

# Singularities in Horava-Lifshitz theory

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Singularities in (3 + 1)-dimensional Horava-Lifshitz (HL) theory of gravity are studied. These singularities can be divided into scalar, non-scalar curvature, and coordinate singularities. Because of the foliation-preserving diffeomorphisms of the theory, the number of scalars that can be constructed from the extrinsic curvature tensor  $K_{ij}$ , the 3-dimensional Riemann tensor and their derivatives is much larger than that constructed from the 4-dimensional Riemann tensor and its derivatives in general relativity (GR). As a result, even for the same spacetime, it may be singular in the HL theory but not in GR. Two representative families of solutions with projectability condition are studied, one is the (anti-) de Sitter Schwarzschild solutions, and the other is the Lu-Mei-Pope (LMP) solutions written in a form satisfying the projectability condition - the generalized LMP solutions. The (anti-) de Sitter Schwarzschild solutions are vacuum solutions of both HL theory and GR, while the LMP solutions with projectability condition satisfy the HL equations coupled with an anisotropic fluid with heat flow. It is found that the scalars  $K$  and  $K_{ij}K^{ij}$  are singular only at the center for the de Sitter Schwarzschild solution, but singular at both the center and  $r = (3M/|\Lambda|)^{1/3}$  for the anti-de Sitter Schwarzschild solution. The singularity at  $r = (3M/|\Lambda|)^{1/3}$  is absent in GR. In addition, all the generalized LMP solutions have two scalar curvature singularities, located at either  $r = 0$  and  $r = r_s > 0$ , or  $r = r_1$  and  $r = r_2$  with  $r_2 > r_1 > 0$ , or  $r = r_s > 0$  and  $r = \infty$ , depending on the choice of the free parameter  $\lambda$ .

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## I. INTRODUCTION

There has been considerable interest recently on a theory of quantum gravity proposed by Horava [1], motivated by Lifshitz theory in solid state physics [2], for which the theory is usually referred to as the Horava-Lifshitz (HL) theory. The HL theory is based on the perspective that Lorentz symmetry should appear as an emergent symmetry at long distances, but can be fundamentally absent at high energies [3]. With such a perspective, Horava considered systems whose scaling at short distances exhibits a strong anisotropy between space and time,

$$\mathbf{x} \rightarrow \ell \mathbf{x}, \quad t \rightarrow \ell^z t. \quad (1.1)$$

In (3 + 1)-dimensional spacetimes, in order for the theory to be power-counting renormalizable, it needs  $z \geq 3$ . At low energies, the theory is expected to flow to  $z = 1$ , whereby the Lorentz invariance is “accidentally restored.” So, the HL theory is non-relativistic, ultra-violet (UV) complete, explicitly breaks Lorentz invariance at short distances, but is expected to reduce to general relativity (GR) in the infrared (IR) limit.

The effective speed of light in this theory diverges in the UV regime, which could potentially resolve the horizon problem without invoking inflation [4]. The spatial

curvature is enhanced by higher-order curvature terms [5–7], and this opens a new approach to the flatness problem and to a bouncing universe [5, 8, 9]. In addition, in the super-horizon region scale-invariant curvature perturbations can be produced without inflation [10, 11], and the perturbations become adiabatic during slow-roll inflation driven by a single scalar field and the comoving curvature perturbation is constant [11]. Due to all these remarkable features, the theory has attracted lot of attention lately [12–14].

To formulate the theory, Horava assumed two conditions – *detailed balance and projectability* (He also considered the case where the detailed balance condition was softly broken) [1]. The detailed balance condition restricts the form of a general potential in a ( $D + 1$ )-dimensional Lorentz action to a specific form that can be expressed in terms of a  $D$ -dimensional action of a relativistic theory with Euclidean signature, whereby the number of independent-couplings is considerably limited. The projectability condition, on the other hand, originates from the fundamental symmetry of the theory – the foliation-preserving diffeomorphisms of the Arnowitt-Deser-Misner (ADM) form,

$$ds^2 = -N^2 c^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (i, j = 1, 2, 3), \quad (1.2)$$

which require coordinate transformations be only of the types,

$$t \rightarrow f(t), \quad x^i \rightarrow \zeta^i(t, \mathbf{x}), \quad (1.3)$$

that is, space-dependent time reparameterizations are no longer allowed, although spatial diffeomorphisms are still

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a symmetry. Then, it is natural, but not necessary, to restrict the lapse function  $N$  to be space-independent, while the shift vector  $N^i$  and the 3-dimensional metric  $g_{ij}$  in general depend on both time and space,

$$N = N(t), \quad N^i = N^i(t, x), \quad g_{ij} = g_{ij}(t, x). \quad (1.4)$$

This is the projectability condition, and clearly is preserved by the foliation-preserving diffeomorphisms (1.3). However, due to these restricted diffeomorphisms, one more degree of freedom appears in the gravitational sector - a spin-0 graviton. This is potentially dangerous, and needs to be highly suppressed in the IR regime, in order to be consistent with observations. Similar problems also raise in other modified theories, such as massive gravity [15].

Under the rescaling (1.1), the dynamical variables  $N$ ,  $N^i$  and  $g_{ij}$  scale as,

$$N \rightarrow N, \quad N^i \rightarrow \ell^{-2} N^i, \quad g_{ij} \rightarrow g_{ij}. \quad (1.5)$$

Note that in [7, 11], the constant  $c$  in the metric (1.2) was absorbed into  $N$ , so that there the lapse function scaled as  $\ell^{-2}$ .

So far most of the work on the HL theory has abandoned the projectability condition but kept the detailed balance [4, 6, 12–14]. One of the main reasons is that the detailed balance condition leads to a very simple action, and the resulted theory is much easier to deal with, while abandoning projectability condition gives rise to local rather than global Hamiltonian constraint and energy conservation. However, with detailed balance a scalar field is not UV stable [5], and gravitational perturbations in the scalar section have ghosts [1] and are not stable for any given value of the dynamical coupling constant  $\lambda$  [16]. In addition, detailed balance also requires a non-zero (negative) cosmological constant, breaks the parity in the purely gravitational sector [17], and makes the perturbations not scale-invariant [18]. Breaking the projectability condition, on the other hand, can cause strong couplings [19] and gives rise to an inconsistency theory [20].

To resolve these problems, various modifications have been proposed [21]. In particular, Blas, Pujolas and Sibiryakov (BPS) [22] showed that the strong coupling problem can be solved without projectability condition (in which the lapse function becomes dependent on both  $t$  and  $x^i$ ), when terms constructed from the 3-vector

$$a_i \equiv \frac{\partial_i N}{N}, \quad (1.6)$$

are included. Contrary claims can be found in [23]. In addition, it is not clear how the inconsistency problem [20] is resolved in such a generalization.

