

# COSMIC PERTURBATIONS WITH RUNNING $G$ AND $\Lambda$

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*Abstract.* Cosmologies with running cosmological term  $\rho_\Lambda$  and gravitational Newton's coupling  $G$  may naturally be expected if the evolution of the universe can ultimately be derived from the first principles of Quantum Field Theory or String Theory. For example, if matter is conserved and the vacuum energy density varies quadratically with the expansion rate as  $\rho_\Lambda(H) = n_0 + n_2 H^2$ , with  $n_0 \neq 0$  (a possibility that has been advocated in the literature within the QFT framework), it can be shown that  $G$  must vary logarithmically (hence very slowly) with  $H$ . In this paper, we derive the general cosmological perturbation equations for models with variable  $G$  and  $\rho_\Lambda$  in which the fluctuations  $\delta G$  and  $\delta \rho_\Lambda$  are explicitly included. We demonstrate that, if matter is covariantly conserved, the late growth of matter density perturbations is independent of the wavenumber  $k$ . Furthermore, if  $\rho_\Lambda$  is negligible at high redshifts and  $G$  varies slowly, we find that these cosmologies produce a matter power spectrum with the same shape as that of the  $\Lambda$ CDM model, thus predicting the same basic features on structure formation. Despite this shape indistinguishability, the free parameters of the variable  $G$  and  $\rho_\Lambda$  models can still be effectively constrained from the observational bounds on the spectrum amplitude.

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# 1 Introduction

The current “standard model” (or “concordance model”) of our universe, being an homogeneous and isotropic FLRW cosmological model, consists of a remarkably small number of ingredients, to wit: matter, radiation and a cosmological constant (CC) term,  $\Lambda$ . The first two ingredients are dynamical and vary (decrease fast) with the cosmic time  $t$  whereas the third,  $\Lambda$ , remains strictly constant. After many years of theoretical insight (cf. e.g. the reviews [1,2] and references therein), the situation of the original FLRW cosmological models has not changed much, in the sense that we have not been able to make any fundamental advance in the comprehension of the relationship between the matter energy density and the CC. Still, we have performed a major accomplishment at the phenomenological level by simultaneously fitting the modern independent data sets emerging from LSS galaxy surveys, supernova luminosities and the cosmic microwave background (CMB) anisotropies [3–7]. On the basis of this successful fit, in which  $\Lambda$  enters as a free parameter, one claims that a non-vanishing and positive cosmological constant has been measured. Nevertheless, we still don’t know what is the true meaning of the fitted parameter  $\Lambda$ . Assuming that Newton’s gravitational coupling  $G$  is strictly constant, the combined set of observational data determine the value

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \simeq (2.3 \times 10^{-3} \text{ eV})^4, \quad (1)$$

which we interpret as the vacuum energy density. It corresponds to  $\Omega_\Lambda^0 = \rho_\Lambda/\rho_c^0 \simeq 0.7$  when the  $\rho_\Lambda$  density is normalized with respect to the current critical density  $\rho_c^0 = (3\sqrt{h} \times 10^{-3} \text{ eV})^4$  – for a reduced Hubble constant value of  $h \simeq 0.70$ . Assuming (in the light of the same observational data) that the universe is spatially flat, this means that the current matter density normalized to the critical density is  $\Omega_m^0 = \rho_m^0/\rho_c^0 \simeq 0.3$ .

All the efforts to deduce the value of the energy density (1) – a very small one for all particle physics standards, except if a very light neutrino mass is the only particle involved [8] – have failed up to now. The main stumbling block to a solution is the fact that any approach based on the fundamental principles of quantum field theory (QFT) or string theory lead to some explicit or implicit form of severe fine-tuning among the parameters of the theory. The reason for this is that these theoretical descriptions are plagued by large hierarchies of energy scales associated to the existence of many possible vacua.

This difficulty became clear already from the first attempts to treat the dark energy component as a dynamical scalar field [9]. The idea was to let such a field select automatically the vacuum state in a dynamical way, especially one with zero value of the energy density. More recently, this approach took the popular form of a “quintessence” field slowly rolling down its potential and has adopted many different faces [10] – for a review, see [2]. But the situation at present is even harder because the quintessence field – or, more generally, a dark energy (DE) field – should be able to naturally choose not zero but the very small number (1) as its ground state.

