations. Notice that in the joint case $\mu\rho\tau = 0$ one has

$$2(\tau\overline{\tau} + \mu\rho) = (x+y)^3 + K(x+y)^2 - \frac{w}{3}, \quad (68)$$

with $K = K^\perp$ for $\tau = 0$ and $K = K^\Sigma$ for $\mu\rho = 0$. When $\tau \neq 0$ or $\mu\rho \neq 0$ we may take x, resp. y as a coordinate, where (51)-(52) and (60) imply

$$(x+y)(\overline{\tau}m_a + \tau\overline{m}_a) = \mathrm{d}_a x, \quad (69)$$
$$(x+y)(\mu k_a - \rho l_a) = \mathrm{d}_a y. \quad (70)$$

In view of (46)-(48) and (60) it then remains to determine suitable complementary coordinates for $g^\perp_{ab}$ or g^Σ_{ab}.

For $\tau \neq 0$, Frobenius's theorem and (69) suggest to examine whether zero-weighted functions ϕ and f exist such that

$$\mathrm{i}\frac{x+y}{2\tau\overline{\tau}}(\tau\overline{m}_a - \overline{\tau}m_a) = f\mathrm{d}_a\phi. \quad (71)$$

This amounts to calculating the integrability conditions of the system

$$\text{Þ}\phi = \text{Þ}'\phi = 0, \quad \overline{\tau}\eth\phi = -\tau\eth'\phi = \mathrm{i}\frac{x+y}{2f} \quad (72)$$

which turn out to be

$$\text{Þ}f = \text{Þ}'f = 0, \quad \alpha_a\nabla^a f = 0. \quad (73)$$

These last equations have the trivial solution $f = 1$, for which a solution ϕ of (72) is determined up to an irrelevant constant. We take ϕ as the coordinate complementary to x. On solving (69) and (71) with $f = 1$ for m_a and $\overline{m}_a$ and using (57) we conclude that

$$\mathrm{d}s^2_\perp = \frac{\mathrm{d}x^2}{2\tau\overline{\tau}} + 2\tau\overline{\tau}\mathrm{d}\phi^2, \quad (74)$$
$$2\tau\overline{\tau} = x^3 + Cx + D \quad (75)$$

for classes IC and ID. Notice that, with the invariant x taken as a coordinate, the metric solutions should be restricted to spacetime regions where $x^3 + Cx + D > 0$ for consistency with Lorentzian signature. Clearly,

$$\partial_\phi{}^a = \mathrm{i}\frac{\tau\overline{m}^a - \overline{\tau}m^a}{x+y} = \frac{2\tau\overline{\tau}}{(x+y)^2}\mathrm{d}^a\phi, \quad (76)$$

is a HO spacelike Killing vector field (KVF).

For $\mu\rho \neq 0$ one analogously considers

$$\eth\psi = \eth'\psi = 0, \quad \mu\text{Þ}\psi = \rho\text{Þ}'\psi = \frac{x+y}{2g} \quad (77)$$

but the integrability conditions of this system are now

$$\eth g = \eth' g = 0, \quad \beta_a\nabla^a g = -gS\frac{\mu^2 + q^2\rho^2}{q\mu\rho}. \quad (78)$$

So $g = 1$ is *only* a solution in the Einstein subcase $S = 0$, for which we then get

$$\mathrm{d}s^2_\Sigma = \frac{\mathrm{d}y^2}{2\mu\rho} + 2\mu\rho\mathrm{d}\psi^2, \quad (79)$$
$$2\mu\rho = y^3 + Cy - D - \frac{\Lambda}{3} \quad (80)$$

with KVF

$$\partial_\psi{}^a = \frac{\mu\kappa^a + \rho l^a}{x+y} = -\frac{2\mu\rho}{(x+y)^2}\mathrm{d}^a\psi, \quad (81)$$

which is timelike for $\mu\rho > 0$ and spacelike for $\mu\rho < 0$. In general, the second vector field in (81) is always HO: the integrability conditions of (78) are checked to be identically satisfied, such that solutions g and a corresponding solution ψ of (77) exist. However, taking ψ as a complementary coordinate of y eventually leads to a very complicated system of coupled partial differential equations for $g = g(y, \psi)$, which is impossible to solve explicitly. We shall remedy this in section 3.1 but now discuss characterizing features of the Einstein space limit cases.

2.3. Characterizations of PE Petrov type D Einstein spaces

All Petrov type D Einstein spaces, constituting the class $\mathcal{D}_0$, are explicitly known. The line elements are obtained by putting the electromagnetic charge parameter Φ_0, resp. $e^2 + g^2$ equal to zero in the $\mathcal{D}$-metrics given by Debever *et al.* [14] or García [15]. These coordinate forms generalize and streamline those found by Kinnersley [36] in the $\Lambda = 0$ case.

Recently, a manifestly invariant treatment of $\mathcal{D}_0$, making use of the GHP formalism, was presented [33]. Within GHP, $\mathcal{D}_0$-metrics are characterized by the existence of a complex null tetrad wrt (11) and $\Phi_{ij}=0$ hold (i.e., the tetrad is a WPNT and (12)-(13) with $S=0$ hold). According to the Goldberg-Sachs theorem, (17) holds and characterizes the WPNT as well. The scalar invariant identities (see [33**?**])

$$\mu\overline{\rho}=\overline{\mu}\rho, \quad \pi\overline{\pi}=\tau\overline{\tau}, \tag{82}$$

just as (19)-(20), (24)-(25) and the first equation of (29) are also valid in general. From these relations it follows that

$$(15) \Leftrightarrow (18), \tag{83}$$

i.e., *a Petrov type D Einstein space is PE if and only if the WPNT directions are HO.* In fact, it can readily be shown by a more detailed analysis than in [33] that if the spacetime belongs to Kundt's class, i.e., if one of the PNDs is moreover non-diverging, one has

$$\mu=0 \Rightarrow \rho=0 \text{ or } \rho-\overline{\rho}\neq 0\neq \pi+\overline{\tau}. \tag{84}$$

Equations (4), (5) and (31) in [33] then imply

$$\rho=\overline{\rho}\neq 0 \quad\Rightarrow\quad \mu=\overline{\mu}\neq 0=\pi+\overline{\tau}, \tag{85}$$

$$\pi=-\overline{\tau}\neq 0 \;\Rightarrow\; \mu=\overline{\mu},\, \rho=\overline{\rho}. \tag{86}$$