On the other hand, Sotiriou, Visser and Weifurtner (SVW) formulated the most general HL theory with projectability but without detailed balance conditions [17]. The total action consists of three parts, kinetic, potential

and matter,

$$S = \zeta^2 \int dt d^3x N \sqrt{g} (\mathcal{L}_K - \mathcal{L}_V + \zeta^{-2} \mathcal{L}_M), \quad (1.7)$$

where  $g = \det g_{ij}$ , and

$$\begin{aligned} \mathcal{L}_K &= \frac{1}{c^2} [K_{ij} K^{ij} - (1 - \xi) K^2], \\ \mathcal{L}_V &= 2\Lambda - R + \frac{1}{\zeta^2} (g_2 R^2 + g_3 R_{ij} R^{ij}) \\ &\quad + \frac{1}{\zeta^4} (g_4 R^3 + g_5 R_{ij} R^{ij} + g_6 R_j^i R_k^j R_i^k) \\ &\quad + \frac{1}{\zeta^4} [g_7 R \nabla^2 R + g_8 (\nabla_i R_{jk}) (\nabla^i R^{jk})]. \end{aligned} \quad (1.8)$$

Here  $\zeta^2 = 1/16\pi G$ , and  $c$  denotes the speed of light. In the “physical” units, one can set  $c = 1$  [17]. The covariant derivatives and Ricci and Riemann terms are all constructed from the three-metric  $g_{ij}$ , while  $K_{ij}$  is the extrinsic curvature,

$$K_{ij} = \frac{1}{2N} (-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i), \quad (1.9)$$

where  $N_i = g_{ij} N^j$ . The constants  $\xi$ ,  $g_I$  ( $I = 2, \dots, 8$ ) are coupling constants, and  $\Lambda$  is the cosmological constant. In the IR limit, all the high order curvature terms (with coefficients  $g_I$ ) drop out, and the total action reduces when  $\xi = 0$  to the Einstein-Hilbert action.

The SVW generalization seems to have the potential to solve the above mentioned problems [24], although it was found that gravitational scalar perturbations either have ghosts ( $0 \leq \xi \leq 2/3$ ) or are not stable ( $\xi < 0$ ) [7, 25]. In order to avoid ghost instability, one needs to assume  $\xi \leq 0$ . Then, the sound speed  $c_\psi^2 = \xi/(2 - 3\xi)$  becomes imaginary, which leads to an IR instability. Izumi and Mukohyama showed that this type of instability does not show up if  $|c_\psi|$  is less than a critical value [26].

It is fair to say, in order to have a viable HL theory, much work needs to be done, and various aspects of the theory ought to be explored, including the renormalization group flows [27], Vainshtein mechanism [15, 28], solar system tests [29], Lorentz violations [30], and its applications to cosmology [7, 11].

In this paper, we shall study another important issue in the HL theory - the problem of singularities, which is closely related to the issue of black holes in this theory [12]. Although we are initially interested in the case with projectability condition, our conclusions can be equally applied to the HL theory without projectability condition. The extrinsic curvature  $K_{ij}$  and the 3-dimensional Riemann tensor  $R_{jkl}^i$  are not tensors under the 4-dimensional Lorentz transformations,

$$x^\mu \rightarrow x'^\mu = \zeta^\mu(t, x^i). \quad (1.10)$$

As a result, in GR one usually does not use them to construct gauge-invariant quantities. However, in the

HL theory, due to the restricted diffeomorphisms, these quantities become tensors, and can be easily used to construct various scalars. If any of such scalars is singular, such a singularity cannot be limited by the restricted coordinate transformations (1.3). Then, we may say that the spacetime is singular. It is exactly in this vein that we study singularities in the HL theory. In particular, we first generalize the definitions of scalar, non-scalar and coordinate singularities in GR to the HL theory in Sec. II, and then in Sec. III we study two representative families of spherical static solutions of the HL theory, and identify scalar curvature singularities using the three quantities  $K$ ,  $K_{ij}K^{ij}$  and  $R$ . In Sec. IV, we present our main conclusions and remarks. There is also an Appendix, in which we show explicitly that the second class of the LMP solutions written in the ADM frame with projectability condition in general satisfy the HL equations coupled with an anisotropic fluid with heat flow.

Before proceeding further, we would like to note that black holes in GR for asymptotically-flat spacetimes are well-defined [31]. However, how to generalize such definitions to more general spacetimes is still an open question [32, 33]. The problem in the HL theory becomes more complicated [26, 34], partially because of the fact that particles in the HL theory can have non-standard dispersion relations, and therefore no uniform maximal speed exists. As a result, the notion of a horizon is observer-dependent.

## II. SINGULARITIES IN HL THEORY

In GR, there are powerful Hawking-Penrose theorems [31], from which one can see that spacetimes with quite “physically reasonable” conditions are singular. Although the theorems did not tell the nature of the singularities, Penrose’s cosmic censorship conjecture states that those formed from gravitational collapse in a “physically reasonable” situation are always covered by horizons [35].

To study further the nature of singularities in GR, Ellis and Schmidt divided them into two different kinds, *spacetime curvature singularities* and *coordinate singularities* [36]. The former is real and cannot be made disappear by any Lorentz transformations (1.10), while the latter is coordinate-dependent, and can be made disappear by proper Lorentz transformations. Spacetime curvature singularities are further divided into two sub-classes, *scalar curvature singularities* and *non-scalar curvature singularities*. If any of the scalars constructed from the 4-dimensional Riemann tensor  $R_{\mu\nu\lambda}^{\sigma}$  and its derivatives is singular, then the spacetime is said singular, and the corresponding singularity is a scalar one. If none of these scalars is singular, spacetimes can be still singular. In particular, tidal forces and/or distortions (which are the double integrals of the tidal forces), experienced by an observer, may become infinitely large [37]. These kinds of singularities are usually referred to as non-scalar cur-

vature singularities.

To generalize these definitions to the HL theory, as mentioned in the Introduction, both the extrinsic curvature  $K_{ij}$  and the 3-dimensional Riemann tensor  $R_{jkl}^i$  can be used to construct gauge-invariant quantities, as now they are all tensors under the restricted transformations (1.3). Then, we can see that now there are three kinds of scalars: one is constructed from  $K_{ij}$  and its derivatives; one is from the 3-dimensional Riemann tensor  $R_{jkl}^i$  and its derivatives; and the other is the mixture of  $K_{ij}$ ,  $R_{jkl}^i$  and their derivatives. Therefore, we may define a scalar curvature singularity in the HL theory as the one where any of the scalars constructed from  $K_{ij}$ ,  $R_{jkl}^i$  and their derivatives is singular. A non-scalar curvature singularity is the one where none of these scalars is singular, but some other physical quantities, such as tidal forces and distortions experienced by observers, become unbounded. Coordinate singularities are the ones that can be limited by the restricted coordinate transformations (1.3).

