Estimation in nonstationary random coefficient autoregressive models

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Abstract

We investigate the estimation of parameters in the random coefficient autoregressive model $X_k = (\varphi + b_k)X_{k-1} + e_k$, where $(\varphi, \omega^2, \sigma^2)$ is the parameter of the process, $Eb_0^2 = \omega^2$, $Ee_0^2 = \sigma^2$. We consider a nonstationary RCA process satisfying $E \log |\varphi + b_0| \geq 0$ and show that σ^2 cannot be estimated by the quasi-maximum likelihood method. The asymptotic normality of the quasi-maximum likelihood estimator for (φ, ω^2) is proven so the unit root problem does not exist in the random coefficient autoregressive model.

Key words and phrases: random coefficient model, quasi–maximum likelihood, asymptotic normality, consistency, law of large numbers.

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1 Introduction

In this paper we are interested in the random coefficient model (RCA) defined by the equations

$$X_k = (\varphi + b_k)X_{k-1} + e_k, \quad -\infty < k < \infty, \tag{1.1}$$

where φ is a real parameter. The RCA process was introduced by Andél (1976) who also studied its properties. For a detailed early study we refer to Nicholls and Quinn (1982). Throughout this paper we assume that

$$\{(b_k, e_k)\} \text{ are independent, identically distributed random vectors.} \tag{1.2}$$

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Let $\log^+ x = \max\{\log x, 0\}$. It follows from Aue et al. (2006) (cf. also Quinn (1980, 1982)) that under condition (1.2) and

$$E \log^+ |e_0| < \infty \quad \text{and} \quad E \log^+ |\varphi + b_0| < \infty, \tag{1.3}$$

equation (1.1) has a stationary, nonanticipating (i.e. X_k is measurable with respect to the σ–algebra generated by $(b_i, e_i), i \leq k$) if and only if

$$-\infty \leq E \log |\varphi + b_0| < 0. \tag{1.4}$$

Quinn and Nicholls (1981) started the study of the estimation of the parameter of the process in (1.1). Let $\boldsymbol{\theta} = (\varphi, \omega^2, \sigma^2)$, where

$$Eb_0 = 0, \quad Eb_0^2 = \omega^2 > 0, \tag{1.5}$$

$$Ee_0 = 0, \quad Ee_0^2 = \sigma^2 > 0 \tag{1.6}$$

and

$$\operatorname{cov}(b_0, e_0) = 0. \tag{1.7}$$

Aue et al. (2006) used the quasi–maximum likelihood method to estimate $\boldsymbol{\theta}$ when (1.4) holds. They established the strong consistency as well as the asymptotic normality of the quasi–maximum likelihood estimator under minimal conditions.

In this paper we consider the case when (1.4) does not hold. We assume

$$X_k = (\varphi + b_k) X_{k-1} + e_k, \quad 1 \leq k \leq n \tag{1.8}$$

and

$$E \log |\varphi + b_0| \geq 0, \tag{1.9}$$

i.e. we start the recursion in (1.8) from the initial value X_0 and (1.9) guarantees that the solutions of (1.8) cannot converge. Throughout this paper we assume that X_0 is a constant. Following the theory developed for the stationary case, we estimate the parameter $\boldsymbol{\theta}$ of the process in (1.8) using the quasi–likelihood method. Assuming that b_0 and e_0 are normally distributed, the conditional log–likelihood function (the constant terms are omitted) is given by

$$L_n(\mathbf{u}) = \sum_{k=1}^{n} \ell_k(\mathbf{u}) \quad \text{with } \ell_k(\mathbf{u}) = -\frac{1}{2}\left(\log(xX_{k-1}^2 + y) + \frac{(X_k - sX_{k-1})^2}{xX_{k-1}^2 + y}\right),$$

where $\mathbf{u} = (s, x, y)$. We show that

$$\frac{1}{n} L_n(\mathbf{u}) \xrightarrow{P} \infty$$

but

$$\frac{1}{n}\big(L_n(\mathbf{u}) - L_n(\boldsymbol{\theta})\big) \xrightarrow{P} f(s, x) \qquad \text{for all } \mathbf{u} \text{ with } x > 0 \text{ and } y > 0,$$

where

$$f(s, x) = \frac{1}{2}\left\{\log \frac{\omega^2}{x} + 1 - \frac{\omega^2}{x} - \frac{(\varphi - s)^2}{x}\right\}. \tag{1.10}$$

Since $f(\cdot)$ does not depend on y, the quasi–maximum likelihood method cannot be used to estimate σ^2. Since $|X_n| \xrightarrow{P} \infty$ $(n \to \infty)$ (cf. Lemma 4.1), so in (1.1) $b_n X_{n-1}$ dominates e_n which is the reason why the variance of e_0 cannot be estimated by the quasi–likelihood method. Hence we are interested in estimating $\boldsymbol{\eta} = (\varphi, \omega^2)$. Now $\widehat{\boldsymbol{\eta}}_n = \widehat{\boldsymbol{\eta}}_n(y) = (\widehat{\eta}_{n,1}(y), \widehat{\eta}_{n,2}(y))$ is defined by

$$\max_{\mathbf{z}\in\Gamma} L_n(\mathbf{z}, y) = L_n(\widehat{\boldsymbol{\eta}}_n, y),$$

$\mathbf{z} = (s, x)$ and the set Γ satisfies

$$\Gamma = \{(s, x) : \ s_* \le s \le s^*, \ x_* \le x \le x^*\} \tag{1.11}$$

with some $s_* < s^*$, $0 < x_* < x^*$. We prove the asymptotic consistency of $\widehat{\boldsymbol{\eta}}_n(y)$ for all y and consider the asymptotic normality of $\widehat{\boldsymbol{\eta}}_n$ under various conditions.

2 Results

First we study the asymptotic consistency of $\widehat{\boldsymbol{\eta}}_n(y)$.

Theorem 2.1. *If* (1.2), (1.5)–(1.9) *and*(1.11) *hold, then*

$$\widehat{\boldsymbol{\eta}}_n(y) \xrightarrow{P} (\varphi, \omega^2) \tag{2.1}$$

for all $y > 0$.

Next we consider the asymptotic normality of $\widehat{\boldsymbol{\eta}}_n(y)$. Let

$$\Omega_0 = \begin{pmatrix} \omega^2 & \omega^2 E b_0^3 \\ \omega^2 E b_0^3 & \mathrm{var}(b_0^2) \end{pmatrix}. \tag{2.2}$$

Theorem 2.2. *If the conditions of Theorem 2.1 are satisfied and*

$$E e_0^4 < \infty \qquad \text{and} \qquad E b_0^4 < \infty, \tag{2.3}$$

then the distribution of $n^{1/2}(\widehat{\boldsymbol{\eta}}_n(\sigma^2) - (\varphi, \omega^2))$ *converges to the bivariate normal distribution with mean* $\mathbf{0}$ *and covariance matrix* Ω_0.

We note that Theorems 2.1 and 2.2 were obtained by Ling and Li (2006) as a preliminary result for the study of non-stationary double AR(1) processes when b_0 and e_0 are normally distributed and independent. Their result implies that in case of normal (b_0, e_0), σ^2 cannot be estimated by the quasi–maximum likelihood method. A similar phenomenon was also observed by Jensen and Rahbek (2004a,b) in nonstationary ARCH models. Theorem 2.2 assumes that σ^2 is known. We show in the next section that $\widehat{\boldsymbol{\eta}}_n(y)$ is asymptotically normal for all $y > 0$ under the condition $E \log|\varphi + b_0| > 0$.