Consequently, (a) the Kundt and Robinson-Trautman subclasses of $\mathcal{D}_0$ have empty intersection, (b) in the Robinson-Trautman case *both* PND's are non-twisting but diverging and (c) *a Petrov type D Einstein space is PE if and only if both PND's are non-twisting.* This result is implicit in [14], where the concerning PE metrics form the Einstein space subclasses of the classes labeled by C^{00}, $C^{0}{}_{+}$, $C^{0}{}_{-}$ and C^*. C^{00} has a flat $\Lambda=0$-limit (cf. (87) below), C^* corresponds to Kinnersley's case IIIA, while C^0_+ and C^0_- correspond to cases I and IV with $l=0$. The static part of C^0_+ (C^0_-) corresponds to class A (B) in the classification by Ehlers and Kundt [5], while that of C^* corresponds to the original static C-metric.

Notice that with $S=0$ and $w=\Lambda=const$, the boost-field q does not appear in the equations (11)-(37) and is not defined by the geometry any more, in contrast to the situation for perfect fluids $S\neq 0$ (for which $q\equiv 4\Phi_{22}/S$). The set (6)-(10), i.e. the requirement that a US given by (5) exists, is decoupled from (11)-(37) and is not needed to derive (19)-(37) from (11)-(18). Hence the integrability of the complete set (9)-(37) precisely tells that *the closed set (11)-(37) characterizes the class $\mathcal{D}_0$ of purely electric Petrov type D Einstein spaces, which are precisely those Einstein spaces belonging to Barnes' class I, all admitting a one-degree freedom of USs in all regions of spacetime [48].* Barnes' boost-isotropic classes IA and IC coincide with C^{00}, resp. $C^{0}{}_{-}$ of Debever et al., while $C^{0}{}_{+}$ and C^* form the Einstein space subclasses of IB, resp. ID.

A static member of $\mathcal{D}_0$ necessarily admits a rigid (i.e. non-expanding) US, such that $\mu\rho\geq 0$, cf. (38). Conversely, when $\mu\rho=0$ or $\mu\rho>0$ for a PE member, it admits the HO timelike KVF (67), resp. (81). Hence *a Petrov type D Einstein space is static if it admits a rigid US. This is precisely the case when both PNDs are non-twisting (i.e. when the spacetime is PE) with positive or zero product $\mu\rho$ of the divergences of aligned null vectors subject to the normalization condition (A1). For $\mu\rho>0$ there is an up to reflection unique rigid US, defined from the geometry by (5) and $q=\mu/\rho$, which is parallel to the unique timelike HO KVF direction. For $\mu\rho=0\Rightarrow\mu=\rho=0$ (classes IA and IC) all USs are rigid USs and they have one degree of freedom, while the timelike HO KVFs are parametrized by two constants.*

Necessary and sufficient conditions for a Petrov type D spacetime with arbitrary energy-momentum tensor to allow for a (rigid) US or to be static are given in appendix B. In particular it provides a proof of the last statement regarding the case $\mu=\rho=0$, which is in fact valid for all boost-isotropic spacetimes with $\pi=-\overline{\tau}$ wrt a WPNT (cf. criterion 6"). One can also check from (11)-(37) that in the PE case with $\mu\rho>0$ criterion 2" is satisfied, providing an alternative proof of the essential uniqueness of the rigid US and KVF for this case. Taken together, these two facts are in accordance with a result by Wahlquist and Estabrook [38]. The existence of a one-dimensional freedom of USs for all PE Einstein spaces is essentially due to the hypersurface-orthogonality (17)-(18) of the WPNT directions (cf. criterion 5), a property which is shared with all LRS II spacetimes exhibiting (pseudo-)spherical or planar symmetry. Also, *all* the above statements remain true in the electrovac class $\mathcal{D}$.

We note that it is claimed in [1] that all vacuum spacetimes admitting a shearfree normal congruence are static, which would generalize Trümper's result [24] to Petrov type D vacua. This is contradicted by the above. Notice that the C^0_+- and C^*-metrics were not explicitly shown to be *all* contained in his classes IB and ID (also those for which $\mu\rho<0$; see further § 3.2).

For completeness we write down the PE Petrov type D Einstein space metrics, as recovered here through (46), (60), (63), (66), (75) and (80):

$$C^{00}:\ \mathrm{d}s^2=\frac{2\mathrm{d}\zeta\mathrm{d}\overline{\zeta}}{1+\frac{\Lambda}{2}\zeta\overline{\zeta}}-\frac{2\mathrm{d}u\mathrm{d}v}{1-\frac{\Lambda}{2}uv}, \tag{87}$$

$$C^0_\pm:\ \mathrm{d}s^2=r^2\left(\mathrm{d}\xi^2+\delta\cos(\sqrt{k}\xi)^2\mathrm{d}\eta^2\right)+\frac{\mathrm{d}r^2}{g(r,k)}-\delta g(r,k)\mathrm{d}\phi^2, \tag{88}$$

$$C^*:\ \mathrm{d}s^2=(x+y)^{-2}\left(\frac{\mathrm{d}x^2}{f(x)}+f(x)\mathrm{d}\phi^2-\frac{\mathrm{d}y^2}{f(-y)+\Lambda/3}+(f(-y)+\Lambda/3)\mathrm{d}\psi^2\right), \tag{89}$$

where $k\in\{-1,0,1\}$, $\delta=1$ for $C^{0}{}_{+}$ and $\delta=-1$ for $C^{0}{}_{-}$, and

$$g(r,k) = k - \frac{2m}{r} - \frac{\Lambda}{3}r^2, \tag{90}$$

$$f(x) = F = x^3 + Cx + D. \tag{91}$$

Regarding C^{00} one deduces from (53)-(54), (61) and (64) that

$$\Psi_2 = -\frac{\Lambda}{3} = 4x^3, \quad K^{\perp} = -3x = -3y = K^{\Sigma},$$

and rescales ζ, u and v by a factor $2x$ to arrive at (87). This line element represents the Einstein space limit $\Phi_0 = 0$ of Bertotti's homogeneous electrovac family with cosmological constant [39, 40], exhibiting spatial rotation and boost isotropy. Notice that the limit $\Lambda = 0$ yields the Minkowski spacetime.