Several comments now are in order. The scalars constructed from the 3-dimensional tensors  $K_{ij}$  and  $R_{jkl}^i$  and their derivatives include all scalars constructed from the 4-dimensional  $R_{\mu\nu\lambda}^{\sigma}$  and its derivatives. Thus, according to the above definitions, all scalar singularities under the general Lorentz transformations (1.10) are also scalar singularities under the restricted transformations (1.3), but not the other way around. In this sense, scalar singularities in the HL theory are more general than those in GR. One simple example is the anti-de Sitter Schwarzschild solutions, which are also solutions of the SVW generalization with  $\xi = 0$ , as in this case the 3-dimensional Ricci tensor  $R_{ij}$  vanishes identically, and the contributions of high order derivatives of curvature to the potential  $\mathcal{L}_V$  are zero, as can be seen from Eq. (1.8). However, as shown in the next section, the corresponding two scalars  $K$  and  $K_{ij}K^{ij}$  all become singular at  $r = (3M/|\Lambda|)^{1/3}$ . This singularity is absent in GR [31].

In 3-dimensional space, the Weyl tensor vanishes identically, and the Riemann tensor is determined algebraically by the curvature scalar and the Ricci tensor:

$$R_{ijkl} = g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk} - \frac{1}{2}(g_{ik}g_{jl} - g_{il}g_{jk})R. \quad (2.1)$$

Therefore, the singular behavior of the scalars made of the 3-dimensional Riemann tensor  $R_{jkl}^i$  may well be represented by the 3-dimensional curvature scalar  $R$ .

## III. SINGULARITIES IN SPHERICAL STATIC SPACETIMES

The metric of general spherically symmetric static spacetimes that preserve the ADM form of Eq. (1.2) with the projectability condition can be cast in the form [38],

$$ds^2 = -dt^2 + e^{2\nu} (dr + e^{\mu-\nu} dt)^2 + r^2 d\Omega^2, \quad (3.1)$$

where  $\mu = \mu(r)$ ,  $\nu = \nu(r)$ . Then, we have

$$\begin{aligned} R &= \frac{2e^{-2\nu}}{r^2} \left[ 2r\nu' - (1 - e^{2\nu}) \right], \\ K &= e^{\mu-\nu} \left( \mu' + \frac{2}{r^2} \right), \\ K_{ij}K^{ij} &= e^{2(\mu-\nu)} \left( \mu'^2 + \frac{2}{r^2} \right), \end{aligned} \quad (3.2)$$

where  $\nu' \equiv d\nu/dr$ , etc. It is interesting to note that for the metric (3.1) we have

$$C_{ij} = 0 = \epsilon^{ijk} R_{il} \nabla_j R_k^l, \quad (3.3)$$

where  $C_{ij}$  is the Cotton tensor, defined as

$$C_{ij} = \epsilon^{ikl} \nabla_k \left( R_l^j - \frac{1}{4} R \delta_l^j \right). \quad (3.4)$$

As a result, the HL theory with detailed balance [1]

$$\begin{aligned} S_{HLd} &= \int dt dx^3 N \sqrt{g} \left\{ \frac{2}{\kappa^2} [K_{ij}K^{ij} - \lambda K^2] \right. \\ &\quad + \frac{\kappa^2 \mu^2 (\Lambda_W R - 3\Lambda_W^2)}{8(1-3\lambda)} + \frac{\kappa^2 \mu^2 (1-4\lambda) R^2}{32(1-3\lambda)} \\ &\quad - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} + \frac{\kappa^2 \mu}{2w^2} \epsilon^{ijk} R_{il} \nabla_j R_k^l \\ &\quad \left. - \frac{\kappa^2}{2w^2} C_{ij} C^{ij} \right\}, \end{aligned} \quad (3.5)$$

can be effectively considered as a particular case of the general SVW action (1.7) with

$$\begin{aligned} G &= \frac{\kappa^2}{32\pi c^2}, \quad c^2 = \frac{\kappa^4 \mu^2 \Lambda_W}{16(1-3\lambda)}, \quad \Lambda = \frac{3\Lambda_W}{2}, \\ g_2 &= \frac{4\lambda-1}{4\Lambda_W} \zeta^2, \quad g_3 = \frac{1-3\lambda}{\Lambda_W} \zeta^2, \\ \xi &= 1 - \lambda, \quad g_4 = g_5 = \dots = g_8 = 0. \end{aligned} \quad (3.6)$$

It should be noted that these relations are valid only when the conditions of Eq. (3.3) hold. In general, these two terms do not vanish and violate parity, while the SVW action always preserves it. It must not be confused with the parameter  $\mu$  used in the action (3.5) and the metric coefficient used in (3.1).

To study singularities in the HL theory, in the rest of this paper we shall restrict ourselves to two representative cases, the (anti-) de Sitter Schwarzschild solutions and the solutions found by Lu, Mei and Pope (LMP) [6].

### A. (Anti-) de Sitter Schwarzschild Solutions

The (anti-) de Sitter Schwarzschild solutions are given by [38]

$$\mu = \frac{1}{2} \ln \left( \frac{M}{r} + \frac{\Lambda}{3} r^2 \right), \quad \nu = 0. \quad (3.7)$$

When  $\Lambda > 0$ , it represents the de Sitter Schwarzschild solutions, and when  $\Lambda < 0$  it represents the anti-de Sitter Schwarzschild solutions. As mentioned previously, they are also solutions of the SVW generalization with  $\xi = 0$ . Inserting the above into Eq.(3.2), we find that

$$\begin{aligned} R &= 0, \\ K &= \left( \frac{3M + \Lambda r^3}{12r^3} \right)^{1/2} \left( \frac{4}{r} - \frac{3M - 2\Lambda r^3}{3M + \Lambda r^3} \right), \\ K_{ij}K^{ij} &= \frac{3M + \Lambda r^3}{12r^3} \left[ 8 + \left( \frac{3M - 2\Lambda r^3}{3M + \Lambda r^3} \right)^2 \right]. \end{aligned} \quad (3.8)$$

Clearly, when  $\Lambda \geq 0$ ,  $K$  and  $K_{ij}K^{ij}$  are singular only at the center  $r = 0$ . However, when  $\Lambda < 0$ , they are also singular at  $r = r_\Lambda \equiv (3M/|\Lambda|)^{1/3}$ . In contrast to GR, this singularity is a scalar one, and cannot be removed by any coordinate transformations given by Eq. (1.3).

In GR, the (anti-) de Sitter Schwarzschild solutions are usually given in the orthogonal gauge,

$$ds^2 = -e^{2\Psi(r)} d\tau^2 + e^{2\Phi(r)} dr^2 + r^2 d\Omega^2, \quad (3.9)$$

with

$$\Psi = -\Phi = \frac{1}{2} \ln \left( 1 - \frac{2M}{r} + \frac{1}{3} \Lambda r^2 \right). \quad (3.10)$$

Clearly, the metric (3.9) does not satisfy the projectability condition, and its coefficient  $g_{rr}$  is singular at  $e^{2\Phi(r_{EH})} = 0$ . But, in GR this is a coordinate singularity, and all scalars made of the 4-dimensional Riemann tensor and its derivatives are finite at  $r = r_{EH}$ .