Usually, the statistical inference is about φ, the expected value of the autoregressive coefficient. We show that $\widehat{\eta}_{n,1}(y)$ is asymptotically normal for all y so there is no need to know σ^2 to get asymptotic statistical inference about φ.

Theorem 2.3. *We assume that the conditions of Theorem 2.1 are satisfied and* (2.3) *holds. Then for any $y > 0$ the distribution of $\sqrt{n}(\widehat{\eta}_{n,1}(y) - \varphi)/\omega$ converges to the standard normal distribution and consequently the distribution of $\sqrt{n}(\widehat{\eta}_{n,1}(y) - \varphi)/\sqrt{\widehat{\eta}_{n,2}(y)}$ converges also to the standard normal distribution.*

Next we are interested in the asymptotic distribution of $\widehat{\boldsymbol{\eta}}_n(\sigma^2) - (\varphi, \omega^2)$ without assuming (2.3). The assumption $Eb_0^4 < \infty$ will be replaced with the requirement that b_0^2 is in the domain of attraction of a stable law. This means that

$$P\{b_0^2 > x\} = x^{-\alpha}L(x), \text{ where } 1 < \alpha < 2 \text{ and } L \text{ is a slowly varying function at } \infty. \quad (2.4)$$

Asumption $\alpha > 1$ guarantees that $Eb_0^2 = \omega^2$ exists. Let

$$a_n = \inf\{x : x^{-\alpha}L(x) \leq 1/n\}.$$

If (2.4) holds, then

$$\frac{1}{a_n} \sum_{1 \leq i \leq n} (b_i^2 - \omega^2) \xrightarrow{\mathcal{D}} \xi, \quad (2.5)$$

where ξ is a stable random variable with characteristic function

$$\exp\{-d|t|^\alpha(1 + \mathbf{i}\text{sign}(t)\tan(\pi\alpha/2))\}, \quad \text{if } 1 < \alpha < 2, \quad (2.6)$$

and d is a positive constant (cf. Breiman (1968, p. 204).

Theorem 2.4. *We assume that the conditions of Theorem 2.1 are satisfied,* (2.4) *and*

$$E|e_0|^\nu < \infty \quad \textit{with some } \nu > 2\alpha/(\alpha - 1) \quad (2.7)$$

hold. Then $n^{1/2}(\widehat{\eta}_{n,1}(\sigma^2) - \varphi)$ and $n(\widehat{\eta}_{n,2}(\sigma^2) - \omega^2)/a_n$ are asymptotically independent, the distribution of $n^{1/2}(\widehat{\eta}_{n,1}(\sigma^2) - \varphi)$ converges to the normal distribution with mean 0 and variance ω^2 and the distribution of $n(\widehat{\eta}_{n,2}(\sigma^2) - \omega^2)/a_n$ converges to the stable distribution with characteristic function given in (2.6).

We note that if $\{e_k\}$ and $\{b_k\}$ are independent sequences, then (2.7) can be replaced with $Ee_0^4 < \infty$.

3 Growth of X_n

We will show in Section 4 (cf. Lemma 4.1) that under the conditions of Theorem 2.1, $X_n \xrightarrow{P} \infty$. Now we find the order of the growth of X_n. To state our results we need further notation. Let

$$\xi_i = \log|\varphi + b_i|, \quad S(i) = \xi_1 + \cdots + \xi_i \quad \text{and} \quad \gamma_i = \prod_{1 \leq j \leq i} \text{sign}\,(\varphi + b_j).$$

In this section we consider the case when

$$E\log|\varphi + b_0| > 0. \quad (3.1)$$

Theorem 3.1. *If* (1.2), (1.3), (1.8) *and* (3.1) *hold, then*

$$e^{-S(n)}\gamma_n X_n \longrightarrow X_0 + Y \quad a.s.$$

where

$$Y = \sum_{1\le i<\infty} e^{-S(i)}\gamma_i e_i.$$

The random normalization $\exp(-S(n))$ is the correct one in Theorem 3.1, if the limit is non–zero with probability one. The next result provide conditions for

$$P\{Y + X_0 \neq 0\} = 1. \tag{3.2}$$

Theorem 3.2. *We assume that* (1.2), (1.3), (1.8) *and* (3.1) *hold.*
(i) If

$$P\{(\varphi + b_0)X_0 + e_0 = c\} = 0 \quad \textit{for all } c, \tag{3.3}$$

then (3.2) *holds.*
(ii) If

$$\{b_k\} \text{ and } \{e_k\} \text{ are independent sequences} \tag{3.4}$$

and

$$P\{e_0 = c\} < 1 \ \textit{ for all } c, \tag{3.5}$$

then (3.2) *holds.*

The first corollary says that under condition (3.1), X_n grows exponentially fast with probability one.

Corollary 3.1. *If* (1.2),(1.3), (1.8) (3.1) *and* (3.3) *or* (3.4) *and* (3.5) *hold, then*

$$e^{-\tau n}|X_n| \longrightarrow \infty \quad a.s. \textit{ for all } 0 < \tau < E\log|\varphi + b_0|$$

and

$$e^{-\tau n}|X_n| \longrightarrow 0 \quad a.s. \textit{ for all } \tau > E\log|\varphi + b_0|.$$

The second corollary is the asymptotic normality of $\widehat{\boldsymbol{\eta}}_n(y)$ without assuming that $y = \sigma^2$.

Corollary 3.2. *If* (1.2), (1.5)–(1.8), (1.11), (2.3), (3.1) *and* (3.3) *or* (3.4) *and* (3.5) *hold, then for all $y > 0$ the distribution of $n^{1/2}(\widehat{\boldsymbol{\eta}}_n(\sigma^2) - (\varphi, \omega^2))$ converges to the bivariate normal distribution with mean* $\mathbf{0}$ *and covariance matrix* Ω_0.

Similarly, in case of $E\log|\varphi + b_0| > 0$, we have the following generalization of Theorem 2.4.

Corollary 3.3. *If* (1.2), (1.5)–(1.8), (1.11), (2.4), (2.7), (3.1) *and* (3.3) *or* (3.4) *and* (3.5) *hold, then for all $y > 0$, $n^{1/2}(\widehat{\eta}_{n,1}(\sigma^2) - \varphi)$ and $n(\widehat{\eta}_{n,2}(\sigma^2) - \omega^2)/a_n$ are asymptotically independent, the distribution of $n^{1/2}(\widehat{\eta}_{n,1}(\sigma^2) - \varphi)$ converges to the normal distribution with mean 0 and variance ω^2 and the distribution of $n(\widehat{\eta}_{n,2}(\sigma^2) - \omega^2)/a_n$ converges to the stable distribution with characteristic function given in* (2.6).

4 Proofs of Theorems 2.1–2.4

The proofs will use the following result:

Lemma 4.1. *If* (1.2) *and* (1.5)–(1.9) *hold, then*

$$|X_n| \xrightarrow{P} \infty, \tag{4.1}$$

Proof. We note that

$$P\{e_0 + c(\varphi + b_0) = c\} < 1 \quad \text{for all } \ c.$$

Indeed, if $e_0 + c(\varphi + b_0) = c$ with probability one, then multiplying this equation with e_0 and taking expected values we get $Ee_0^2 + c\varphi Ee_0 + cEb_0e_0 = cEe_0$. Since $Ee_0 = Ee_0b_0 = 0$, we get $Ee_0^2 = 0$, which contradicts $Ee_0^2 = \sigma^2 > 0$ (c.f. (1.6)). Since (1.9) implies $P\{\varphi + b_0 = 0\} = 0$, the result follows immediately from Remark 2.8 and Corollary 4.1 of Goldie and Maller (2000). □

We start with the study of the log likelihood function.