In order to obtain (88) one makes use of (68), replaces in the C^0_+ (C^0_-) case the coordinate y (x) by $r = -(2m)^{1/3}/(x+y)$ and rescales the remaining coordinates by a factor $(2m)^{1/3}$, where the constant curvature of $g^{\perp}_{ab}$ (g^{Σ}_{ab}) is related to m by $K = k(2m)^{-2/3}$ ($Y = m^{1/3}$). These solutions have a total group G_4 of isometries acting on three-dimensional orbits. The subcase C^0_+, $k = 1$, reproduces after $\xi \mapsto \pi/2 - \xi$ the well-known forms of the spherically symmetric Schwarzschild [41] and Kottler [42] metrics.

The form (89), with $\Lambda = 0$, is the form of the C-metric obtained by Levi-Civita and recovered by Ehlers and Kundt. It has the abelian group G_2 of isometries generated by $\partial_\phi{}^a$ and $\partial_\psi{}^a$. Kinnersley and Walker gave a modified form with $x + y$ replaced by $\alpha(x + y)$ and $x^3 + Cx + D$ by $1 - x^2 - 2m\alpha x^3$. Recently, Hong and Teo [43] introduced a factored form of the cubic in the case where it has three distinct real roots (corresponding to $m\alpha < 1/\sqrt{27}$), which greatly simplifies certain analyses of the C-metric. A further coordinate transformation can be made such that the Schwarzschild metric is comprised as the subcase $\alpha = 0$. In fact, this was performed for the charged C-metric. This technique was applied in extensive form in [19].

3. PERFECT FLUID GENERALIZATIONS OF THE C-METRIC

3.1. Line element

We resume the integration of class I started at the end of § 2.3. We thereby focus on the subclass ID characterized by $\tau \neq 0 \neq \mu\rho$. Let us first summarize what we did so far. We started off with the closed set (9)-(37) of first-order GHP equations in the seven (weighted) real variables Ψ, S, w, μ, ρ, q, $\eth'\tau$ and the complex variable τ. These variables are equivalent to two dimensionless spin and boost gauge fields, e.g. $\tau/\overline{\tau}$ and μ/ρ, and seven real scalar invariants. The boost and spin gauge fields could serve to invariantly fix the tetrad – the ID members being therefore anisotropic – but can be further ignored. For the C^*-Einstein spaces, $S = 0$ and $w = \Lambda = const$, and we remarked that q is not a part of the intrinsic describing set of variables. Hence we end up with four real scalar invariants in this subcase. These invariants are equivalent to the two constants C and D and two independent functions x and y, which we took as coordinates and in terms of which, on adding two coordinates ϕ and ψ related to the symmetries, the corresponding C^*-metric can be expressed. In the perfect fluid case $S \neq 0$, the four invariants and their use persist, just as the coordinate ϕ. However, ψ is no longer a suitable coordinate and the scalar invariants S and w are no longer constant, while $q = 4\Phi_{22}/S$ now fixes the invariantly defined fluid velocity vector u^a at each point by (5). Thus we need one more scalar invariant for our description and one remaining coordinate complementary to y.

For the first purpose it is natural to look at the kinematics of the fluid, which are fully determined by

$$b = 2\nabla_{(c}u_{a)}m^a\overline{m}^c = \nabla_c u_a v^a v^b = \frac{\theta}{3}, \tag{92}$$

$$\dot{u}_{\parallel} \equiv v_a \dot{u}^a, \quad \dot{u}^{\perp}_a \equiv 2\overline{m}_{(a}m_{c)}\dot{u}^c, \tag{93}$$

where v^a is the intrinsic normalized spacelike vector field defined in (16), specifying at each point to the up to reflection unique vector orthogonal to u^a and lying in Σ, while $\dot{u}_{\parallel}$ and $\dot{u}^{\perp}_a$ are the component along v^a, resp. projection onto $\Sigma^{\perp}$ of the acceleration $\dot{u}^a$. In analogy with (92) we define the invariant $\tilde{b}$ by

$$\tilde{b} = 2\nabla_{(c}v_{a)}m^a\overline{m}^c. \tag{94}$$

The relation with GHP quantities is

$$b = \frac{\mu - q\rho}{\sqrt{2q}}, \quad \tilde{b} = -\frac{\mu + q\rho}{\sqrt{2q}}, \tag{95}$$

$$\dot{u}_{\parallel} = (2q)^{-3/2}(\text{Þ}'q + q\text{Þ}q) = \frac{\text{Þ}q}{\sqrt{2q}} - b, \tag{96}$$

$$-\dot{u}^{\perp}_a = \overline{\tau}m_a + \tau\overline{m}_a \equiv \alpha_a = \frac{\mathrm{d}_a x}{x+y}. \tag{97}$$

Notice that (95) is equivalent to

$$b\,u_a - \tilde{b}\,v_a = \mu k_a - \rho l_a \equiv \beta_a = \frac{\mathrm{d}_a y}{x+y} \tag{98}$$

and, in conjunction with (58), implies

$$2\mu\rho = \tilde{b}^2 - b^2 = y^3 + Cy - D - \frac{w}{3}. \tag{99}$$

We choose b as the final describing invariant and use $\tilde{b}$ and $\dot{u}_{\parallel}$ as auxiliary variables. In view of (95)-(97) one deduces that the differential information for S, w and b

comprised in (9)-(37) is precisely

$$\mathrm{d}_a w = 3bSu_a, \quad D_a S = -S\dot{u}_a, \tag{100}$$

$$\mathrm{d}_a b = -\mathbf{u}(b) u_a,$$

$$\mathbf{u}(b) = -\mathbf{v}(\tilde{b}) + \tilde{b}(\dot{u}_{\|} - \tilde{b}) + \frac{S}{2}, \tag{101}$$

$$\mathbf{v}(\tilde{b}) = -\frac{x+y}{2}(3y^2 + C). \tag{102}$$

From the last equation it follows that $\tilde{b}$ is non-constant, such that we may see the second part of (101) as a definition of $\dot{u}_{\|}$. The relations (100) are nothing but the energy resp. momentum conservation equations for a perfect fluid subject to $D_a w = 0$. The first part of equation (101) confirms that $D_a\theta = 0$ [1, 23], whilst the second implies again that the expansion scalar does not vanish in general, contrary to Barnes' conclusion in [1] (cf. below).