Despite the fact that a working dynamical mechanism able to choose the correct vacuum state has to be found yet, another important motivation for the quintessence models is that they aim at explaining the puzzling coincidence between the present value of  $\rho_\Lambda$  and the value of the matter density,  $\rho_m^0$ . In other words, why  $\Omega_\Lambda^0/\Omega_m^0 = \mathcal{O}(1)$ ? Arguably, a dynamical DE should be a starting

point to understand this puzzle. Detailed analyses of these models exist in the literature, including their confrontation with the data [11–14].

On the other hand, the possibility that the cosmological term is a running quantity, which could be sensitive to the quantum matter effects, seems a more appealing ansatz, as it could provide an interface between QFT and cosmology [8, 15]. This fundamental possibility has been recently emphasized in [16]– for a review, see [17]. Actually, the analysis of the various observational data show (see Ref. [18, 19]) that a wide class of dynamical CC models are indeed able to fit the combined observations to a level of accuracy comparable to the standard  $\Lambda$ CDM model. In some cases, the dynamical nature of  $\Lambda$  allows these models to provide a clue to the coincidence problem [20, 21], and maybe eventually to the full cosmological constant problem [22].

In general, in this kind of dynamical CC scenarios we have  $\Lambda = \Lambda(\xi)$  where  $\xi = \xi(t)$  is a cosmological variable that evolves with time, and therefore ultimately  $\Lambda = \Lambda(t)$  is also time varying. Originally, these models were purely phenomenological, with no relation to QFT [23]. Nowadays we know that not all models of this kind are allowed, and the fact that the observational data can discriminate which of them are good candidates and which are not so good gives some insight into the function  $\xi = \xi(t)$  [18]. A particularly interesting class of variable CC models are those in which the gravitational coupling  $G$  changes very slowly (logarithmically) with the expansion of the universe [24, 25]. This scenario is possible e.g. if matter is covariantly conserved.

In practice,  $\xi$  could be the expansion rate  $H = H(t)$ , the scale factor  $a = a(t)$ , the matter energy density, etc. In this paper, we shall assume that  $\xi = H$ , similarly as it was done for models with a variable  $\Lambda$  interacting with matter [15, 26, 27]. With this ansatz, we shall study the impact on the structure formation, showing that these models predict the same shape for the matter power spectrum as the  $\Lambda$ CDM model, a fact that would not apply e.g. for variable CC models in which the CC decays into matter. We find this feature remarkable and we shall explore its possible consequences and also some possible phenomenological tests of this kind of models.

The paper is organized as follows. In the next section, we classify the possible scenarios with variable  $\rho_\Lambda$  and  $G$ . In section 3, we present the general set of coupled perturbation equations involving  $\delta\rho_\Lambda$  and  $\delta G$ , showing that the late growth of matter fluctuations becomes independent of the scale  $k$ . A class of models with variable cosmological term as a series function of the Hubble rate is introduced in section 4. In section 5, we concentrate on a particular model in this class, which is well motivated from the QFT point of view, and analyze the constraints imposed by primordial nucleosynthesis and structure formation. In the last section, we present and discuss our conclusions. Finally, an appendix is included at the end where we discuss gauge issues.

## 2 Generic models with variable cosmological parameters

We start from Einstein’s equations in the presence of the cosmological constant term,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} + g_{\mu\nu}\rho_\Lambda), \quad (2)$$

where  $T_{\mu\nu}$  is the ordinary energy-momentum tensor associated to isotropic matter and radiation, and  $\rho_\Lambda$  represents the vacuum energy density associated to the CC. Let us now contemplate the

possibility that  $G = G(t)$  and  $\Lambda = \Lambda(t)$  can be both functions of the cosmic time within the context of the FLRW cosmology. It should be clear that the very precise measurements of  $G$  existing in the literature refer only to distances within the solar system and astrophysical systems. In cosmology these scales are immersed into much larger scales (galaxies and clusters of galaxies) which are treated as point-like (and referred to as “fundamental observers”, comoving with the cosmic fluid). Therefore, the variations of  $G$  at the cosmological level could only be observed at much larger distances where at the moment we have never had the possibility to make direct experiments. In practice, the potential variation of  $G = G(t)$  and  $\Lambda = \Lambda(t)$  should be expressed in terms of a possible cosmological redshift dependence of these functions,  $G = G(z)$  and  $\Lambda = \Lambda(z)$ .