Lemma 4.2. *If* (1.2), (1.3) *and* (1.5)–(1.9) *are satisfied, then*

$$\sup_{\mathbf{u}\in\Gamma^*}\left|\frac{1}{n}\big(L_n(\mathbf{u}) - L_n(\boldsymbol{\theta})\big) - f(s,x)\right| \xrightarrow{P} 0, \tag{4.2}$$

where $f(\cdot)$ *is defined in* (1.10) *and*

$$\Gamma^* = \big\{\mathbf{u} = (s,x,y): \ s_* \le s \le s^*, \ x_* \le x \le x^*, \ y_* \le y \le y^*\big\},$$

with $0 < x_*$ *and* $0 < y_*$.

Proof. We write

$$\begin{aligned} L_n(\mathbf{u}) - L_n(\boldsymbol{\theta}) = {} & \frac{1}{2}\sum_{1\le k\le n}\log\frac{\omega^2X_{k-1}^2+\sigma^2}{xX_{k-1}^2+y} \\ & + \frac{1}{2}\sum_{1\le k\le n}\frac{(X_{k-1}b_k+e_k)^2}{\omega^2X_{k-1}^2+\sigma^2} - \frac{1}{2}\sum_{1\le k\le n}\frac{\big((\varphi-s)X_{k-1}+X_{k-1}b_k+e_k\big)^2}{xX_{k-1}^2+y}. \end{aligned} \tag{4.3}$$

Using the mean value theorem we conclude

$$\begin{aligned} \left|\log\frac{\omega^2X_{k-1}^2+\sigma^2}{xX_{k-1}^2+y} - \log\frac{\omega^2}{x}\right| &\le c_1\left(\frac{x^*}{\omega^2} + \frac{x^*X_{k-1}^2+y^*}{\omega^2X_{k-1}^2+\sigma^2}\right)\frac{1}{x_*X_{k-1}^2+y_*} \\ &\le c_2\frac{1}{x_*X_{k-1}^2+y_*}. \end{aligned}$$

By (4.1) we have that

$$E\frac{1}{x_*X_n^2+y_*} \to 0 \tag{4.4}$$

and therefore by the Markov inequality

$$\frac{1}{n}\sum_{1\le k\le n}\sup_{\mathbf{u}\in\Gamma^*}\left|\log\frac{\omega^2X_{k-1}^2+\sigma^2}{xX_{k-1}^2+y}-\log\frac{\omega^2}{x}\right|\xrightarrow{P}0. \tag{4.5}$$

Also,

$$\begin{aligned}\sum_{1\le k\le n}&\left\{\frac{(X_{k-1}b_k+e_k)^2}{\omega^2X_{k-1}^2+\sigma^2}-1\right\}\\ &=\sum_{1\le k\le n}(b_k^2-\omega^2)\frac{X_{k-1}^2}{\omega^2X_{k-1}^2+\sigma^2}+\sum_{1\le k\le n}\frac{e_k^2}{\omega^2X_{k-1}^2+\sigma^2}\\ &\quad+\sum_{1\le k\le n}b_ke_k\frac{2X_{k-1}}{\omega^2X_{k-1}^2+\sigma^2}-\sum_{1\le k\le n}\frac{\sigma^2}{\omega^2X_{k-1}^2+\sigma^2}.\end{aligned}$$

Similarly to (4.4) we obtain

$$\frac{1}{n}\sum_{1\le k\le n}\frac{\sigma^2}{\omega^2X_{k-1}^2+\sigma^2}\xrightarrow{P}0$$

Since by (4.1) and the independence of e_n and X_{n-1} we have

$$E\frac{e_n^2}{\omega^2X_{n-1}^2+\sigma^2}\to 0,$$

thus we get

$$\frac{1}{n}\sum_{1\le k\le n}\frac{e_k^2}{\omega^2X_{k-1}^2+\sigma^2}\xrightarrow{P}0.$$

Now we write

$$\begin{aligned}\sum_{1\le k\le n}(b_k^2-\omega^2)\frac{X_{k-1}^2}{\omega^2X_{k-1}^2+\sigma^2}&=\sum_{1\le k\le n}(b_k^2-\omega^2)\frac{1}{\omega^2}-\sum_{1\le k\le n}b_k^2\frac{\sigma^2}{\omega^2(\omega^2X_{k-1}^2+\sigma^2)}\\ &\quad+\sum_{1\le k\le n}\frac{\omega^2\sigma^2}{\omega^2(\omega^2X_{k-1}^2+\sigma^2)}.\end{aligned}$$

The weak law of large numbers yields

$$\frac{1}{n}\sum_{1\le k\le n}(b_k^2-\omega^2)\frac{1}{\omega^2}\xrightarrow{P}0. \tag{4.6}$$

Using now the independence of b_k and X_{k-1} with (4.1) we obtain

$$E\frac{1}{n}\sum_{1\le k\le n}b_k^2\frac{\sigma^2}{\omega^2(\omega^2X_{k-1}^2+\sigma^2)}\to 0\ \text{ and }\ E\frac{1}{n}\sum_{1\le k\le n}\frac{\omega^2\sigma^2}{\omega^2(\omega^2X_{k-1}^2+\sigma^2)}\to 0$$

and therefore by the Markov inequality and (4.6) we conclude

$$\frac{1}{n}\sum_{1\le k\le n}(b_k^2-\omega^2)\frac{X_{k-1}^2}{\omega^2X_{k-1}^2+\sigma^2}\xrightarrow{P}0. \tag{4.7}$$

By the independence of (b_k, e_k) and X_{k-1} we get

$$E\left|\frac{1}{n}\sum_{1\le k\le n}b_ke_k\frac{X_{k-1}}{\omega^2X_{k-1}^2+\sigma^2}\right|\le\frac{1}{n}\sum_{1\le k\le n}E|b_ke_k|E\left|\frac{X_{k-1}}{\omega^2X_{k-1}^2+\sigma^2}\right|\to 0$$

on account of (4.1), resulting in

$$\frac{1}{n}\sum_{1\le k\le n}b_ke_k\frac{X_{k-1}}{\omega^2X_{k-1}^2+\sigma^2}\xrightarrow{P}0.$$

Hence we proved that

$$\frac{1}{n}\sum_{1\le k\le n}\left\{\frac{(X_{k-1}b_k+e_k)^2}{\omega^2X_{k-1}^2+\sigma^2}-1\right\}\xrightarrow{P}0. \tag{4.8}$$

Next we write

$$\begin{aligned}
&\frac{\big(X_{k-1}(\varphi-s)+X_{k-1}b_k+e_k\big)^2}{xX_{k-1}^2+y}\\
&\quad=(\varphi-s)^2\frac{X_{k-1}^2}{xX_{k-1}^2+y}+b_k^2\frac{X_{k-1}^2}{xX_{k-1}^2+y}+e_k^2\frac{1}{xX_{k-1}^2+y}\\
&\qquad+2(\varphi-s)b_k\frac{X_{k-1}^2}{xX_{k-1}^2+y}+2(\varphi-s)e_k\frac{X_{k-1}}{xX_{k-1}^2+y}+2b_ke_k\frac{X_{k-1}}{xX_{k-1}^2+y}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
E\sup_{\mathbf{u}\in\Gamma^*}\left|\sum_{1\le k\le n}e_kb_k\frac{X_{k-1}}{xX_{k-1}^2+y}\right|&\le\sum_{1\le k\le n}E\left|b_ke_k\frac{X_{k-1}}{x_*X_{k-1}^2+y_*}\right|\\
&=E|e_0b_0|\sum_{1\le k\le n}E\frac{|X_{k-1}|}{x_*X_{k-1}^2+y_*}
\end{aligned}$$

and since by (4.1)

$$E\frac{|X_n|}{x_*X_n^2+y_*}\to 0,$$

the Markov inequality yields

$$\sup_{\mathbf{u}\in\Gamma^*}\left|\frac{1}{n}\sum_{1\le k\le n}e_kb_k\frac{X_{k-1}}{xX_{k-1}^2+y}\right|\xrightarrow{P}0.$$