For the second purpose we rely on the hypersurface-orthogonality of u^a by assumption: zero-weighted real scalar fields t and I exist such that

$$\mathrm{d}_a t = I u_a. \tag{103}$$

The integrability condition hereof is

$$D_a I = -I\dot{u}_a = -I(\dot{u}_a^{\perp} + \dot{u}_{\|} v_a), \tag{104}$$

which is equivalent to

$$\eth I = \tau I, \quad \mathbf{v}(I) = -\dot{u}_{\|} I. \tag{105}$$

We see from (98) and (103) that t is functionally independent of y (and of x and ϕ) and we take it as the fourth coordinate. On using (103)-(104), (100) and the first part of (101) are equivalent to

$$b = b(t), \quad w = w(t), \quad A \equiv \frac{S}{2I} = A(t), \tag{106}$$

$$\mathrm{d}_a w = 6b\, A \mathrm{d}_a t = 3b\, S u_a, \tag{107}$$

while (99) yields

$$\tilde{b} = \tilde{b}(y,t) = \sqrt{y^3 + Cy - D + b(t)^2 - w(t)/3}. \tag{108}$$

This, (97)-(98) and the first part of (105) entail that the scalar field

$$J \equiv \frac{x+y}{I\tilde{b}} \tag{109}$$

only depends on y and t. Inverting (98) and (103) we get

$$(x+y)u_a = \tilde{b}J\mathrm{d}_a t, \quad (x+y)v_a = bJ\mathrm{d}_a t - \frac{\mathrm{d}_a y}{\tilde{b}}, \tag{110}$$

or dually

$$-\frac{u^a}{x+y} = \frac{\partial_t{}^a}{\tilde{b}J} + b\partial_y{}^a, \quad -\frac{v^a}{x+y} = \tilde{b}\partial_y{}^a. \tag{111}$$

The only equation still to be satisfied results from eliminating $\dot{u}_{\|}$ between the second parts of (101) and (105), which gives

$$\tilde{b}^2\mathbf{v}(J) = J\tilde{b}\mathbf{u}(b) + A(x+y), \tag{112}$$

and in the chosen coordinates translates to

$$J_{,y} = \tilde{b}^{-3}\left(\mathrm{d}b/\mathrm{d}t - A\right). \tag{113}$$

As $g^{\Sigma}_{ab} = (v_a v_b - u_a u_b)/(x+y)^2$, and by assembling the above pieces, we obtain

$$\mathrm{d}s^2 = (x+y)^{-2}[\mathrm{d}s^2_{\perp} + \mathrm{d}s^2_{\Sigma}], \tag{114}$$

$$\mathrm{d}s^2_{\Sigma} = \left(bJ\mathrm{d}t - \frac{\mathrm{d}y}{\tilde{b}}\right)^2 - \left(\tilde{b}J\mathrm{d}t\right)^2, \tag{115}$$

with $\mathrm{d}s^2_{\perp}$ given by (75) and where the scalar fields occurring in (115) are given or related by (106)-(108) and

$$J = J(y,t) = [b'(t) - A(t)]\int_0^y \frac{\mathrm{d}\chi}{\tilde{b}(\chi,t)^3} + L(t), \tag{116}$$

a prime denoting ordinary derivation wrt t and $L(t)$ being a free function of integration. Notice that we nowhere used $S \neq 0$ explicitly in the above integration. Therefore, the above line element describes the *complete* class ID, including the C^*-Einstein space limits.

We neither used $\tau \neq 0$. This implies that the line element of the complete class IB, characterized by $\mu\rho \neq 0 = \tau$ and constituted by all LRS II Einstein spaces and shear-free perfect fluids with $D_a w = 0$, is obtained as well by assembling (46), (60), (63), (114)-(116) and (106)-(108). A coordinate form for this class was first given by Kustaanheimo [44]. By a coordinate transformation $y \mapsto r = -(y+x)^{-1}$ and redefinition of $L(t)$ the form (16.49), (16.51) mentioned in [16] is obtained.

As an alternative to the above, one checks that v^a is HO: $(x+y)v_a = \mathrm{d}_a z$, and the coordinate z could be used instead of y. Put $J\tilde{b} \equiv e^Z$, $Z = Z(y,t)$. By (98) we now have

$$y = Y(z,t), \quad Y_{,z} = -\tilde{b}, \quad \theta = 3Y_{,t}e^{-Z}. \tag{117}$$

The metric can be written as

$$\mathrm{d}s^2 = (x+Y)^{-2}\left(f^{-1}\mathrm{d}x^2 + f\mathrm{d}\phi^2 + \mathrm{d}z^2 - e^{2Z}\mathrm{d}t^2\right) \tag{118}$$

with $f = f(x)$ given by (91). This is exactly the form Barnes [1] obtains. We now discuss the more familiar integration procedure of his paper, namely a direct attack of the field equations, in these coordinates and wrt a Weyl principal orthonormal tetrad naturally associated with (118). One can check that only four of the field equations are not identically satisfied (the indices 1 to 4 label the tetrad vectors):

$$G_{34} = -Y_{,tz} + Y_{,t}Z_{,z} = 0 \quad (119)$$

$$G_{11} - G_{33} = -2Y_{,zz} + \frac{df}{dx} + (x+Y)\left(Z_{,z}^2 + Z_{,zz} - \frac{1}{2}\frac{d^2f}{dx^2}\right) = 0 \quad (120)$$

$$G_{11} = \left(Z_{,zz} + Z_{,z}^2\right)(x+Y)^2 + \left(e^{-2Z}\left(Y_{,tt} - Y_{,t}Z_{,t}\right) - Y_{,z}Z_{,z} - \frac{1}{2}\frac{df}{dx}\right) 2(x+Y) + 3\left(Y_{,z}^2 + f - Y_{,t}^2 e^{-2Z}\right) = p \quad (121)$$

$$G_{33} + G_{44} = 2(x+Y)\left(Y_{,zz} - Y_{,z}Z_{,z} + e^{-2Z}\left(Y_{,tt} - Y_{,t}Z_{,t}\right)\right) = S \quad (122)$$

Hence, *if supplemented with* $\theta \sim Y_{,t} = 0$, these equations are the ones obtained in Barnes [1]: equation (119)$\equiv$ $\mathbf{v}(\theta) = 0$ was missed out, and both equations (121) and (122) differ from equations (4.23), resp. (4.24) in [1] by a term $2(x+Y)Y_{,tt}e^{-2Z}$. Thus, it is clear that with these differences a correct non-expanding solution can be found, but the analysis of expanding solutions will be incorrect.