Let us consider the various possible scenarios for variable cosmological parameters that appear when we solve Einstein’s equations (2) in the flat FLRW metric,  $ds^2 = dt^2 - a^2(t)d\mathbf{x}^2$ . To start with, one finds Friedmann’s equation with non-vanishing  $\rho_\Lambda$ , which provides Hubble’s expansion rate  $H = \dot{a}/a$  ( $\dot{a} \equiv da/dt$ ) as a function of the matter and vacuum energy densities:

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda). \quad (3)$$

On the other hand, the general Bianchi identity of the Einstein tensor in (2) leads to

$$\nabla^\mu [G(T_{\mu\nu} + g_{\mu\nu} \rho_\Lambda)] = 0. \quad (4)$$

Using the FLRW metric explicitly, the last equation results into the following “mixed” local conservation law:

$$\frac{d}{dt} [G(\rho_m + \rho_\Lambda)] + 3G H (\rho_m + p_m) = 0. \quad (5)$$

If  $\dot{\rho}_\Lambda \neq 0$ , then  $\rho_m$  is not generally conserved as there may be transfer of energy from matter-radiation into the variable  $\rho_\Lambda$  or vice versa (including a possible contribution from a variable  $G$ , if  $\dot{G} \neq 0$ ). Thus this law mixes, in general, the matter-radiation energy density with the vacuum energy  $\rho_\Lambda$ . However, the following particular scenarios are possible:

- i)  $G = \text{const.}$  and  $\rho_\Lambda = \text{const.}$  This is the standard case of  $\Lambda$ CDM cosmology, implying the local covariant conservation law of matter-radiation:

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0; \quad (6)$$

- ii)  $G = \text{const}$  and  $\dot{\rho}_\Lambda \neq 0$ , in which case Eq.(5) leads to the mixed conservation law

$$\dot{\rho}_\Lambda + \dot{\rho}_m + 3H(\rho_m + p_m) = 0; \quad (7)$$

- iii)  $\dot{G} \neq 0$  and  $\rho_\Lambda = \text{const}$ , implying  $\dot{G}(\rho_m + \rho_\Lambda) + G[\dot{\rho}_m + 3H(\rho_m + p_m)] = 0$ ;
- iv)  $\dot{G} \neq 0$  and  $\dot{\rho}_\Lambda \neq 0$ . In this case, if we assume the standard local covariant conservation of matter-radiation, i.e Eq. (6), it is easy to see that Eq. (5) boils down to

$$(\rho_m + \rho_\Lambda)\dot{G} + G\dot{\rho}_\Lambda = 0. \quad (8)$$

Notice that only in cases i) and iv) matter is covariantly self-conserved, meaning that matter evolves according to the solution of Eq. (6):

$$\rho_m(a) = \rho_m^0 a^{-\alpha_m} = \rho_m^0 (1+z)^{\alpha_m}, \quad \alpha_m = 3(1+\omega_m), \quad (9)$$

where  $\rho_m^0$  is the current matter density and  $\omega_m = p_m/\rho_m = 0, 1/3$  are the equation of state (EOS) parameters for cold and relativistic matter, respectively. We have expressed the result (9) in terms of the scale factor  $a = a(t)$  and the cosmological redshift  $z = (1-a)/a$ .

In cases ii) and iii), instead, matter is not conserved (if one of the two parameters  $\rho_\Lambda$  or  $G$  indeed is to be variable, respectively). Explicit cosmological models with variable parameters as in case ii) have been investigated in detail in [15, 19, 27]. Cosmic perturbations of this model have been considered in [28–32]. Case iii) has been studied at the background level in [33]. Finally, the background evolution of case iv) has been studied in different contexts in Ref. [24, 25]<sup>1</sup>. In the next section, we shall address the calculation of cosmic perturbations in a general model where  $\rho_\Lambda$  and  $G$  can evolve with the expansion, and we will then specialize the set of equations for the type iv) model in which matter is covariantly conserved. Only in section 4 we will further narrow down the obtained set of perturbation equations for a concrete model with running cosmological parameters [24].