Similar arguments give

$$\sup_{\mathbf{u}\in\Gamma^*}\left|\frac{1}{n}|\varphi-s|\sum_{1\le k\le n} e_k\frac{X_{k-1}}{xX_{k-1}^2+y}\right| \xrightarrow{P} 0.$$

Next we observe that

$$\begin{aligned}\sup_{\mathbf{u}\in\Gamma^*}\left|\sum_{1\le k\le n} b_k\frac{X_{k-1}^2}{xX_{k-1}^2+y}\right| &\le \sup_{\mathbf{u}\in\Gamma^*}\frac{1}{x}\left|\sum_{1\le k\le n} b_k\right| + \sup_{\mathbf{u}\in\Gamma^*}\left|\sum_{1\le k\le n} b_k\frac{1}{x}\frac{y}{xX_{k-1}^2+y}\right|\\ &\le \frac{1}{x_*}\left|\sum_{1\le k\le n} b_k\right| + \frac{y^*}{x_*}\sum_{1\le k\le n}|b_k|\frac{1}{x_*X_{k-1}^2+y_*}.\end{aligned}$$

By the law of large numbers we have

$$\frac{1}{n}\sum_{1\le k\le n} b_k \xrightarrow{P} 0$$

and the Markov inequality with (4.1) gives

$$\frac{1}{n}\sum_{1\le k\le n}|b_k|\frac{1}{x_*X_{k-1}^2+y_*} \xrightarrow{P} 0.$$

Similarly,

$$\sup_{\mathbf{u}\in\Gamma^*}\frac{1}{n}\left|\sum_{1\le k\le n} e_k^2\frac{1}{xX_{k-1}^2+y}\right| \le \frac{1}{n}\sum_{1\le k\le n} e_k^2\frac{1}{x_*X_{k-1}^2+y_*} \xrightarrow{P} 0.$$

Now,

$$b_k^2\frac{X_{k-1}^2}{xX_{k-1}^2+y}-\frac{\omega^2}{x} = -b_k^2\frac{y}{x(xX_{k-1}^2+y)}+\frac{1}{x}(b_k^2-\omega^2),$$

and therefore, arguing as above, we get

$$\begin{aligned}&\sup_{\mathbf{u}\in\Gamma^*}\frac{1}{n}\left|\sum_{1\le k\le n}\left(b_k^2\frac{X_{k-1}^2}{xX_{k-1}^2+y}-\frac{\omega^2}{x}\right)\right|\\ &\quad\le \frac{1}{x_*}\frac{y^*}{n}\sum_{1\le k\le n} b_k^2\frac{1}{x_*X_{k-1}^2+y_*}+\frac{1}{x_*}\left|\sum_{1\le k\le n}(b_k^2-\omega^2)\right| \xrightarrow{P} 0.\end{aligned}$$

Similarly,

$$\sup_{\mathbf{u}\in\Gamma^*}\left|\frac{1}{n}\sum_{1\le k\le n}\left(\frac{X_{k-1}^2}{xX_{k-1}^2+y}-\frac{1}{x}\right)\right| \xrightarrow{P} 0.$$

Thus we proved

$$\sup_{\mathbf{u}\in\Gamma^*}\left|\frac{1}{n}\sum_{1\le k\le n}\frac{\big((\varphi-s)X_{k-1}+X_{k-1}b_k+e_k\big)^2}{xX_{k-1}^2+y}-\left(\frac{(\varphi-s)^2}{x}+\frac{\omega^2}{x}\right)\right| \xrightarrow{P} 0. \tag{4.9}$$

The result in Lemma 4.2 follows from (4.3), (4.5),(4.8) and (4.9). □

Lemma 4.3. *If the conditions of Lemma 4.2 are satisfied and* $\boldsymbol{\eta} = (\varphi, \omega^2) \in \Gamma$, *then*

$$\sup_{y_* \le y \le y^*} \left| \widehat{\boldsymbol{\eta}}_n(y) - \boldsymbol{\eta} \right| \xrightarrow{P} 0 \quad \textit{for all } \ 0 < y_* < y^*.$$

Proof. It is easy to see that

$$f(s,x) \le f(\boldsymbol{\eta}) \qquad \text{for all } \ (s,x)$$

and we have equality if and only if $(s,x) = \boldsymbol{\eta}$. Since

$$\max_{\mathbf{z} \in \Gamma} \big(L_n(\mathbf{z}, y) - L_n(\boldsymbol{\theta}) \big) = L_n(\widehat{\boldsymbol{\eta}}_n, y) - L_n(\boldsymbol{\theta}),$$

$L_n(\mathbf{u})$, $\mathbf{u} \in \Gamma^*$ is continuous on Γ^*, it converges uniformly to $f(s,x)$, standard arguments provide the result (cf. Pfanzangl (1969)). □

Lemma 4.4. *If the conditions of Theorem 2.2 are satisfied, then for all* $0 < y$ *we have*

$$\left| g_{1,n}(y) - \sum_{1 \le k \le n} \frac{b_k}{\omega^2} \right| = o_P(n^{1/2}) \tag{4.10}$$

and

$$\left| g_{2,n}(\sigma^2) - \sum_{1 \le k \le n} \frac{1}{2\omega^4} (b_k^2 - \omega^2) \right| = o_P(n^{1/2}), \tag{4.11}$$

where $g_{1,n}(y)$ *and* $g_{2,n}(y)$ *are the partial derivatives of* $L_n(\mathbf{u})$ *with respect to* s *and* x *at* (φ, ω^2, y).

Proof. Elementary calculations yield

$$\frac{\partial \ell_k(\mathbf{u})}{\partial s} = \frac{(X_k - sX_{k-1})X_{k-1}}{xX_{k-1}^2 + y}$$

and

$$\frac{\partial \ell_k(\mathbf{u})}{\partial x} = -\frac{1}{2} \left[\frac{X_{k-1}^2}{xX_{k-1}^2 + y} - \frac{(X_k - sX_{k-1})^2 X_{k-1}^2}{(xX_{k-1}^2 + y)^2} \right]$$

and therefore

$$g_{1,n}(y) = \sum_{1 \le k \le n} \left\{ \frac{b_k X_{k-1}^2}{\omega^2 X_{k-1}^2 + y} + \frac{e_k X_{k-1}}{\omega^2 X_{k-1}^2 + y} \right\},$$

$$g_{2,n}(y) = \sum_{1 \le k \le n} -\frac{1}{2} \left\{ \frac{X_{k-1}^2}{\omega^2 X_{k-1}^2 + y} - \frac{(X_{k-1} b_k + e_k)^2 X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} \right\}.$$

Using the independence of (e_k, b_k) and X_{k-1} we get

$$\operatorname{var}\left(\frac{1}{n^{1/2}} \sum_{1 \le k \le n} \frac{e_k X_{k-1}}{\omega^2 X_{k-1}^2 + y} \right) = \frac{\sigma^2}{n} \sum_{1 \le k \le n} E \frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} \longrightarrow 0.$$