Differentiating (120) twice wrt x yields $d^4f(x)/dx^4 = 0$, whence

$$f(x) = ax^3 + bx^2 + cx + d. \quad (123)$$

Substituting this in equation (120), and equating coefficients of powers of x, leads to

$$3aY(z,t) = Z_{,zz}(z,t) + Z_{,z}^2(z,t) + b \quad (124)$$

$$Y_{,zz}(z,t) = \frac{c}{2} - Z(z,t)b + \frac{3}{2}aY^2(z,t) \quad (125)$$

This last equation can be solved for z in terms of Y:

$$\int_0^{Y(z,t)} \frac{dr}{\sqrt{ar^3 - br^2 + cr + f_1(t)}} - z + f_2(t) = 0 \quad (126)$$

suggesting eventually to transform coordinates from (z,t) into (y,t), with $y = Y(z,t)$. Rescaling and translating coordinates allows us to set $a = 1$ and $b = 0$. One can check that the remaining equations lead exactly to equations (107) and (113), recovering solution (114)-(116).

3.2. Properties

Consider the metric (114)-(116), with (75) and (106)-(108).

Let us discuss the intrinsic freedom we have when the metric is assumed to describe a perfect fluid ($S \neq 0$). By (107) and (110) we have that $A(t)\mathrm{d}_at = S/2u_a$ and $J(y,t)\mathrm{d}_at = (x+y)u_a/\tilde{b}$ are invariantly defined one-forms, and hence so is $L(t)\mathrm{d}_at$ because of (116). It follows that $\frac{L}{A}(t)$ is a scalar invariant. Moreover, as $A(t)\mathrm{d}_at$ is exact we may remove the only remaining coordinate freedom on t by putting $A(t) = 1$, such that the conservation of energy equation (107) can be considered as a definition $\theta(t) = w'(t)/2$, cf. (92). Hence, in this most general and natural picture for $S \neq 0$, *the scalar constants* C, D *and invariants* $\frac{L}{A}(t)$, $w(t)$ *stand in biunivocal correspondence with the ID perfect fluid spacetimes, locally.* Notice that the presence of two invariantly defined, distinguishing free functions could have been predicted, since after elimination of $\dot{u}_{||}$, there are two scalar invariants $\mathbf{u}(b)$ and $\mathbf{u}(S)$ remaining unprescribed in the system of equations (100)-(102).

In this fashion however, the physical implications remain obscure: it would be nice to have a well-known free function in stead of L/A. Spacetimes with $L(t) = 0$ have $w(t)$ as the only free function. If $L(t) \neq 0$, $L(t)$ can alternatively be fixed to 1 by a coordinate transformation $T(t)$. The remaining free functions display the variation $\theta'(t)$ of the expansion scalar, the energy density and the pressure (since $A\mathrm{d}_at = \frac{1}{2}Su_a$). These are related by energy conservation (107). $w(t)$ and $(Su_a)(t)$ can be chosen freely. Alternatively, one can subdivide further in $\theta = 0$ and $\theta \neq 0$. In the case $\theta = 0$, w is constant because of (107) and can be chosen freely, just as $A(t)$. In the most interesting case $\theta \neq 0$, $w(t)$ and $\theta(t)$ can be chosen freely, determining Su_a. Thus *class ID provides a class of anisotropic cosmological models with arbitrary evolution of matter density and (non-zero) expansion.*

Regarding symmetry, all perfect fluid ID models admit at least one KVF $\partial_\phi{}^a$ given by (76), which at each point amounts to an invariantly defined spacelike vector orthogonal to $\dot{u}^{\perp a}$ and lying in $\Sigma^\perp$. If ϕ is chosen to be a periodic coordinate, with range given by $[-\pi E, \pi E)$, the spacetime is cyclically symmetric. We will then refer to the region $F(x) = 0$, where the norm of $\partial_\phi{}^a$ vanishes, as the axis of symmetry [45] [49]. Finding the complete group of isometries and their nature is trivial in our approach. The functions x, y, w and L/A are invariant scalars, such that $K^a\mathrm{d}_ax = K^a\mathrm{d}_ay = K^a\mathrm{d}_aw = K^a\mathrm{d}_a\frac{L}{A} = 0$ for any KVF K^a. As the ID models are anisotropic, it follows that the complete isometry group is at most G_2, and if it is G_2, both w and L/A are constant. Conversely, when w and L/A are constant we have $\theta \equiv 3b = 0$ from (107), $\tilde{b} = \tilde{b}(y)$ from (108) and $J(y,t) = -A(t)F_2(y)$ from (116). By redefining the time coordinate such that $A(t) = 1$ one sees from (114)-(116) that $\partial_t{}^a$ is a HO timelike KVF. We conclude that *the ID perfect fluid models have at least one spacelike KVF* $\partial_\phi{}^a$*, which may be interpreted as the generator of cyclic symmetry. They admit a second independent KVF if and only if both scalar invariants* w *and* L/A *are constant, in which case the spacetimes are static and the complete group of isometries is abelian* G_2*, generated by* $\partial_\phi{}^a$ *and* $\partial_t{}^a$*.*

Consider the case where F has 3 real non-degenerate roots x_i, i.e. $C < -3\left(\frac{D}{2}\right)^{2/3}$. If $x_1 < x_2 < x_3$ then $F(x) > 0$ for all $x \in (x_1, x_2)$. Furthermore, we let ϕ be a periodic coordinate. The ratio between circumference and radius of a small circle around the axis, $x = x_1$ or $x = x_2$, is given by

$$\lim_{\substack{x\to x_2\\ <}} \frac{2\pi E\sqrt{F(x)}}{\int_x^{x_2}\sqrt{F^{-1}(x)}\mathrm{d}x} = -\pi E\left(3x_2^2 + C\right) \tag{127}$$

respectively

$$\lim_{x\to x_1} \frac{2\pi E\sqrt{F(x)}}{\int_{x_1}^{x}\sqrt{F^{-1}(x)}\mathrm{d}x} = \pi E\left(3x_1^2 + C\right). \tag{128}$$

It is only possible to choose the parameter E such that the complete axis is regular, if $3x_1^2 + C = -(3x_2^2 + C)$. However, eliminating C and D between this equation and $F(x_1) = F(x_2) = 0$ implies $x_1 = x_2$. Consequently, if $F(x)$ has three real non-degenerate roots, the spacetime contains a conical singularity. This echoes the properties of the C-metric [6, 45], but further research is needed to interpret the metric (114).