It is interesting to note that in the orthogonal gauge (3.9),  $K$  and  $K_{ij}K^{ij}$  all vanish, as can be seen from Eqs. (1.9), while the 3-dimensional Ricci scalar is given by  $R_{orth} = -2\Lambda$ , where  $R_{orth}$  denotes the quantity calculated in the orthogonal gauge (3.9).

In [38], it was showed explicitly that metric (3.9) is related to metric (3.1) by the coordinate transformations,

$$\tau = t - \int^r \sqrt{e^{-2\Psi} - 1} e^\Phi dr, \quad (3.11)$$

under which we have

$$\begin{aligned} \Phi(r) &= \nu(r) - \frac{1}{2} \ln \left( 1 - e^{2\mu} \right), \\ \Psi(r) &= \frac{1}{2} \ln \left( 1 - e^{2\mu} \right), \end{aligned} \quad (3.12)$$

or inversely,

$$\mu = \frac{1}{2} \ln \left( 1 - e^{2\Psi} \right), \quad \nu = \Phi(r) + \Psi(r). \quad (3.13)$$

Note that the coordinate transformations (3.11) are not allowed by the foliation-preserving diffeomorphisms (1.3). In addition, since  $K$ ,  $K_{ij}K^{ij}$  and  $R$  are not scalars under these transformations, it explains why we have completely different physical interpretations in the two

different gauges, defined, respectively, by Eqs. (3.1) and (3.11). In fact, in the framework of the HL theory the two different gauges represent two different theories - metric (3.1) represents a HL theory with projectability condition, while metric (3.11) represents a HL theory without projectability condition.

### B. The LMP Solutions

The LMP solutions were originally found for the HL theory with detailed balance but without projectability conditions in the form (3.9). There are two classes of solutions, given, respectively, by

$$\Phi = -\frac{1}{2} \ln(1+x^2), \quad (3.14)$$

for any  $\Psi(r)$ , and

$$\begin{aligned} \Phi &= -\frac{1}{2} \ln(1+x^2 - \alpha x^{\alpha_{\pm}}), \\ \Psi &= -\beta_{\pm} \ln(x) + \frac{1}{2} \ln(1+x^2 - \alpha x^{\alpha_{\pm}}), \end{aligned} \quad (3.15)$$

where  $x \equiv \sqrt{-\Lambda_W} r$ , and

$$\begin{aligned} \alpha_- &\equiv \frac{2\lambda - \sqrt{6\lambda - 2}}{\lambda - 1} = \frac{2(2\lambda - 1)}{2\lambda + \sqrt{6\lambda - 2}}, \\ \alpha_+ &\equiv \frac{2\lambda + \sqrt{6\lambda - 2}}{\lambda - 1}, \quad \beta_{\pm} \equiv 2\alpha_{\pm} - 1, \end{aligned} \quad (3.16)$$

with  $\alpha$  being an arbitrary real constant. Fig. 1 schematically shows the curves of  $\alpha_{\pm}$  vs  $\lambda$ , which shows that there is no solution for the  $\alpha_+$  branch when  $\lambda = 1$ .

Solutions given by Eqs. (3.14) and (3.15) shall be referred to, respectively, as Class A and B solutions. Particular values of  $\alpha_{\pm}$  are

$$\begin{aligned} \alpha_-(\lambda) &= \begin{cases} -1, & \lambda = 1/3, \\ 0, & \lambda = 1/2, \\ 1/2, & \lambda = 1, \\ 2/3, & \lambda \simeq 1.375, \\ 1, & \lambda = 3, \\ 2, & \lambda = \infty, \end{cases} \\ \alpha_+(\lambda) &= \begin{cases} -1, & \lambda = 1/3, \\ -\infty, & \lambda = 1^{-0}, \\ +\infty, & \lambda = 1^{+0}, \\ 2, & \lambda = \infty, \end{cases} \end{aligned} \quad (3.17)$$

which are useful in the following discussions.

In the orthogonal gauge (3.9), as mentioned previously, the extrinsic curvature  $K_{ij}$  vanishes identically, while the 3-dimensional curvature for the above two classes of solutions can be written as

$$R_{orth} = \frac{2}{r^2} \left\{ \alpha(1 + \alpha_{\pm})x^{\alpha_{\pm}} - 3x^2 \right\}. \quad (3.18)$$

When  $\alpha = 0$ , the corresponding  $R_{orth}$  is for the solution (3.14), which reduces to a constant. That is, in this case

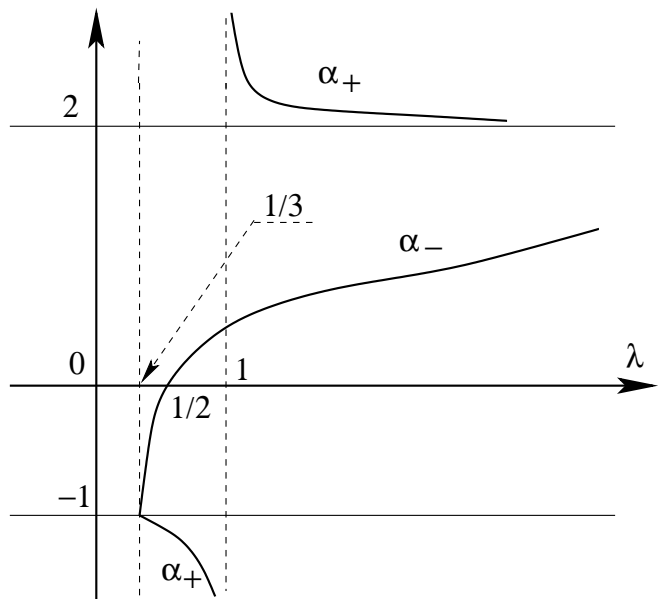


FIG. 1: The functions  $\alpha_{\pm}(\lambda)$  defined by Eq. (3.16).

all the three scalars  $K$ ,  $K_{ij}K^{ij}$  and  $R$  are finite. When  $\alpha \neq 0$ , it is for the solution (3.15), which is singular only when  $\alpha_{\pm} \leq 2$  at  $r = 0$ . From Fig.1 we can see that  $\alpha_-$  is always less than two for a finite  $\lambda$ . Therefore, this branch of solutions is always singular at the center.  $\alpha_+$  is always less than two for  $1/3 \leq \lambda < 1$  and greater than two for  $\lambda > 1$ . Therefore,  $R_{orth}$  is finite for the  $\alpha_+$  solutions with  $\lambda > 1$  for any  $r$ , including the center  $r = 0$ , while it is singular at the center for the  $\alpha_+$  solutions with  $1/3 \leq \lambda < 1$ .

#### 1. Class A Solutions

Transforming the above solutions into the canonical ADM form (3.1), we find that for the solution (3.14), we have

$$\mu = -\infty, \quad \nu = -\frac{1}{2} \ln(1 - \Lambda_W r^2), \quad (3.19)$$

where in writing the above expressions, we had chosen  $\Psi = 0$ . Then, the corresponding three scalars are given by

$$K = K_{ij}K^{ij} = 0, \quad R = 6\Lambda_W, \quad (3.20)$$

which are all finite. As shown in [38], this solution is also a vacuum solution of the SVW generalization (1.7) with a non-vanishing constant curvature  $k = 6\Lambda_W$ . Since it does not depend explicitly on the coupling constant  $\xi$ , it is a vacuum solution for any given  $\xi$ , including  $\xi = 0$ . For detail, we refer readers to Sec. V of [38].