Similarly,

$$\begin{aligned}
&\mathrm{var}\left(n^{-1/2}\sum_{1\le k\le n} b_k\left\{\frac{X_{k-1}^2}{\omega^2 X_{k-1}^2+y}-\frac{1}{\omega^2}\right\}\right)\\
&=\mathrm{var}\left(n^{-1/2}\sum_{1\le k\le n} b_k\frac{y}{\omega^2(\omega^2 X_{k-1}^2+y)}\right)\\
&=\frac{y^2}{\omega^2}\frac{1}{n}\sum_{1\le k\le n} E\frac{1}{(\omega^2 X_{k-1}^2+y)^2}\longrightarrow 0,
\end{aligned}$$

and thus an application of the Markov inequality completes the proof of (4.10).
Write

$$\begin{aligned}
&\frac{X_{k-1}^2}{\omega^2 X_{k-1}^2+y}-\frac{(X_{k-1}b_k+e_k)^2 X_{k-1}^2}{(\omega^2 X_{k-1}^2+y)^2} \qquad (4.12)\\
&=(\omega^2-b_k^2)\frac{X_{k-1}^4}{(\omega^2 X_{k-1}^2+y)^2}+\frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2+y)^2}(y-2e_k b_k X_{k-1}-e_k^2)\\
&=(\omega^2-b_k^2)\frac{X_{k-1}^4}{(\omega^2 X_{k-1}^2+y)^2}+\frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2+y)^2}(\sigma^2-e_k^2)\\
&\quad-\frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2+y)^2}2e_k b_k X_{k-1}\\
&\quad+\frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2+y)^2}(y-\sigma^2).
\end{aligned}$$

One can easily verify

$$E\left(n^{-1/2}\sum_{1\le k\le n}(\omega^2-b_k^2)\left(\frac{X_{k-1}^4}{(\omega^2 X_{k-1}^2+y)^2}-\frac{1}{\omega^4}\right)\right)^2\longrightarrow 0, \qquad (4.13)$$

$$E\left(n^{-1/2}\sum_{1\le k\le n}\frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2+y)^2}(\sigma^2-e_k^2)\right)^2\longrightarrow 0, \qquad (4.14)$$

$$E\left(n^{-1/2}\sum_{1\le k\le n}\frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2+y)^2}2e_k b_k X_{k-1}\right)^2\longrightarrow 0, \qquad (4.15)$$

and since $y=\sigma^2$ is assumed

$$\frac{1}{n^{1/2}}|y-\sigma^2|E\sum_{1\le k\le n}\frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2+y)^2}=0, \qquad (4.16)$$

and therefore (4.11) is proven. □

Lemma 4.5. *If the conditions of Lemma 4.2 are satisfied, then*

$$\sup_{\mathbf{u}\in\Gamma^*}\big|g_{ij,n}(\mathbf{u})-g_{ij}(\mathbf{u})\big| \xrightarrow{P} 0 \quad 1\le i,j\le 2,$$

where

$$\begin{aligned}
g_{11,n}(\mathbf{u}) &= \frac{\partial^2}{\partial s^2}\frac{1}{n}L_n(\mathbf{u}) = -\frac{1}{n}\sum_{1\le k\le n}\frac{X_{k-1}^2}{xX_{k-1}^2+y};\\
g_{12,n}(\mathbf{u}) &= g_{21,n}(\mathbf{u}) = \frac{\partial^2}{\partial s\partial x}\frac{1}{n}L_n(\mathbf{u}) = \frac{\partial^2}{\partial x\partial s}\frac{1}{n}L_n(\mathbf{u})\\
&= \frac{1}{n}\sum_{1\le k\le n} -\frac{(X_k-sX_{k-1})X_{k-1}^3}{(xX_{k-1}^2+y)^2},\\
g_{22,n}(\mathbf{u}) &= \frac{\partial^2}{\partial x^2}\frac{1}{n}L_n(\mathbf{u}) = \frac{1}{n}\sum_{1\le k\le n}\left\{\frac{X_{k-1}^4}{2(xX_{k-1}^2+y)^2}-\frac{(X_k-sX_{k-1})^2X_{k-1}^4}{(xX_{k-1}^2+y)^3}\right\}
\end{aligned}$$

and

$$g_{11}(\mathbf{u}) = -\frac{1}{x},\qquad g_{12}(\mathbf{u}) = g_{21}(\mathbf{u}) = -\frac{(\varphi-s)}{x^2},\qquad g_{22}(\mathbf{u}) = \frac{1}{2x^2}-\frac{(\varphi-s)^2+\omega^2}{x^3}.$$

Proof. It can be proven along the lines of the proof of Lemma 4.2 and therefore the details are omitted. □

Proof of Theorem 2.2. Combining the central limit theorem for independent identically distributed random vectors with Lemma 4.4, we get that

$$n^{-1/2}\big(g_{1,n}(\sigma^2), g_{2,n}(\sigma^2)\big) \xrightarrow{\mathcal{D}} N_2(\mathbf{0},\Omega_*), \tag{4.17}$$

where

$$\Omega_* = \begin{pmatrix} \dfrac{1}{\omega^2} & \dfrac{Eb_0^3}{2\omega^4} \\ \dfrac{Eb_0^3}{2\omega^4} & \dfrac{\operatorname{var} b_0^2}{4\omega^8} \end{pmatrix}.$$

Let $\|\cdot\|$ denote the maximum norm of vectors. Let $\nabla h(\mathbf{u}) = (\partial h(\mathbf{u})/\partial u_1, \partial h(\mathbf{u})/\partial u_2)^T$. Applying the mean value theorem to the coordinates of $\nabla L_n(\mathbf{u},\sigma^2)$, there are random vector $\boldsymbol{\xi}_{n,1}$ and $\boldsymbol{\xi}_{n,2}$ such that $\|\boldsymbol{\xi}_{n,j}-\boldsymbol{\eta}\| \le \|\widehat{\boldsymbol{\eta}}_n-\boldsymbol{\eta}\|, j=1,2$ and

$$0 = \frac{\partial L_n(\boldsymbol{\eta},\sigma^2)}{\partial u_j} + \left(\nabla\frac{\partial L_n(\boldsymbol{\xi}_{n,j},\sigma^2)}{\partial u_j}\right)^T(\widehat{\boldsymbol{\eta}}_n-\boldsymbol{\eta}),\quad j=1,2. \tag{4.18}$$

Lemma 4.5 and Theorem 2.1 give that for all $y>0$

$$\left(\frac{1}{n}\nabla\frac{\partial L_n(\boldsymbol{\xi}_{n,1},y)}{\partial u_1}, \frac{1}{n}\nabla\frac{\partial L_n(\boldsymbol{\xi}_{n,2},y)}{\partial u_2}\right) \xrightarrow{P} \Omega_{**}, \tag{4.19}$$

where

$$\Omega_{**} = \begin{pmatrix} -\dfrac{1}{\omega^2} & 0 \\ 0 & -\dfrac{1}{2\omega^4} \end{pmatrix}.$$

Putting together (4.17)–(4.19) we conclude

$$n^{1/2}\big(\widehat{\boldsymbol{\eta}}_n(\sigma^2) - \boldsymbol{\eta}\big) \xrightarrow{\mathcal{D}} N_2\big(\mathbf{0}, \Omega_{**}^{-1}\Omega_*\Omega_{**}^{-1}\big).$$

Since $\Omega_0 = \Omega_{**}^{-1}\Omega_*\Omega_{**}^{-1}$, the proof of Theorem 2.2 is complete. □

The proof of Theorem 2.3 uses the following lemma.