4. CONCLUSION

A new Petrov type D exact solution of Einstein's field equation in a spatially homogeneous perfect fluid has been presented. The solution admits in general a group G_1 of isometries, and contains a static class where the isometry group is G_2. This new exact solution thus provides a class of cosmological models with fewer then 3 killing vector fields, where the evolution of matter density and (non-zero) expansion can be chosen freely. Whether a thermodynamic interpretation of the perfect fluid can be made, is however a matter subject to further research. [?] It is certainly not possible to prescribe a barotropic equation of state $w(p)$. Due to its relation with the C-metric, the examination of the properties of this low-symmetry cosmological model becomes tractable.

Appendix A: GHP formalism

The GHP formalism bares a large resemblance to the Newman-Penrose formalism. Use is made of a complex null tetrad $(m^a, \overline{m}^a, l^a, k^a)$, with

$$k^a l_a = -1, \quad m^a \overline{m}_a = 1 \tag{A1}$$

and all other inner products vanishing. To put it in other words, at each point one takes a timelike plane, two vectors k^a and l^a lying along its real null directions, and two vectors m^a and $\overline{m}^a$ lying along the complex conjugate null directions of the spacelike plane orthogonal to it. The basic variables in the formalism are the spin coefficients

$$\kappa = -\Gamma_{144}, \quad \nu = \Gamma_{233}, \tag{A2}$$
$$\rho = -\Gamma_{142}, \quad \mu = \Gamma_{231}, \tag{A3}$$
$$\sigma = -\Gamma_{141}, \quad \lambda = \Gamma_{232}, \tag{A4}$$
$$\tau = -\Gamma_{143}, \quad \pi = \Gamma_{234}, \tag{A5}$$

the 9 independent components of the traceless part of the Ricci tensor $S_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}$

$$\begin{aligned}
\Phi_{00} &= \frac{1}{2}S_{ab}k^ak^b, \quad \Phi_{11} = \frac{1}{2}S_{ab}\left(k^al^b + m^a\overline{m}^b\right),\\
\Phi_{01} &= \frac{1}{2}S_{ab}k^am^b, \quad \Phi_{12} = \frac{1}{2}S_{ab}l^am^b,\\
\Phi_{02} &= \frac{1}{2}S_{ab}m^am^b, \quad \Phi_{22} = \frac{1}{2}S_{ab}l^al^b,
\end{aligned}$$

with $\Phi_{ji} = \overline{\Phi_{ij}}$, and the 10 independent components of the Weyl tensor

$$\begin{aligned}
\Psi_0 &= C_{abcd}k^am^bk^cm^d,\\
\Psi_1 &= C_{abcd}k^al^bk^cm^d,\\
\Psi_2 &= C_{abcd}m^ak^bl^c\overline{m}^d,\\
\Psi_3 &= C_{abcd}l^ak^bl^c\overline{m}^d,\\
\Psi_4 &= C_{abcd}l^a\overline{m}^bl^c\overline{m}^d.
\end{aligned}$$

Changes of the tetrad leaving the null directions spanned by k^a, l^a, m^a and $\overline{m}^a$ invariant, and at the same time preserving the normalization conditions (A1), consist of boosts

$$k^a \to Ak^a, \; l^a \to A^{-1}l^a \tag{A6}$$

and spatial rotations

$$m^a \to e^{i\theta}m^a. \tag{A7}$$

Quantities transforming under (A6)-(A7) as

$$\eta \to A^{\frac{\mathrm{w}_p+\mathrm{w}_q}{2}} e^{i\frac{\mathrm{w}_p-\mathrm{w}_q}{2}\theta}\eta$$

are called *well-weighted of type* (w_p, w_q). They have boost-weight $\mathrm{w}_B(\eta) = \frac{\mathrm{w}_p+\mathrm{w}_q}{2}$ and spin-weight $\mathrm{w}_S(\eta) = \frac{\mathrm{w}_p-\mathrm{w}_q}{2}$. One can check that the scalars defined above are weighted scalars (they are all spin coefficients of the NP formalism which are well-weighted). The following derivative operators are defined such that a well-weighted quantity η is transformed in a well-weighted quantity:

$$D_a\eta = \nabla_a\eta + \mathrm{w}_B(\eta)\Gamma_{34a}\eta + \mathrm{w}_S(\eta)\Gamma_{12a}\eta, \tag{A8}$$

and one uses the notation

$$D_1 = ð, \; D_2 = ð', \; D_3 = Þ', \; D_4 = Þ. \tag{A9}$$

One can check that, if η is a weighted scalar of type (p, q), $\mathrm{w}_B(D_a\eta) = \mathrm{w}_B(\eta) + \tilde{w}_B(a)$ and $\mathrm{w}_S(D_a\eta) = \mathrm{w}_S(\eta) + \tilde{w}_S(a)$, with

$$\tilde{w}_B(a) = \begin{cases} 0 & a = 1,2\\ -1 & a = 3\\ 1 & a = 4 \end{cases}$$

$$\tilde{w}_S(a) = \begin{cases} 1 & a = 1\\ -1 & a = 2\\ 0 & a = 3,4. \end{cases}$$