### 3 Perturbations with variable $\Lambda$ and $G$

For the analysis of the cosmic perturbations in the general running  $\rho_\Lambda$  and  $G$  model iv) of the previous section, we have to perturb all parts of Einstein's equations that may evolve with time; namely the metric, the energy-momentum tensor for both matter and vacuum, and finally we must perturb also the gravitational constant. Einstein's equations (2) can be conveniently cast as follows:

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda \right). \quad (10)$$

As a background metric, we use the flat FLRW metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) \delta_{ij} dx^i dx^j. \quad (11)$$

In perturbing it,  $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$ , we adopt here the synchronous gauge<sup>2</sup> ( $h_{00} = h_{0i} = 0$ ). The total energy momentum tensor of the cosmological fluid can be written as the sum of the matter and vacuum contributions:

$$T_\nu^\mu = T_\nu^\mu(\text{matter}) + T_\nu^\mu(\Lambda) = -p_T \delta_\nu^\mu + (\rho_T + p_T) U^\mu U_\nu, \quad (12)$$

where

$$\begin{aligned} \rho_T &= \rho_m + \rho_\Lambda \\ p_T &= p_m + p_\Lambda = \omega_m \rho_m - \rho_\Lambda, \end{aligned} \quad (13)$$

<sup>1</sup>Some potential astrophysical implications of scenario iv) have been addressed first in [24] and recently in [34].

<sup>2</sup>See the Appendix for an alternative calculation in the Newtonian gauge.

(notice that  $\omega_\Lambda \equiv p_\Lambda/\rho_\Lambda = -1$  even if the CC is running). As for the density and pressure, we introduce the perturbations in the usual form, except that we have to include also the perturbation for the vacuum energy density because, in the present context,  $\Lambda$  is an evolving variable. Thus, we have  $\rho_i \rightarrow \rho_i + \delta\rho_i$ ,  $p_i \rightarrow p_i + \delta p_i$ , where  $i = m, \Lambda$ . As warned above, if we allow for a variable gravitational coupling  $G$ , we must also consider perturbations for it:

$$G \rightarrow G + \delta G. \quad (14)$$

Obviously, the perturbations  $\delta G$  and  $\delta\rho_\Lambda$  are the two most distinguished dynamical components of the present cosmic perturbation analysis, as they are both absent in the  $\Lambda$ CDM model.

Finally, we have to perturb the 4-velocity of the matter particles,  $U^\mu \rightarrow U^\mu + \delta U^\mu$ . For an observer moving with the fluid (i.e. a comoving observer) it reads:  $U^\mu = (1, \delta U^i)$ .

We are now ready to find the perturbed equations of motion for this running model. As the fundamental equations describing the perturbations for our model, we will take: i) the 00 component of the Einstein equations (10); and ii) the generalized Bianchi identity (4), which splits into energy conservation (notice that  $\dot{G} \neq 0$  in the present framework)

$$\nabla_\mu (GT_0^\mu) = \partial_\mu (GT_0^\mu) + G [\Gamma_{\sigma\mu}^\mu T_0^\sigma - \Gamma_{\mu 0}^\sigma T_\sigma^\mu] = 0, \quad (15)$$

and momentum conservation:

$$\nabla_\mu (GT_i^\mu) = \partial_\mu (GT_i^\mu) + G [\Gamma_{\sigma\mu}^\mu T_i^\sigma - \Gamma_{\mu i}^\sigma T_\sigma^\mu] = 0. \quad (16)$$

As for the 00 component of the Einstein equations, and taking into account that  $G$  gets perturbed as in (14), we have:

$$\begin{aligned} 0th \text{ order: } \quad -3\frac{\ddot{a}}{a} &= 4\pi G(\rho_T + 3p_T), \\ 1st \text{ order: } \quad \dot{\hat{h}} + 2H\hat{h} &= 8\pi [G(\delta\rho_T + 3\delta p_T) + \delta G(\rho_T + 3p_T)], \end{aligned} \quad (17)$$

where we have defined  $\hat{h} = \frac{\partial}{\partial t} \left( \frac{h_{kk}}{a^2} \right)$  and repeated Latin indices are understood to be summed over the values 1, 2, 3. As energy-density components, we have matter and the running cosmological constant, with EOS parameters indicated in (13), and thus the perturbed equation takes on the form:

$$\dot{\hat{h}} + 2H\hat{h} = 8\pi \left\{ G[(1 + 3\omega_m)\delta\rho_m + \delta\rho_\Lambda + 3\delta p_\Lambda] + \delta G[(1 + 3\omega_m)\rho_m - 2\rho_\Lambda] \right\}. \quad (18)$$