Lemma 4.6. *If the conditions of Lemma 4.2 are satisfied, then for all* $y > 0$

$$\sup_{\mathbf{u}\in\Gamma}\left| g_{12,n}(\mathbf{u}) - (s-\varphi)\frac{1}{n}\sum_{1\le k\le n}\frac{X_{k-1}^4}{(xX_{k-1}^2+y)^2}\right| = \mathcal{O}_P(n^{-1/2}).$$

Proof. Using the expression for $g_{12,n}(\mathbf{u})$ in Lemma 4.5 we get that

$$g_{12,n}(\mathbf{u}) = -\frac{1}{n}\sum_{1\le k\le n}\frac{(\varphi-s)X_{k-1}^4}{(xX_{k-1}^2+y)^2} - \frac{1}{n}\sum_{1\le k\le n}\frac{b_kX_{k-1}^4}{(xX_{k-1}^2+y)^2} - \frac{1}{n}\sum_{1\le k\le n}\frac{e_kX_{k-1}^3}{(xX_{k-1}^2+y)^2}.$$

Also,

$$\frac{1}{n}\sum_{1\le k\le n}\frac{b_kX_{k-1}^4}{(xX_{k-1}^2+y)^2} = \frac{1}{n}\frac{1}{x^2}\sum_{1\le k\le n}b_k - \frac{2y}{xn}\sum_{1\le k\le n}\frac{b_kX_{k-1}^2}{(xX_{k-1}^2+y)^2} - \frac{1}{n}\frac{y^2}{x^2}\sum_{1\le k\le n}\frac{b_k}{(xX_{k-1}^2+y)^2}.$$

The central limit theorem yields

$$\sup_{x_*\le x\le x^*}\left|\frac{1}{n}\frac{1}{x^2}\sum_{1\le k\le n}b_k\right| = \mathcal{O}_P(n^{-1/2}).$$

Next we show that

$$\sup_{x_*\le x\le x^*}|A_n(x)| = o_P(1), \tag{4.20}$$

where

$$A_n(x) = \frac{1}{n^{1/2}}\sum_{1\le k\le n}\frac{b_kX_{k-1}^2}{(xX_{k-1}^2+y)^2}.$$

Since for any $x \in [x_*, x^*]$

$$EA_n(x)^2 = \frac{\omega^2}{n}\sum_{1\le k\le n}E\left(\frac{X_{k-1}^2}{(xX_{k-1}^2+y)^2}\right)^2 \to 0,$$

the finite dimensional distributions of $A_n(x)$ converge to 0. Similarly, for all $x, x' \in [x_*, x^*]$ we have by the mean value theorem that

$$\begin{aligned} E(A_n(x) - A_n(x'))^2 &= \frac{\omega^2}{n} \sum_{1 \le k \le n} E\left(X_{k-1}^2 \left[\frac{1}{(xX_{k-1}^2 + y)^2} - \frac{1}{(x'X_{k-1}^2 + y)^2} \right] \right)^2 \\ &\le (x - x')^2 \frac{\omega^2}{n} \sum_{1 \le k \le n} E\left(2X_{k-1}^4 \frac{1}{(x_* X_{k-1}^2 + y_*)^3} \right)^2 \\ &\le (x - x')^2, \end{aligned}$$

for all n large enough. By Billingsley (1968, p. 96), the sequence $A_n(x)$ is tight, and therefore $A_n(x)$ converges in $\mathcal{C}[x_*, x^*]$ to 0. Hence the proof of (4.20) is complete.
Repeating the arguments leading to (4.20), we conclude

$$\sup_{x_* \le x \le x^*} \left| \frac{1}{n} \frac{y^2}{x^2} \sum_{1 \le k \le n} \frac{b_k}{(xX_{k-1}^2 + y)^2} \right| + \sup_{x_* \le x \le x^*} \left| \frac{1}{n} \sum_{1 \le k \le n} \frac{e_k X_{k-1}^3}{(xX_{k-1}^2 + y)^2} \right| = \mathcal{O}_P(n^{-1/2}).$$

The proof of Lemma 4.6 is established now. □

Proof of Theorem 2.3. Similarly to (4.18) we have

$$0 = \frac{\partial L_n(\boldsymbol{\eta}, y)}{\partial u_j} + \left(\triangledown \frac{\partial L_n(\boldsymbol{\xi}_{n,j}, y)}{\partial u_j} \right)^T (\widehat{\boldsymbol{\eta}}_n(y) - \boldsymbol{\eta}), \quad j = 1, 2,$$

where $\boldsymbol{\xi}_{n,j}$ satisfies $\|\boldsymbol{\xi}_{n,j} - \boldsymbol{\eta}\| \le \|\widehat{\boldsymbol{\eta}}_n(y) - \boldsymbol{\eta}\|$, $j = 1, 2$. This gives

$$\widehat{\eta}_{n,1}(y) - \varphi = -\left(c_{11}(n) \frac{1}{n} g_{1,n}(y) + c_{12}(n) \frac{1}{n} g_{2,n}(y) \right), \tag{4.21}$$

where $c_{ij}(n)$ are defined by

$$\left(\frac{1}{n} \triangledown \frac{\partial L_n(\boldsymbol{\xi}_{n,1}, y)}{\partial u_1}, \frac{1}{n} \triangledown \frac{\partial L_n(\boldsymbol{\xi}_{n,2}, y)}{\partial u_2} \right)^{-1} = \begin{pmatrix} c_{11}(n) & c_{12}(n) \\ c_{21}(n) & c_{22}(n) \end{pmatrix}. \tag{4.22}$$

Using (4.12)–(4.15) we get that

$$g_{2,n}(y) = o_P(n). \tag{4.23}$$

Now (4.19) gives that $c_{11}(n) \to -\omega^2$ in probability. Applying Lemma 4.6 and (4.19) we get that

$$|c_{12}(n)| = |\widehat{\eta}_{1,n}(y) - \varphi| \mathcal{O}_P(1) + \mathcal{O}_P(n^{-1/2}). \tag{4.24}$$

By (4.21)–(4.24) we conclude

$$\begin{aligned} \widehat{\eta}_{n,1}(y) - \varphi &= \left(-\omega^2 + o_P(1) \right) \frac{1}{n} g_{1,n}(y) + [|\widehat{\eta}_{n,1}(y) - \varphi| \mathcal{O}_P(1) + \mathcal{O}_P(n^{-1/2})] o_P(1) \\ &= \left(-\omega^2 + o_P(1) \right) \frac{1}{n} g_{1,n}(y) + |\widehat{\eta}_{n,1}(y) - \varphi| o_P(1) + o_P(n^{-1/2}), \end{aligned}$$

which yields

$$\widehat{\eta}_{n,1}(y)-\varphi=(1+o_P(1))^{-1}\left((-\omega^2+o_P(1))\frac{1}{n}g_{n,1}(y)+o_P(n^{-1/2})\right)$$

Now the first part of Theorem 2.3 follows from (4.10).
The second part is an immedaite consequence of the first part Theorem 2.1and Slutsky's lemma. □

The proof of Theorem 2.4 is based on the following modification of Lemma 4.4.

Lemma 4.7. *If the conditions of Theorem 2.4 are satisfied, then for all* $0<y$ *we have*

$$\left|g_{1,n}(y)-\sum_{1\le k\le n}\frac{b_k}{\omega^2}\right|=o_P(n^{1/2}) \tag{4.25}$$

and

$$\left|g_{2,n}(\sigma^2)-\sum_{1\le k\le n}\frac{1}{2\omega^4}(b_k^2-\omega^2)\right|=o_P(a_n), \tag{4.26}$$

where $g_{1,n}(y)$ *and* $g_{2,n}(y)$ *are the partial derivatives of* $L_n(\mathbf{u})$ *with respect to* s *and* x *at* (φ,ω^2,y).