Notice that the differential of zero-weighted scalars f can be expressed as

$$\begin{aligned} \mathrm{d}_a f &= -\text{Þ}' f k_a - \text{Þ} f l_a + \eth' f m_a + \eth f \overline{m}_a & \text{(A10)} \\ &= -\mathbf{l}(f) k_a - \mathbf{k}(f) l_a + \overline{\mathbf{m}}(f) m_a + \mathbf{m}(f)\overline{m}_a & \text{(A11)} \\ &= -\mathbf{u}(f) u_a + \mathbf{v}(f) v_a + \overline{\mathbf{m}}(f) m_a + \mathbf{m}(f)\overline{m}_a, & \text{(A12)} \end{aligned}$$

where u^a and v^a are related to k^a and l^a according to (5), resp. (**??**). The equations to be solved are the Ricci identities

$$\nabla_d \Gamma_{abc} - \nabla_c \Gamma_{abd} = R_{abcd} + 2\Gamma_{ae[c|}\Gamma^e_{\ b|d]} + 2\Gamma_{abe}\Gamma^e_{\ [cd]},$$

for $(a, b) = (1, 4)$ and $(a, b) = (2, 3)$, and the Bianchi identities

$$\nabla_{[f|} R_{ab|cd]} = -2R_{abe[c}\Gamma^e_{\ df} + \Gamma^e_{\ a[c} R_{df]eb} - \Gamma^e_{\ b[c} R_{df]ea},$$

written out explicitly in terms of the derivative operators (A8). These equations are insufficient: as compared to the Newman-Penrose formalism there are 6 Ricci identities missing (those identities involving spin coefficients which are not well-weighted). These Ricci identities are absorbed in the commutators between the derivative operators, which are given by

$$\begin{aligned} [D_a, D_b]\,\eta = -2\Gamma^c_{\ [ab]} D_c \eta + \mathrm{w}_B(\eta)\left(R_{34ab} - 2\Gamma_{3c[a}\Gamma^c_{\ |4|b]}\right)\eta \\ +\mathrm{w}_S(\eta)\left(R_{12ab} - 2\Gamma_{1e[a}\Gamma^e_{\ |2|b]}\right)\eta + \tilde{w}_B(b)\Gamma_{34a} D_b \eta \\ +\tilde{w}_S(b)\Gamma_{12a} D_b \eta - \tilde{w}_B(a)\Gamma_{34b} D_a \eta - \tilde{w}_S(a)\Gamma_{12b} D_a \eta. \end{aligned}$$

One can show that, after expressing ∇_a in terms of D_a, all spin coefficients that appear explicitly in these equations can be expressed in terms of the spin coefficients (A2-A5). The equations can be shown to be consistent and complete. To find a particular class of solutions, invariant conditions are imposed on the scalar quantities, and consistency with the above equations generates a set of scalar equations to be solved.

The GHP formalism is especially suited for situations where two null directions are naturally singled out, such that k^a and l^a can be chosen along these directions. E.g., for any given Petrov type D spacetime the Weyl tensor has precisely two principal null directions (PND's) which can be covariantly determined. Choosing k^a and l^a along them is equivalent to condition (11) for the Weyl tensor components. A complex null tetrad realizing this condition is called a *Weyl principal null tetrad* (WPNT) in the literature.

Appendix B: (Rigid) synchronizability and staticity of Petrov type D spacetimes:

Consider a Petrov type D spacetime and let $B_n \equiv (k^a, l^a, m^a, \overline{m}^a)$ be an arbitrary WPNT, i.e. (11) holds. For complex $(\mathrm{w}_p, \mathrm{w}_p)$-weighted scalars $z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$ we mean with $z > 0$ ($z < 0$) that z is real and strict positive (negative) in the sequel.

The spacetime will admit a unit timelike vector field u^a satisfying (1) - corresponding to an umbilical synchronization or forming the tangent field of a shearfree and vorticity-free cloud of test particles - if and only if a real non-negative (-2,-2)-weighted field q exists such that (5) and (6)-(10) hold wrt B_n. On using ghp6 and ghp12 it turns out that, regardless of the structure of the energy-momentum tensor, the only integrability condition of (9) is (15), i.e., the Weyl tensor is purely electric wrt u^a. It follows that *a given Petrov type D spacetime admits a shearfree normal unit timelike vector field if and only if, wrt an arbitrary B_n, Ψ_2 is real and one of the following sets of spin-boost invariant conditions holds*:

1. the scalar invariant $\lambda\sigma > 0$ and $q_0 \equiv \lambda/\overline{\sigma}$ satisfies (7) and (8)-(10);

2. the real scalar invariant $(\mu - \overline{\mu})(\rho - \overline{\rho}) > 0$, and $q_0 \equiv -(\mu - \overline{\mu})/(\rho - \overline{\rho})$ satisfies (6) and (8)-(10);

3. $\lambda = \sigma = \mu - \overline{\mu} = \rho - \overline{\rho} = 0$, the scalar invariant $\kappa\nu \neq 0$ and one of the following situations occurs, where q_0, defined in each subcase, satisfies (9)-(10) and where $b \equiv (\pi + \overline{\tau})/\overline{\kappa}$, $c \equiv \nu/\overline{\kappa}$:

3a. $\mathrm{Im}(b)\mathrm{Im}(c) > 0$ and $q_0 \equiv \mathrm{Im}(c)/Im(b)$, with also $q_0^2 - \mathrm{Re}(b)q_0 + \mathrm{Re}(c) = 0$;

3b. b is real, $c < 0$ and $q_0 \equiv (b + \sqrt{b^2 - 4c})/2$;

3c. $b > 0$, $c > 0$, $b^2 \geq 4c$, and $q_0 \equiv (b + \sqrt{b^2 - 4c})/2$ or $q_0 \equiv (b - \sqrt{b^2 - 4c})/2$;

4. $\lambda = \sigma = \mu - \overline{\mu} = \rho - \overline{\rho} = 0$, and either $\kappa = 0 \neq \nu$, $(\overline{\pi} + \tau)\nu > 0$ and $q_0 \equiv \nu/(\pi + \overline{\tau})$ satisfies (9)-(10), or $\kappa \neq 0 = \nu$, $(\overline{\pi} + \tau)\overline{\kappa} > 0$ and $q_0 \equiv (\pi + \overline{\tau})/\overline{\kappa}$ satisfies (9)-(10);

5. $\lambda = \sigma = \mu - \overline{\mu} = \rho - \overline{\rho} = 0$, $\kappa = \nu = \pi + \overline{\tau} = 0$.