## 2. Class B Solutions

For Class B solutions (3.15), they can be written in the canonical ADM form (3.1) with

$$\mu = \frac{1}{2} \ln \Delta, \quad \nu = (1 - 2\alpha_{\pm}) \ln(x), \quad (3.21)$$

where

$$\Delta \equiv 1 - x^{2(1-2\alpha_{\pm})} - x^{4(1-\alpha_{\pm})} + \alpha x^{2-3\alpha_{\pm}}. \quad (3.22)$$

Clearly, to have real solutions, we must assume  $\Delta \geq 0$ . It should be noted that unlike Class A solutions, this class of solutions do not satisfy the vacuum equations of the HL theory with projectability condition, due to the fact that the HL actions are not invariant under the coordinate transformations (3.11). As shown in Appendix, they can be interpreted as solutions of the HL theory with projectability condition coupled with a spherical anisotropic fluid with heat flow. The properties of singularities of these quantities can be well represented by the three scalars  $K$ ,  $K_{ij}K^{ij}$  and  $R$ , as can be seen from Appendix, so in the following we shall not consider them specifically.

Inserting the above solutions into Eq. (3.2) we find that

$$\begin{aligned} R &= \frac{2}{r^2} \left\{ 1 - (1 + 2\beta_{\pm}) x^{2\beta_{\pm}} \right\}, \\ K &= \frac{\sqrt{-\Lambda_W}}{2\Delta^{1/2} x^{3-2\alpha_{\pm}}} \left( 4\sqrt{-\Lambda_W} \Delta - x^2 \delta \right), \\ K_{ij}K^{ij} &= -\frac{\Lambda_W}{4\Delta x^{4(1-\alpha_{\pm})}} (8\Delta - x^2 \delta^2), \end{aligned} \quad (3.23)$$

where

$$\begin{aligned} \delta &\equiv 2(1 - 2\alpha_{\pm}) x^{1-4\alpha_{\pm}} + 4(1 - \alpha_{\pm}) x^{3-4\alpha_{\pm}} \\ &\quad - \alpha(2 - 3\alpha_{\pm}) x^{1-3\alpha_{\pm}}. \end{aligned} \quad (3.24)$$

To study these solutions further, it is found convenient to consider the two branches  $\alpha_{\pm}$  separately. We shall use  $\beta = \beta_{\pm}$  to denote the  $\alpha_{\pm}$  branches.

**Case i)**  $\beta = \beta_+$ ,  $1/3 \leq \lambda < 1$ : In this case we have  $\alpha_+ \leq -1$  and  $\beta_+ \leq -3$ . Then, from the expression (3.22) we find that

$$\Delta = \begin{cases} 1, & x = 0, \\ -\infty, & x \rightarrow \infty. \end{cases} \quad (3.25)$$

Thus, there must exist a point  $x = x_s$  at which we have  $\Delta(x_s) = 0$ . Since  $\Delta \geq 0$ , we must restrict the solutions to the range  $0 \leq x \leq x_s$  (or equivalently,  $0 \leq r \leq r_s$ , where  $r_s \equiv x_s/\sqrt{-\Lambda_W}$ ). Fig. 2 shows this case. From the expressions (3.23) we find that all these three scalars diverge at the center  $r = 0$ , while  $K$  and  $K_{ij}K^{ij}$  diverge at  $r = r_s$ . Therefore, in the present case there are two scalar curvature singularities, located, respectively, at  $r = 0$  and  $r = r_s$ .

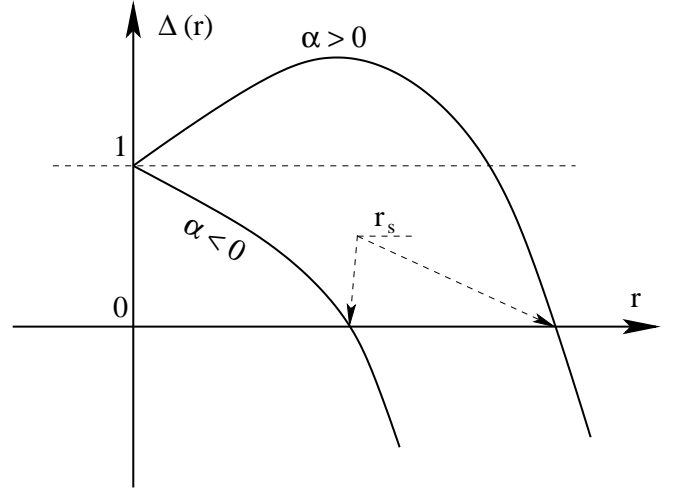


FIG. 2: The function  $\Delta(x)$  defined by Eq. (3.22) for the  $\alpha_+$  branch with  $1/3 \leq \lambda < 1$ . All three scalars  $R$ ,  $K$  and  $K_{ij}K^{ij}$ , defined by Eq. (3.23), become unbounded at  $r = 0$ , while only  $K$  and  $K_{ij}K^{ij}$  diverge at  $r = r_s$ .

**Case ii)**  $\beta = \beta_+$ ,  $\lambda > 1$ : In this case we have  $\alpha_+ \geq 2$  and  $\beta_+ \geq 3$ , where the equality holds only when  $\lambda = \infty$ . Then, from the expression (3.22) we find that

$$\Delta = \begin{cases} -\infty, & x = 0, \\ +1, & x \rightarrow \infty. \end{cases} \quad (3.26)$$

Fig.3 shows this case, from which we find that for any given  $\lambda > 1$ , there always exists a minimum  $r_s > 0$  so that  $\Delta(r) \geq 0$  for  $r \geq r_s$ , where  $r_s$  is the solution of  $\Delta(r) = 0$ . Then, from the expressions (3.23) we find that  $K$  and  $K_{ij}K^{ij}$  diverge at  $r = r_s$ , while  $R$  diverges as  $r \rightarrow \infty$ . Therefore, in the present case there are also two scalar curvature singularities, located, respectively, at  $r = r_s$  and  $r = \infty$ .

**Case iii)**  $\beta = \beta_-$ ,  $1/3 \leq \lambda < 1$ : In this case we have  $-1 \leq \alpha_- < 1/2$  and  $-3 \leq \beta_- < 0$ , where the equality holds only for  $\lambda = 1/3$ . Then, from the expression (3.22) we find that

$$\Delta = \begin{cases} +1, & x = 0, \\ -\infty, & x \rightarrow \infty. \end{cases} \quad (3.27)$$

from Fig.4, we find that for any given  $\lambda$  in this range, there always exists a maximum  $r_s > 0$  so that  $\Delta(r) \geq 0$  for  $0 \leq r \leq r_s$ . Then, From the expressions (3.23) we find that all three scalars  $R$ ,  $K$  and  $K_{ij}K^{ij}$  become unbounded at the center, while only  $K$  and  $K_{ij}K^{ij}$  diverge at  $r = r_s$ . Therefore, in the present case there are two scalar curvature singularities, located, respectively, at  $r = 0$  and  $r = r_s$ .