Let us now work out the equation of local covariant conservation of the energy, Eq.(15). After a straightforward calculation, the final equations read as follows. On the one hand,

$$0th \text{ order: } \quad \dot{G}(\rho_m + \rho_\Lambda) + G(\dot{\rho}_m + \dot{\rho}_\Lambda) + 3GH\rho_m(1 + \omega_m) = 0. \quad (19)$$

If we assume local covariant conservation of matter,  $\nabla_\mu T_0^\mu(\text{matter}) = 0$ , i.e. Eq.(6), we see that (19) indeed reduces to Eq.(8), as it should be. On the other hand,

$$\begin{aligned} 1st \text{ order: } \quad G \left[ \delta\dot{\rho}_m + \delta\dot{\rho}_\Lambda + \rho_m(1 + \omega_m) \left( \theta - \frac{\hat{h}}{2} \right) + 3H[(1 + \omega_m)\delta\rho_m + (\delta\rho_\Lambda + \delta p_\Lambda)] \right] + \\ + \dot{G}(\delta\rho_m + \delta\rho_\Lambda) + [\dot{\rho}_m + \dot{\rho}_\Lambda + 3H(1 + \omega_m)\rho_m] \delta G + (\rho_m + \rho_\Lambda) \delta\dot{G} = 0. \end{aligned} \quad (20)$$

In the previous equation, we have introduced the variable

$$\theta \equiv \nabla_\mu \delta U^\mu = \partial_i \delta U^i, \quad (21)$$

which represents the perturbation on the matter velocity gradient, and we have used  $\delta U^0 = 0$ . If we invoke once more the local covariant conservation of matter, both at 0th order – see Eq. (6) – and at 1st order,

$$\delta \dot{\rho}_m + \rho_m(1 + \omega_m) \left( \theta - \frac{\hat{h}}{2} \right) + 3H(1 + \omega_m) \delta \rho_m = 0, \quad (22)$$

equation (20) can be further reduced to the simpler form:

$$G[\delta \dot{\rho}_\Lambda + 3H(\delta \rho_\Lambda + \delta p_\Lambda)] + \dot{G}(\delta \rho_m + \delta \rho_\Lambda) + \dot{\rho}_\Lambda \delta G + (\rho_m + \rho_\Lambda) \delta \dot{G} = 0. \quad (23)$$

Finally, let us work out the equation of local covariant conservation of momentum, Eq.(16). In this instance, there are no 0th order terms (in the background, the momentum conservation is trivial). The perturbed result reads

$$a^2 \partial_t [G \rho(1 + \omega) \delta U^i] + 5a \dot{a} G \rho(1 + \omega) \delta U^i + G \partial_i (\delta p) + p \partial_i (\delta G) = 0, \quad (24)$$

where it is understood that we should sum over all the energy components, in this case, matter and  $\Lambda$ . By applying the operator  $\partial_i$  and transforming to the Fourier space, we get:

$$(1 + \omega) \left[ \dot{G} \rho \theta + G(\dot{\rho} \theta + \rho \dot{\theta} + 5H \rho \theta) \right] - \frac{k^2}{a^2} [G \delta p + \omega \rho \delta G] = 0, \quad (25)$$

which, after summing over matter and CC, becomes:

$$(1 + \omega_m) \left[ \dot{G} \rho_m \theta + G(\dot{\rho}_m \theta + \rho_m \dot{\theta} + 5H \rho_m \theta) \right] = \frac{k^2}{a^2} [G(\delta p_\Lambda + \omega_m \delta \rho_m) + (\omega_m \rho_m - \rho_\Lambda) \delta G]. \quad (26)$$