The subdivision of case 3 stems from a straightforward analysis of equation (8). In cases 1, 2, 3a, 3b and 4 there is a *unique* shearfree normal unit timelike vector field u^a, whereas there may be one or two such u^a's in case 3c. Case 5 precisely corresponds to the hypersurface-orthogonality of the B_n tetrad vectors. Moreover, the imaginary part of ghp5 − ghp11 yields (15) in this case. In conjunction with (9)-(10) we conclude that *any Petrov type D spacetime with HO Weyl principal null tetrad directions admits a one-degree freedom of shearfree normal timelike congruences.* Important examples of such spacetimes are the Petrov type D purely electric Einstein spaces and their 'electrovac' generalizations (see [37?] and § 2.3) and all spacetimes with (pseudo-)spherical or planar symmetry (which constitute the locally rotationally symmetric (LRS) class II Lorentzian spaces, see [34]). These examples all satisfy (12) on top of (17)-(18) and are further characterized by $\Phi_{00} = \Phi_{22} = (\Phi_{11} =)\, 0$, resp. $\pi = \tau = \eth R = 0$ (cf. [46]).

The spacetime will admit a unit timelike vector field u^a satisfying

$$u_{a;b} = -\dot{u}_a u_b, \tag{B1}$$

corresponding to a rigid umbilical synchronization or modeling a rigid non-rotating cloud of test particles, when $\theta = 0$ or, in GHP language, (38) holds additionally to the above. Notice that (7) is then identically satisfied. Thus *a given Petrov type D spacetime admits a shear-free normal and non-expanding unit timelike vector field if and only if, wrt B_n, Ψ_2 is real and one of the following holds*:

1'. condition 1 with (7) replaced by (38);

2'. the scalar invariant $\mu\rho > 0$ and $q_0 \equiv \mu/\overline{\rho}$ satisfies (6) and (8)-(10);

3'-5'. conditions 3-5 with $\mu - \overline{\mu} = \rho - \overline{\rho} = 0$ replaced by $\mu = \rho = 0$.

Here case 5' yields that *any Petrov type D spacetime with geodesic, shearfree and non-diverging PNDs and HO Weyl principal complex null directions admits a one-degree freedom of shearfree normal and non-expanding timelike congruences.*

The spacetime is static if and only if it admits a HO timelike Killing vector field. An equivalent characterization was given by Ehlers and Kundt [5]: the spacetime is static if and only if a unit timelike vector field u^a exists for which shear, vorticity and expansion scalar vanish, i.e. (B1) holds, and for which the acceleration $\dot{u}^a$ is Fermi-propagated along the integral curves of u^a:

$$\ddot{u}_{[a} u_{b]} = 0. \tag{B2}$$

The field u^a is then parallel to a (HO and timelike) Killing vector field. By a long but straightforward calculation, thereby simplifying expressions by means of (6)-(9), (38), ghp1, ghp7 and the $[\text{Þ}, \text{Þ}'](q)$ commutator relation, one shows that the extra condition (B2) is equivalent to

$$\begin{aligned}(q\overline{\kappa} + q^{-1}\nu)(\text{Þ}q + \sqrt{2q}) - 2\text{Þ}\nu + 2q\text{Þ}\overline{\tau}&\\ +\Phi_{21} - q\Phi_{10} = 0,& \qquad (\text{B3})\\ \text{Þ}\text{Þ}q = \pi\tau + \overline{\pi\tau} - q(\kappa\pi + \overline{\kappa\pi}) - q^{-1}(\nu\overline{\pi} + \overline{\nu}\pi)&\\ +2\Phi_{11} - \frac{R}{12} + 2\Psi_2.& \qquad (\text{B4})\end{aligned}$$

In case 5' above, the Ricci equations ghp1, ghp3 and ghp9 yield $\text{Þ}\overline{\tau} = \Phi_{10}$ and $\Phi_{00} = \Phi_{22} = 0$, while (B3)-(B4) reduces to

$$\Phi_{12} + q\Phi_{01} = 0, \tag{B5}$$

$$\text{Þ}\text{Þ}q = -2\tau\overline{\tau} + 2\Phi_{11} - \frac{R}{12} + 2\Psi_2. \tag{B6}$$

In the subcase $\Phi_{01} = \Phi_{12} = 0$, the $[\text{Þ}, \text{Þ}']$, $[\text{Þ}, \eth]$ and $[\text{Þ}, \eth']$ commutators applied to q yield

$$\text{Þ}'\text{Þ}q = -q\text{Þ}\text{Þ}q + (\text{Þ}q)^2, \tag{B7}$$

$$\eth\text{Þ}q = \tau\text{Þ}q, \quad \eth'\text{Þ}q = \overline{\tau}\text{Þ}q. \tag{B8}$$

The compatibility requirement of (B6)-(B8) with the commutator relations for $\text{Þ}q$ gives the single condition

$$\text{Þ}'R + q\text{Þ}R = 0. \tag{B9}$$

According to the Sach's star dual [27] of the LRS criterium in [46], the subcase $\text{Þ}'R = \text{Þ}R = 0$ hereof precisely corresponds to a boost isotropic spacetime with $\pi + \overline{\tau} = 0$. From the above we conclude: *A Petrov type D spacetime is static if and only if, wrt an arbitrary B_n, one of the following sets of conditions holds:*

1"-4". Ψ_2 is real, conditions 1'-4' hold and q_0 additionally satisfies (B3)-(B4);

5"a. condition 5' holds, the scalar invariant $\Phi_{01}\Phi_{21} < 0$ and $q_0 \equiv -\Phi_{12}/\Phi_{01}$ satisfies (9)-(10) and (B4);

5"b. condition 5' holds, $\Phi_{01} = \Phi_{21} = 0$, the scalar invariant $(\text{Þ}'R)(\text{Þ}R) < 0$ and $q_0 \equiv -\text{Þ}'R/\text{Þ}R$ satisfies (9)-(10) and (B6);

6". the spacetime is (locally) boost isotropic and $\pi + \overline{\tau} = 0$.

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[47] There are typo's in these equations: the integers 3 and 9 should be omitted in (50) and (51)-(52), respectively, while there should be $9k$ instead of k in (53).
[48] This means that USs are biunivocally related to smooth fields of vectors defined on a certain curve in the considered spacetime region.
[49] Strictly speaking, with the terminology of [16], the spacetime has in general only a cyclic symmetry, as the axis will be shown to be irregular, and consequently not part of the spacetime.