Proof. We follow the proof of Lemma 4.4. Since the proof of (4.10) required only that $Eb_0^2<\infty$, we have (4.25).
To prove (4.26), we use (4.12). It is assumed that $Ee_0^4<\infty$ and therefore (4.14) holds. Assumption (2.4) yields that $E|b_0|^{2\tau}<\infty$ for all $0<\tau<\alpha$, and therefore condition (2.7) with Hölder's inequality gives $Ee_0^2b_0^2<\infty$. Hence

$$E\left(n^{-1/2}\sum_{1\le k\le n}\frac{X_{k-1}^2}{(\omega^2X_{k-1}^2+y)^2}2e_kb_kX_{k-1}\right)^2=\frac{4}{n}E(e_0b_0)^2\sum_{1\le k\le n}\left(\frac{X_{k-1}^3}{(\omega^2X_{k-1}^2+y)^2}\right)^2\longrightarrow 0.$$

Clearly, (4.16) is satisfied. Thus it is enough to show that

$$\sum_{1\le k\le n}(\omega^2-b_k^2)\left(\frac{X_{k-1}^4}{(\omega^2X_{k-1}^2+y)^2}-\frac{1}{\omega^4}\right)=o_P(a_n). \tag{4.27}$$

Let

$$\epsilon_k=b_k^2-\omega^2\ \text{ and }\ z_{k-1}=\frac{X_{k-1}^4}{(\omega^2X_{k-1}^2+y)^2}-\frac{1}{\omega^4}.$$

It is clear that $|z_k|\le c_1$ with some constant c_1. Also, according to Lemma 4.1, $|z_k|\to 0$ in probability, as $k\to\infty$, and therefore

$$\delta_k=Ez_k^2\to 0\ \ (k\to\infty). \tag{4.28}$$

Fix n and define

$$\epsilon_k^* = \epsilon_k I\{|\epsilon_k| \le \tau_n a_n\} \quad \text{and} \quad \epsilon_k^{**} = \epsilon_k^* - E\epsilon_k^*, \quad 1 \le k \le n,$$

where τ_n is a numerical sequence (to be chosen later) tending to ∞ and $I\{\cdot\}$ denotes the indicator function. Let

$$A(t) = \int_{-t}^{t} x^2 dF(x),$$

where F denotes the distribution function of ϵ_0. By the classical theory of the domain of attraction of stable laws (cf. Feller (1966, pp. 574–577)) we have that

$$\lim_{t\to\infty} \frac{A(t)}{t^{2-\alpha}L(t)} = c_2 \tag{4.29}$$

with some $0 < c_2 < \infty$. Also, we note that by the definition of a_n and the properties of regularly varying functions we get that

$$nL(a_n)/a_n^\alpha \to 1 \quad (n \to \infty). \tag{4.30}$$

We also need that for any $\kappa > 0$ there is a constant $0 < c_3 < \infty$ such that

$$\frac{L(\lambda x)}{L(x)} \le c_3 \lambda^\kappa \quad \text{for all} \quad \lambda \ge 1 \;\text{ and }\; x \ge 1. \tag{4.31}$$

The assertion in (4.31) is an immediate consequence of the monotone equivalence theorems in Bingham et al (1987, p. 23). Indeed, there is a non–increasing regularly varying function ψ such that

$$\lim_{x\to\infty} \frac{x^{-\kappa}L(x)}{\psi(x)} = 1,$$

and since $\psi(\lambda x) \le \psi(x)$ for all $\lambda \ge 1$ and $x \ge 1$, so (4.31) is proven.
Using the independence of ϵ_k^{**} and z_{k-1} we conclude

$$E(\epsilon_k^{**} z_{k-1})^2 = E(\epsilon_k^{**})^2 E z_{k-1}^2 \le E(\epsilon_k^*)^2 E z_{k-1}^2 = A(\tau_n a_n)\delta_{k-1}$$

and the orthogonality of $\{\epsilon_k^{**} z_{k-1}, k \ge 1\}$ yields

$$\operatorname{var}\left(\frac{1}{a_n}\sum_{1\le k\le n} \epsilon_k^{**} z_{k-1}\right) = \frac{1}{a_n^2}\sum_{1\le k\le n} E(\epsilon_k^{**} z_{k-1})^2 \le \frac{1}{a_n^2} A(\tau_n a_n) \sum_{1\le k\le n} \delta_{k-1}.$$

Combining (4.29)–(4.31) we get that

$$\begin{aligned}\operatorname{var}\left(\frac{1}{a_n}\sum_{1\le k\le n} \epsilon_k^{**} z_{k-1}\right) &= \mathcal{O}(1) a_n^{-2} (\tau_n a_n)^{2-\alpha} L(\tau_n a_n) \sum_{1\le k\le n} \delta_{k-1} \\ &= \mathcal{O}(1) \tau_n^{2-\alpha} \frac{L(\tau_n a_n)}{L(a_n)} \frac{1}{n} \sum_{1\le k\le n} \delta_{k-1}\end{aligned}$$

$$= \mathcal{O}(1)\tau_n^{2-\alpha+\kappa}\frac{1}{n}\sum_{1\le k\le n}\delta_{k-1}.$$

By (4.28), if $\tau_n \to \infty$ slowly enough, then $\tau_n^{2-\alpha+\kappa}\sum_{1\le k\le n}\delta_{k-1}/n \to 0$, showing that

$$\operatorname{var}\left(\frac{1}{a_n}\sum_{1\le k\le n}\epsilon_k^{**}z_{k-1}\right) = o(1).$$

Using the definitions of ϵ_k^* and a_n together with (2.4), (4.30) and (4.31), we obtain that

$$\sum_{1\le k\le n} P\{\epsilon_k^* \ne \epsilon_k\} = n(1 - F(\tau_n a_n)) \to 0 \quad (n\to\infty).$$

Next we observe that

$$\frac{n}{a_n}\int_{-\tau_n a_n}^{\tau_n a_n} x dF(x) \to 0 \quad (n\to\infty),$$

if $\tau_n \to \infty$ slowly enough, so using $z_k \to 0$ $(k\to\infty)$ we conclude that

$$\sum_{1\le k\le n} z_{k-1}E\epsilon_k^* = \sum_{1\le k\le n} z_{k-1}\int_{-\tau_n a_n}^{\tau_n a_n} x dF(x) = o(1)n\int_{-\tau_n a_n}^{\tau_n a_n} x dF(x) = o(a_n).$$

Now the proof of (4.27) is complete. □

Proof of Theorem 2.4. Using (4.18) we get

$$\widehat{\eta}_{n,1}(\sigma^2) - \varphi = -\left(c_{11}(n)\frac{1}{n}g_{1,n}(\sigma^2) + c_{12}(n)\frac{1}{n}g_{2,n}(\sigma^2)\right), \tag{4.32}$$

and

$$\widehat{\eta}_{n,2}(\sigma^2) - \omega^2 = -\left(c_{21}(n)\frac{1}{n}g_{1,n}(\sigma^2) + c_{22}(n)\frac{1}{n}g_{2,n}(\sigma^2)\right), \tag{4.33}$$

where c_{ij} are defined in (4.22). By Lemma 4.7, (4.19) and (4.22) we get that

$$\frac{n}{a_n}(\widehat{\eta}_{n,2}(\sigma^2) - \omega^2) = \frac{1}{a_n}\sum_{1\le k\le n}(b_k^2 - \omega^2) + o_P(1). \tag{4.34}$$