The part corresponding to matter conservation ( $\nabla_\mu T^\mu_{i(\text{matter})} = 0$ ) is

$$(1 + \omega_m) \left[ \dot{\rho}_m \theta + \rho_m \dot{\theta} + 5H \rho_m \theta \right] = \frac{k^2}{a^2} \omega_m \delta \rho_m, \quad (27)$$

so that Eq.(26) simplifies as follows:

$$(1 + \omega_m) \dot{G} \rho_m \theta = \frac{k^2}{a^2} [G \delta p_\Lambda + (\omega_m \rho_m - \rho_\Lambda) \delta G]. \quad (28)$$

Summarizing, our final set of equations is given by (18),(22),(23),(27) and (28). It is particularly relevant for our purposes to rewrite these equations in the matter dominated epoch ( $\omega_m = 0$ ) and assuming adiabatic perturbations for the CC ( $\delta p_\Lambda = -\delta \rho_\Lambda$ ). In a nutshell, we find:

$$\dot{\hat{h}} + 2H \hat{h} = 8\pi [\rho_m - 2\rho_\Lambda] \delta G + 8\pi G [\delta \rho_m - 2\delta \rho_\Lambda] \quad (29)$$

$$\delta \dot{\rho}_m + \rho_m \left( \theta - \frac{\hat{h}}{2} \right) + 3H \delta \rho_m = 0 \quad (30)$$

$$\dot{\theta} + 2H \theta = 0 \quad (31)$$

$$\delta \dot{G} (\rho_m + \rho_\Lambda) + \delta G \dot{\rho}_\Lambda + \dot{G} (\delta \rho_m + \delta \rho_\Lambda) + G \delta \dot{\rho}_\Lambda = 0 \quad (32)$$

$$k^2 [G \delta \rho_\Lambda + \rho_\Lambda \delta G] + a^2 \rho_m \dot{G} \theta = 0. \quad (33)$$



Since Eq. (31) will be important for the subsequent considerations, let us note that it follows from Eq. (27) upon using the matter conservation equation (6).

We are now ready to derive an important result that applies to all cosmological models of the type iv) in section 2, i.e. models with variable  $\Lambda$  and  $G$  in which matter is covariantly conserved, to wit: for all these models, the perturbation equations that we have just derived do *not* depend in fact on the wavenumber  $k$ . To prove this remarkable result is very easy at this stage of the discussion, as it ensues immediately from equations (31) and (33). Indeed, if we change the variable from cosmic time  $t$  to the scale factor  $a$  (which is readily done by using  $\dot{f} \equiv df/dt = aHdf/da \equiv aHf'$ ), equation (31) can be cast as

$$\theta' + \frac{2}{a}\theta = 0. \quad (34)$$

Therefore,

$$\theta = \theta_0 a^{-2}. \quad (35)$$

It follows that the second term on the *l.h.s.* of Eq. (33) behaves as  $a^2 \rho_m \dot{G} \theta = \theta_0 \rho_m \dot{G}$ . Notice that  $\theta_0$  is the perturbation of the matter velocity at present, which is of course much smaller than any value  $\theta_i$  that this variable can take early on, more specifically at any time after the transfer function regime has finished (see below). Obviously,  $\theta_i = \theta_0 a_i^{-2} \ll \theta_0$  (where  $a_i \ll a_0 = 1$ ). Thus, taking into account that the matter perturbations  $\delta\rho_m$  are growing rather than decaying,  $\theta$  will be comparatively negligible. Let us also note that the  $\theta$ -term in Eq. (33) is multiplied by the time derivative of  $G$ , which in all reasonable models (in particular, the one we explore in section 5) should be small. Finally, if we divide Eq. (33) by  $k^2$  and take into account that in practice we are interested in deep sub-horizon scales, i.e. scales  $\lambda$  that satisfy  $\lambda a \ll H^{-1}$  (or, equivalently,  $k \gg Ha$ ), it follows that the  $\theta$ -term in Eq. (33) is entirely negligible. Thus, in all practical respects, this is tantamount to set  $\theta(a) = 0$  ( $\forall a$ ). The outcome is that the full set of perturbation equations becomes independent of  $k$ , as announced<sup>3</sup>. To better assess the physical significance of this important result, we have repeated the calculation in another gauge (the Newtonian or longitudinal gauge) and we have obtained the same result for scales well below the horizon (see the Appendix at the very end for a summarized presentation and discussion).