**Case iv)**  $\beta = \beta_- = 0$ ,  $\lambda = 1$ : In this case we have  $\alpha_- = 1/2$ , and

$$\Delta = \alpha x^{1/2} - x^2. \quad (3.28)$$

Thus, to have  $\Delta$  non-negative, we must assume  $\alpha > 0$ . Then,  $\Delta \geq 0$  for  $0 \leq r \leq r_s \equiv \alpha^{2/3}$ , where  $\Delta(x) = 0$  at

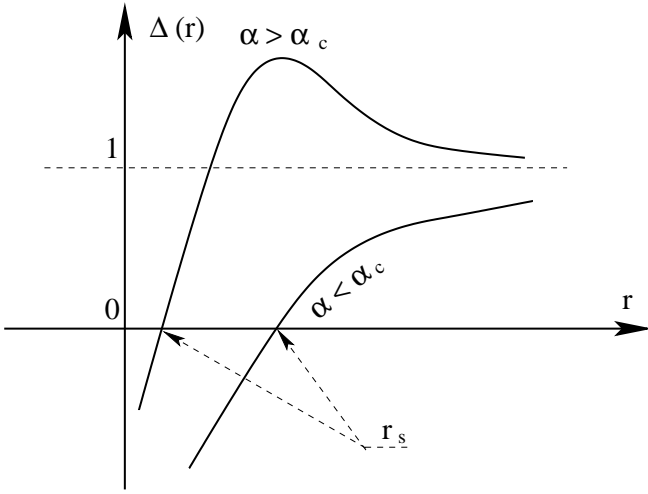


FIG. 3: The function  $\Delta(x)$  defined by Eq. (3.22) for the  $\alpha_+$  branch with  $\lambda > 1$ , where  $\alpha_c$  is the value of  $\alpha$ , for which  $\Delta'(x) > 0$  for  $\alpha < \alpha_c$ .  $K$  and  $K_{ij}K^{ij}$  diverge at  $r = r_s$ , while  $R$  diverges as  $r \rightarrow \infty$ .

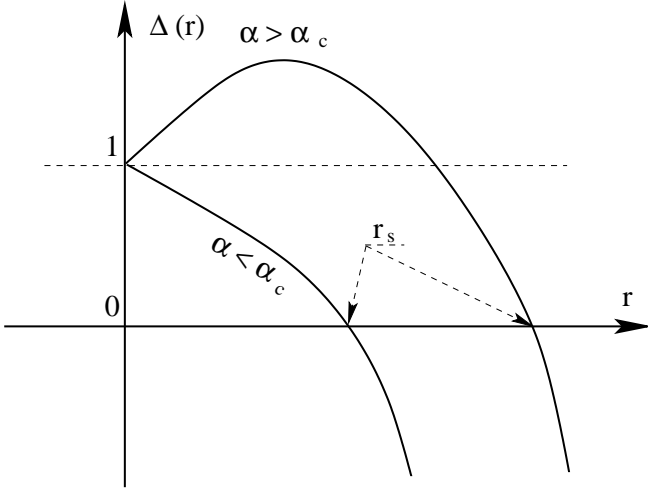


FIG. 4: The function  $\Delta(x)$  defined by Eq. (3.22) for the  $\alpha_-$  branch with  $1/3 \leq \lambda < 1$ , where  $\alpha_c$  is the value of  $\alpha$ , for which  $\Delta'(x) < 0$  for  $\alpha < \alpha_c$ .  $R$ ,  $K$  and  $K_{ij}K^{ij}$  diverge at the center  $r = 0$ , while only  $K$  and  $K_{ij}K^{ij}$  diverges at  $r = r_s$ .

both  $r = 0$  and  $r = r_s$ . At the center, all three scalars  $R$ ,  $K$  and  $K_{ij}K^{ij}$  become unbounded, while only  $K$  and  $K_{ij}K^{ij}$  diverge at  $r = r_s$ .

**Case v)**  $\beta = \beta_-$ ,  $1 < \lambda < 3$ : In this case we have  $1/2 \leq \alpha_- < 1$ ,  $0 \leq \beta_- < 1$ , and

$$\Delta = \begin{cases} -\infty, & x = 0, \\ -\infty, & x \rightarrow \infty. \end{cases} \quad (3.29)$$

Fig.5 shows the general properties of  $\Delta(r)$ , from which we can see that for any given  $\lambda$ , there always exists a critical value  $\alpha_c$ , for which  $\Delta(r) \geq 0$  is possible only when  $\alpha > \alpha_c$ . In the latter case,  $\Delta(r) = 0$  always has two positive roots, say,  $r_1$  and  $r_2$ . Without loss of generality, we assume  $r_2 > r_1 > 0$ . When  $r_1 \leq r \leq r_2$ ,  $\Delta(r)$

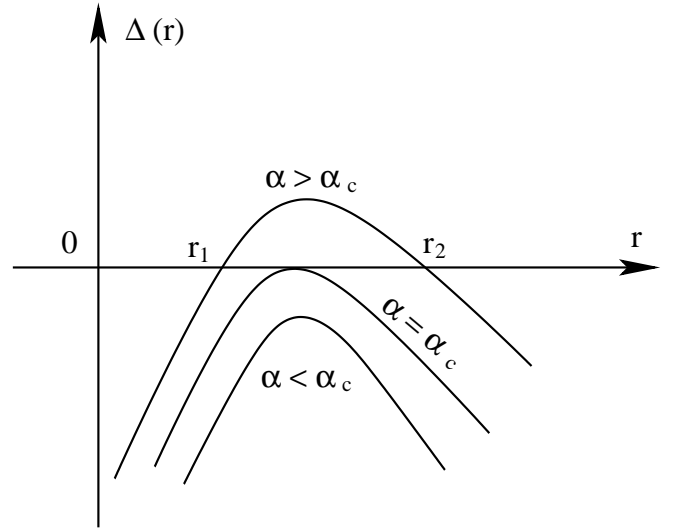


FIG. 5: The function  $\Delta(x)$  defined by Eq. (3.22) for the  $\alpha_-$  branch with  $1 < \lambda < 3$ , where  $\alpha_c$  is the value of  $\alpha$ , for which  $\Delta(x) \geq 0$  is possible only when  $\alpha > \alpha_c$ .  $K$  and  $K_{ij}K^{ij}$  diverge at the two points  $r = r_{1,2}$ , while  $R$  remains finite there.

is non-negative. At the two points  $r = r_1$  and  $r_2$  we have  $\Delta(r_i) = 0$ , and Eq. (3.23) shows that both  $K$  and  $K_{ij}K^{ij}$  become unbounded at these points, while  $R$  remains finite. Therefore, in the present case there are also two scalar curvature singularities, located, respectively, at  $r = r_1$  and  $r = r_2$  for  $\alpha > \alpha_c$ . Solutions with  $\alpha \leq \alpha_c$  are not physical.