Since (4.23) clearly holds, we also have (4.24) and from (4.26) we obtain that

$$\widehat{\eta}_{n,1}(\sigma^2) - \varphi = c_{11}(n)\frac{1}{n}g_{1,n}(\sigma^2) + (|\widehat{\eta}_{n,1} - \varphi|\mathcal{O}_P(1) + \mathcal{O}_P(n^{-1/2}))\mathcal{O}_P(a_n/n).$$

Hence by (4.19) and (4.25) we have

$$n^{1/2}(\widehat{\eta}_{n,1}(\sigma^2) - \varphi) = n^{-1/2}\sum_{1\le k\le n} b_k + o_P(1). \tag{4.35}$$

The convergence in distribution of $n^{1/2}(\widehat{\eta}_{n,1}(\sigma^2)-\varphi)$ and $n(\widehat{\eta}_{n,2}(\sigma^2)-\omega^2)/a_n$ now follows from (4.34) and (4.35); only the asymptotic independence must be established. Note that the vector $(\sum_{1\le k\le} b_k/n^{1/2}, \sum_{1\le k\le n}(b_k^2-\omega^2)/a_n)$ converges in distribution (cf. Section 10.1 in Meerschaert and Scheffler (2001)). The first coordinate of the limit is normal, the second does not contain normal component and therefore the coordinates of the limit distribution are independent (Meerschaert and Scheffler (2001, p. 41)). □

5 Proofs of Theorem 3.1 and Corollaries 3.1–3.3

Using (1.8) one can easily verify that

$$X_\ell = \sum_{i=1}^{\ell} e_i \prod_{j=i+1}^{\ell}(\varphi+b_j) + X_0 \prod_{j=1}^{\ell}(\varphi+b_j),$$

and therefore

$$\begin{aligned}\left(\prod_{j=1}^{\ell}(\varphi+b_j)\right)^{-1} X_\ell &= \sum_{i=1}^{\ell} e_i \left(\prod_{j=1}^{i}(\varphi+b_j)\right)^{-1} + X_0 \qquad (5.1)\\ &= \sum_{i=1}^{\ell} e_i e^{-S(i)}\gamma_i + X_0.\end{aligned}$$

Proof of Theorem 3.1. First we note that assumption (1.3) yields

$$|e_i| = \mathcal{O}(e^{ic_1}) \ \text{ a.s. for any } \ c_1>0$$

(cf. Berkes at al (2003)) and therefore by the strong law of large numbers

$$e^{-S(i)} = o(e^{-ic_2}) \ \text{ a.s. for any } 0<c_2<E|\xi_0|.$$

Hence Y is absolutely convergent with probability one and the result follows immediately from (5.1). □

The proof of the second part of Theorem 3.2 is based on the following lemma:

Lemma 5.1. *If* (1.2),(1.3), (1.8) (3.1), (3.4) *and* (3.5) *hold, then*

$$P\{Y=c\}=0 \quad \text{for any } \ c.$$

Proof. First we show that for any sequence a_n

$$\sum_{1\le i<\infty} P\left\{e^{-S(i)}\gamma_i e_i \ne a_i \,\middle|\, \xi_j, -\infty<j<\infty\right\} = \infty \quad \text{a.s.} \qquad (5.2)$$

Since $E\xi_0$ exists, we get $P\{\xi_0 = 0\} = 0$, so γ_i can be 0 only with probability 0. Hence (5.2) holds, if for any sequence b_n

$$\sum_{1 \le i < \infty} P\{e_i \ne b_i\} = \infty. \tag{5.3}$$

By (3.4), we have (5.3) if and only if

$$\sum_{1 \le i < \infty} P\{e_0 \ne b_i\} = \sum_{1 \le i < \infty} \big(1 - P\{e_0 = b_i\}\big) = \infty. \tag{5.4}$$

If $P\{e_0 = b_i\} \to 1$, then e_0 must be a constant with probability 1, contradicting (3.5). Using (5.2) we get that for any sequence a_n

$$\sum_{1 \le i < \infty} P\big\{e^{-S(i)}\gamma_i e_i \ne a_i\big\} = \infty, \tag{5.5}$$

and therefore Lemma 5.1 follows from Lévy (1931) (cf. also Breiman (1968, p. 51)). □

Lemma 5.2. *If* (1.2), (1.5)–(1.8), (1.11), (3.1) *and* (3.3) *or* (3.4) *and* (3.5) *hold, then*

$$\left| g_{1,n}(y) - \sum_{1 \le k \le n} \frac{b_k}{\omega^2} \right| = \mathcal{O}(1) \quad a.s. \tag{5.6}$$

and

$$\left| g_{2,n}(y) - \sum_{1 \le k \le n} \frac{1}{2\omega^4}(b_k^2 - \omega^2) \right| = \mathcal{O}(1) \quad a.s. \tag{5.7}$$

where $g_{1,n}(y)$ and $g_{2,n}(y)$ are the partial derivatives of $L_n(\mathbf{u})$ with respect to s and x at (φ, ω^2, y).

Proof of Lemma 5.2. We return to the decompositions of $g_{1,n}(y)$ and $g_{2,n}(y)$ used in the proof of Lemma 4.4. Using Theorem 3.1 and Lemma 5.1 we get that

$$\begin{aligned}
&\left| \sum_{1 \le k \le n} \frac{e_k X_{k-1}}{\omega^2 X_{k-1}^2 + y} \right| \\
&\qquad \le \sum_{1 \le k \le n} |e_k| \frac{|X_{k-1}|}{\omega^2 X_{k-1}^2 + y} \\
&\qquad = \sum_{1 \le k \le n} |e_k| e^{-S(k-1)} \big(e^{-S(k-1)}|X_{k-1}|\big)\big(\omega^2 (e^{-S(k-1)}X_{k-1})^2 + e^{-2S(k-1)}y\big)^{-1} \\
&\qquad \le \left\{ \max_{1 \le k < \infty} \big(e^{-S(k-1)}|X_{k-1}|\big)\big(\omega^2 (e^{-S(k-1)}X_{k-1})^2 + e^{-2S(k-1)}y\big)^{-1} \right\} \sum_{1 \le k \le n} |e_k| e^{-S(k-1)} \\
&\qquad = \mathcal{O}(1) \qquad \text{a.s.}
\end{aligned}$$

since by Berkes et al (2003), $\sum_{1\le k\le n}|e_k|e^{-S(k-1)}$ is finite with probability one. Similar arguments give

$$\left|\sum_{1\le k\le n} b_k\left\{\frac{X_{k-1}^2}{\omega^2 X_{k-1}^2+y}-\frac{1}{\omega^2}\right\}\right|\le\sum_{1\le k\le n}|b_k|\frac{1}{\omega^2}\frac{|y|}{\omega^2 X_{k-1}^2+y}=\mathcal{O}(1)\qquad\text{a.s.},$$

completing the proof of (5.6).
The proof of (5.7) goes along the same lines and hence it is omitted. □

Proof of Corollary 3.1. It is an immediate consequence of the strong law of large numbers and Theorems 3.1 and 3.2.

□

Proof of Corollary 3.2. The proof of Theorem 2.2 can be repeated; only Lemma 4.4 must be replaced with Lemma 5.2. □

Proof of Corollary 3.3. Minor modifications of the proof of Theorem 2.4 are required only. Namely, one must use Lemma 5.2 instead of Lemma 4.7. □

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