Having shown that the shape of the spectrum is not distorted at late times in our model, we may ask ourselves if, in contrast, there are additional sources of wavenumber dependence at early times. Recall that the transfer function  $T(k)$  parameterizes the important dynamical effects that the various  $k$ -modes undergo at the early epochs [35]. More specifically, it accounts for the non-trivial evolution of the primordial perturbations through the epochs of horizon crossing and radiation/matter transition (the latter occurs at the so-called “equality time”,  $t_{\text{eq}}$ ). A most important feature in the  $\Lambda$ CDM case is that the scale dependence is fully encoded in  $T(k)$ , and so the spectrum of the standard model evolves without any further distortion during most of the matter epoch till the present time.

We may ask ourselves if the evolution of our model with running  $\Lambda$  and  $G$  at the early epochs follows also the same pattern as the  $\Lambda$ CDM, or if its dynamics could imprint some significant modification on the structure of  $T(k)$ . In such a case, a distortion of the power spectrum with

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<sup>3</sup>This result is similar to the model of Ref. [21], where matter is also covariantly conserved.



respect to the  $\Lambda$ CDM would be generated at early times and it would freely propagate (i.e. without further modifications) till the present days and act as a kind of signature of the model. However, we deem that this possibility is not likely to be the case, for in the kind of models under consideration there is no production/decay of matter or radiation. Thus, the value of the scale factor at equality,  $a_{\text{eq}} = a(t_{\text{eq}})$ , must be identical to that of the standard  $\Lambda$ CDM model (cf. more details in section 5.1). On the other hand, neither the value of  $\rho_\Lambda$  nor of the perturbation in this variable are expected to play an important role in the past. Finally, for reasons that will become apparent later, it is important to restrict our discussion to models wherein the time variation of  $G$  is very small, as this may also affect the previous consideration on the transfer function (and on top of this there are important bounds from primordial nucleosynthesis to be satisfied, see section 5.2).

After insuring that these conditions are fulfilled by the admitted class of variable  $G$  models, we trust that the  $k$ -dependence of the transfer function for these models should be the same as that for the  $\Lambda$ CDM model. Combining this fact with the above proven scale independence of the late time evolution of the coupled set of perturbations for  $\delta\rho_\Lambda$  and  $\delta G$ , we reasonably infer that the matter power spectrum of a model with variable  $\rho_\Lambda$  and slowly variable  $G$  – and with self-conserved matter components – must generally have the very same spectral shape as that of the  $\Lambda$ CDM model. Fortunately, in spite of their shape invariance, such models can still be constrained and distinguished from the standard cosmological model by means of the spectrum amplitude (i.e. from the normalization of the matter fluctuation power spectrum). We shall further elaborate on these points in section 5, where a concrete model exhibiting these properties will be analytically and numerically analyzed.

Let us now retake our analysis of cosmic perturbations by showing that the perturbations in  $G$  are actually tightly linked to the perturbations in  $\rho_\Lambda$ . This feature is another consequence of the aforementioned scale invariance of the late time evolution of the perturbations. Indeed, from (33) and the neglect of the velocity perturbations (35), we obtain:

$$\delta_\Lambda \equiv \frac{\delta\rho_\Lambda}{\rho_\Lambda} = -\frac{\delta G}{G}. \quad (36)$$

It follows that the two kind of perturbations are not ultimately independent. Furthermore, using Eq. (36), a straightforward calculation from (32) renders the simple result:

$$\delta_m \equiv \frac{\delta\rho_m}{\rho_m} = -\frac{\delta\dot{G}}{\dot{G}} = -\frac{(\delta G)'}{G'}. \quad (37)$$

The last two equations show clearly that the perturbations in  $G$  and  $\rho_\Lambda$  may not be consistently set to zero, since they will be generated even if they are assumed to vanish at some initial instant of time.<sup>4</sup> From (30), and using the matter conservation (6), it is easy to see that

$$\hat{h} = 2\dot{\delta}_m. \quad (38)$$

We can now use this last expression and Eq. (29) to produce a second order differential equation