**Case vi)**  $\beta = \beta_- = 1$ ,  $\lambda = 3$ : In this case we have  $\alpha_- = \beta_- = 1$ , and

$$\Delta = \frac{\alpha}{x} - \frac{1}{x^2} = \begin{cases} -\infty, & x = 0, \\ 0, & x \rightarrow \infty. \end{cases} \quad (3.30)$$

Clearly, to have  $\Delta \geq 0$ , we must assume  $\alpha > 0$ . Then, for  $r \geq r_s \equiv 1/(\sqrt{-\Lambda_W} \alpha)$ ,  $\Delta$  is non-negative [See Fig.6, Curve (a)]. At  $r = r_s$ ,  $\Delta(r) = 0$ , and Eq. (3.23) shows that both  $K$  and  $K_{ij}K^{ij}$  become unbounded at this point, while  $R$  remains finite. As  $r \rightarrow \infty$ ,  $\Delta(r) \rightarrow 0$ , and  $K$  and  $K_{ij}K^{ij}$  become unbounded again, while  $R$  still remains finite. Therefore, in the present case there are also two scalar curvature singularities, located, respectively, at  $r = r_s$  and  $r = \infty$ .

**Case vii)**  $\beta = \beta_-$ ,  $\lambda > 3$ : In this case we have  $\alpha_- > 1$ ,  $\beta_- > 1$ , and

$$\Delta = \begin{cases} -\infty, & x = 0, \\ +1, & x \rightarrow \infty. \end{cases} \quad (3.31)$$

Fig.6(b) shows the general properties of  $\Delta(r)$ , from which we can see that for any given  $\lambda$ , there always exists a point  $r = r_s$  where  $\Delta(r_s) = 0$ . When  $r > r_s$ ,  $\Delta$  is always positive. Eq. (3.23) shows that now both  $K$  and  $K_{ij}K^{ij}$  diverge at  $r = r_s$ , while all three scalars become unbounded as  $r \rightarrow \infty$ . Thus, in the present case there are two scalar curvature singularities, located, respectively, at  $r = r_s$  and  $r = \infty$ .

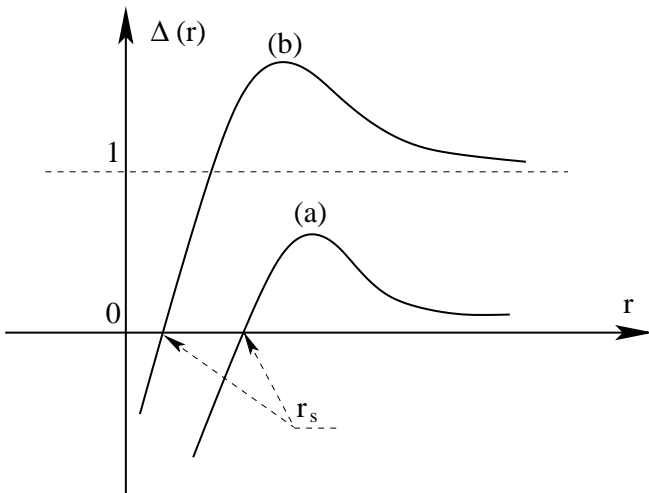


FIG. 6: The function  $\Delta(x)$  defined by Eq. (3.22) for the  $\alpha$ -branch with (a)  $\lambda = 3$ ,  $\alpha > 0$ ; and (b)  $\lambda > 3$  for any  $\alpha$ .

#### IV. CONCLUSIONS

In this paper, we have studied singularities in the HL theory, and classified them into three different kinds, the scalar, non-scalar, and coordinate singularities, following the classification given in GR [36]. Due to the restricted diffeomorphisms (1.3), the number of the scalars that can be constructed from the extrinsic curvature tensor  $K_{ij}$ , the 3-dimensional Riemann tensor  $R_{jkl}^i$  and their derivatives is much larger than that constructed from the 4-dimensional Riemann tensor  $R_{\mu\nu\lambda}^\sigma$  and its derivatives. The latter is invariant under the general Lorentz transformations (1.10). As a result, even for the same spacetime, it may be singular in the HL theory, but not singular in GR. One simple example is the anti-de Sitter Schwarzschild solution written in the ADM form (3.1). This solution is a solution of both HL theory and GR. However, in the HL theory, there are two scalar singularities located, respectively, at the origin  $r = 0$  and  $r = (3M/|\Lambda|)^{1/3}$ , as the two scalars  $K$  and  $K_{ij}K^{ij}$  become singular at these points. It is well-known that the scalar singularity at  $r = (3M/|\Lambda|)^{1/3}$  is absent in GR. This is because both  $K$  and  $K_{ij}K^{ij}$  are not scalars under the general Lorentz transformations (1.10). Thus, even they are singular at this point, it only represents a coordinate singularity, and all scalars constructed from the 4-dimensional Riemann tensor and its derivatives are finite. On the other hand, due to the restricted transformations (1.3),  $K$  and  $K_{ij}K^{ij}$  are scalars in the HL theory, and once they are singular, the resulting singularity cannot be transferred away by the restricted transformations. As a result, in the HL theory it represents a real spacetime singularity.

With the above in mind, we have studied the LMP solutions [6], and found that their singularity behavior in the orthogonal frame defined by (3.9) is different from that in the ADM frame defined by (3.1) for the second

class of the LMP solutions. In particular, in the orthogonal frame,  $K_{ij}$  vanishes, so do  $K$  and  $K_{ij}K^{ij}$ , while the 3-dimensional Ricci scalar  $R$  [cf. Eq. (3.18)] can be singular at the origin or infinity, depending on the choice of the parameter  $\lambda$ . However, in the ADM frame at least one of the three scalars  $K$ ,  $K_{ij}K^{ij}$  and  $R$  is always singular at two different points, either  $r = 0$  and  $r = r_s > 0$ , or  $r = r_1$  and  $r = r_2$  with  $r_2 > r_1 > 0$ , or  $r = r_s > 0$  and  $r = \infty$ , depending on the choice of the free parameter  $\lambda$ , where  $r_s$  is a finite non-zero positive constant. This different singular behavior originates from the fact that the two frames are related by the coordinate transformations (3.11), which is not allowed by the foliation-preserving diffeomorphisms (1.3), or in other words,  $K$ ,  $K_{ij}K^{ij}$  and  $R$  are not scalars under such transformations. In fact, in the framework of the HL theory, the two frames actually represent two different HL theories, one is with the projectability condition, while the other is without. In particular, the second class of the LMP solutions in the orthogonal frame satisfy the vacuum HL equations, while in the ADM frame they satisfy the HL equations coupled with an anisotropic fluid with heat flow, as shown explicitly in the Appendix.

Our above results show clearly that the problem of singularities in the HL theory is a very delicate problem, due to the restricted diffeomorphisms (1.3), which preserve the ADM foliations (1.2). Further investigations are needed, in particular, in terms of the strength of these singularities. In the examples studied in this paper, all singularities indicated by the two scalars  $K$  and  $K_{ij}K^{ij}$  at  $r = r_s > 0$ , including that of the anti-de Sitter Schwarzschild solution, seems weak in the sense of tidal forces and distortions experiencing by observers. Therefore, it is not clear whether or not the spacetime is extendable across such a singularity [37].

Finally, we would like to note that the ADM form (3.1) can be considered as a particular case of the HL theory without projectability condition. Therefore, restricting ourselves only to the HL theory without projectability condition does not solve the singularity problem occurring at  $r = r_s$ .

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#### Appendix: The Generalized LMP Solutions with Projectability Condition

The second class of the LMP solutions with projectability condition takes the form of Eq. (3.1) with  $\mu$  and  $\nu$  given by Eq. (3.21). Unlike the first class, this one does not satisfy the HL vacuum equations with projectability condition, due to the fact that HL actions



are not invariant under the coordinate transformations (3.11). In particular, under these transformations, we have

$$R_{ij} \rightarrow R_{ij} + \delta R_{ij}. \quad (\text{A.1})$$

For spherically symmetric static solutions, the extra term  $\delta R_{ij}$  in general gives rise to an anisotropic fluid with heat flow [38], possibly subjected to some energy conditions [31]. In particular, for the LMP solutions of Eqs. (3.1) and (3.21), the energy density  $J^t$ , defined by

$$J^t = 2 \left( N \frac{\delta \mathcal{L}_M}{\delta N} + \mathcal{L}_M \right), \quad (\text{A.2})$$

is given by the Hamiltonian equation,

$$\int d^3x \sqrt{g} (\mathcal{L}_K + \mathcal{L}_V) = 8\pi G \int d^3x \sqrt{g} J^t, \quad (\text{A.3})$$

where

$$\begin{aligned} \mathcal{L}_K &= -\frac{\Lambda_W e^{2(\mu-\nu)}}{4x^2 \Delta} \left\{ \xi x^2 \Delta'^2 - 8(1-\xi)x\Delta\Delta' \right. \\ &\quad \left. - 8(1-2\xi)x\Delta^2 \right\}, \\ \mathcal{L}_V &= \frac{\Lambda_W}{x^2} \left[ 2 + 3x^2 + 2(1-4\alpha_\pm)x^{2\beta_\pm} \right] \\ &\quad + \frac{\Lambda_W}{x^{8(1-\alpha_\pm)}} \left\{ [8\xi\alpha_\pm^2 - 8(1-\xi)(1-\alpha_\pm) + 3] \right. \\ &\quad \left. + 2[4(1-\xi)(1-\alpha_\pm) - 1]x^{2(1-2\alpha_\pm)} \right. \\ &\quad \left. + (1-2\xi)x^{4(1-2\alpha_\pm)} \right\}, \end{aligned} \quad (\text{A.4})$$

where  $\Delta$  is given by Eq. (3.22), and  $\Delta' \equiv d\Delta/dx$ . The quantity  $v$ , defined by

$$J^i \equiv -N \frac{\delta \mathcal{L}_M}{\delta N_i} = e^{-(\mu+\nu)}(v, 0, 0), \quad (\text{A.5})$$

which is related to heat flow [38], is given by

$$\begin{aligned} v &= \frac{\Lambda_W e^{2(\mu-\nu)}}{32\pi G x^2 \Delta^2} \left\{ \xi x \left[ 2x\Delta\Delta'' + x\Delta'^2 - 2\beta_\pm\Delta\Delta' \right] \right. \\ &\quad \left. + 4\Delta \left[ \xi x\Delta' - 2(1-\xi)\beta_\pm\Delta \right] - 8\xi\Delta^2 \right\}. \end{aligned} \quad (\text{A.6})$$

On the other hand, the corresponding stress part  $\tau^{ij}$  defined by,

$$\tau^{ij} = \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L}_M)}{\delta g_{ij}}, \quad (\text{A.7})$$

can be written in the form,

$$\tau_{ij} = e^{2\nu} p_r \delta_r^i \delta_j^r + r^2 p_\theta \Omega_{ij}, \quad (\text{A.8})$$

where  $\Omega_{ij} \equiv \delta_i^\theta \delta_j^\theta + \sin^2 \theta \delta_i^\phi \delta_j^\phi$ , and

$$\begin{aligned} p_r &= \frac{\Lambda_W e^{2(\mu-\nu)}}{64\pi G x^2 \Delta^2} \left\{ \xi x \left[ 4x\Delta\Delta'' - 3x\Delta'^2 + 4\beta_\pm\Delta\Delta' \right] \right. \\ &\quad \left. + 8\Delta \left[ x\Delta' - 2(1-\xi)\beta_\pm\Delta \right] + 8(1-4\xi)\Delta^2 \right\} \\ &\quad - \frac{e^{-2\nu}}{8\pi G} F_{rr}, \\ p_\theta &= \frac{\Lambda_W e^{2(\mu-\nu)}}{16\pi G x^2 \Delta^2} \left\{ (1-\xi)x^2\Delta\Delta'' + \frac{1}{4}\xi x^2\Delta'^2 \right. \\ &\quad \left. + (1-\xi)\beta_\pm x\Delta\Delta' + 2(1-2\xi)(x\Delta' + \beta_\pm\Delta)\Delta \right\} \\ &\quad + \frac{\Lambda_W}{8\pi G x^2} F_{\theta\theta}, \end{aligned} \quad (\text{A.9})$$

with

$$\begin{aligned} F_{rr} &= \frac{\Lambda_W}{2x^2} \left[ 2 - 2x^{-2\beta_\pm} - 3x^{2(1-\beta_\pm)} \right] \\ &\quad - \frac{\Lambda_W}{4x^{4-2\beta_\pm}} \left[ (19 - 32\xi) \right. \\ &\quad \left. + 4(3 - 5\xi)(2 - 3\beta_\pm)\beta_\pm \right. \\ &\quad \left. - 2(7 - 12\xi)x^{-2\beta_\pm} - (5 - 8\xi)x^{-4\beta_\pm} \right], \\ F_{\theta\theta} &= \frac{3}{2}x^2 - \beta_\pm x^{2\beta_\pm} + \frac{1}{2x^{2-4\beta_\pm}} \left\{ (1 + 2\xi) \right. \\ &\quad \left. + 2(11 - 16\xi)\beta_\pm - 2\xi(7 - 6\beta_\pm)\beta_\pm^2 \right. \\ &\quad \left. - 2(1 - \beta_\pm)x^{-2\beta_\pm} + (1 - 2\xi)x^{-4\beta_\pm} \right\}. \end{aligned} \quad (\text{A.10})$$

Clearly, to interpret the above as an anisotropic fluid with heat flow, some energy conditions [31] might need to be imposed. Since in this paper we are mainly concerned with the nature of singularities, we shall not study these conditions here.

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