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<sup>4</sup>This is analogous to what happens in models with self-conserved DE, where DE perturbations cannot be consistently neglected [36]

for  $\delta_m$ . Using again differentiation with respect to the scale factor, we find

$$\delta_m'' + A(a)\delta_m' = B(a) \left( \delta_m + \frac{\delta G}{G} \right), \quad (39)$$

where we have defined:

$$A = \frac{3}{a} + \frac{H'(a)}{H(a)} \quad (40)$$

$$B = \frac{3}{2a^2} \tilde{\Omega}_m(a) \quad (41)$$

$$\tilde{\Omega}_m(a) = \frac{\rho_m(a)}{\rho_c(a)} = \frac{8\pi G(a)}{3H^2(a)} \rho_m(a). \quad (42)$$

Here,  $\tilde{\Omega}_m(a)$  is the ‘‘instantaneous’’ normalized matter density at the time where the scale factor is  $a = a(t)$ . It is also convenient to define the corresponding instantaneous normalized CC density  $\tilde{\Omega}_\Lambda(a) = \rho_\Lambda(a)/\rho_c(a)$ , and it is easy to check that the following sum rule

$$\tilde{\Omega}_m(z) + \tilde{\Omega}_\Lambda(z) = 1 \quad (43)$$

is satisfied for all  $z$ . Clearly, the sum rule is a simple but convenient rephrasing of Eq. (3).

If we would neglect the perturbations in  $G$  ( $\delta G \sim 0$ ) and hence  $\delta\rho_\Lambda \sim 0$  too, owing to Eq. (36), it is immediate to see that Eq. (39) would reduce to the standard differential equation [35] for the growth factor for DE models with self-conserved matter under the assumption of negligible DE perturbations, i.e. explicitly Eq.(90) of Ref. [21]. In particular, if we take for  $H$  the standard form (3) with  $\rho_\Lambda = \text{const.}$ , we arrive to the equation for the  $\Lambda$ CDM model. However, Eq. (39) with  $\delta G \neq 0$  tells us precisely how to take into account the effect of non-vanishing perturbations in the gravitational constant  $G$ . We could now solve the coupled system formed by (39) and (37), giving initial conditions for  $\delta_m'$ ,  $\delta_m$  and  $\delta G$ , or use those two equations to get a third order differential equation for  $\delta_m$ , in which case we would need to give the initial condition for  $\delta_m''$  instead. This seems to be the natural thing to do, because it involves boundary conditions on  $\delta_m$  and its derivatives only, and does not mix with the boundary conditions on other variables.

In order to get the aforesaid third order differential equation for  $\delta_m$ , we have to differentiate Eq. (39) and use this same equation to eliminate  $\delta G/G$ . Using also (37), we obtain

$$\left( \frac{\delta G}{G} \right)' = \frac{\delta G'}{G} - \frac{G'}{G} \frac{\delta G}{G} = -\frac{G'}{G} \delta_m - \frac{G'}{G} \left( \frac{\delta_m'' + A\delta_m'}{B} - \delta_m \right) = -\frac{G'}{GB} (\delta_m'' + A\delta_m'), \quad (44)$$

and hence we finally arrive at the desired third order equation:

$$\delta_m''' + \left( A - \frac{B'}{B} + \frac{G'}{G} \right) \delta_m'' + \left[ A' - B + A \left( \frac{G'}{G} - \frac{B'}{B} \right) \right] \delta_m' = 0. \quad (45)$$

Now we have to compute the coefficients of this equation. From the definition of  $\tilde{\Omega}_m$ , Eq. (42), we get the useful relation:

$$\frac{\tilde{\Omega}_m'}{\tilde{\Omega}_m} = \frac{G'}{G} - 2\frac{H'}{H} - \frac{3}{a}. \quad (46)$$

On the other hand, differentiating Friedmann's equation (3),

$$2HH' = \frac{8\pi}{3}[G'(\rho_m + \rho_\Lambda) + G(\rho'_m + \rho'_\Lambda)] = \frac{8\pi G}{3}\rho'_m = -\frac{8\pi G}{a}\rho_m, \quad (47)$$

where in the last two steps we have taken into account the energy-conservation law (8) and (9) for  $\alpha_m = 3$  (matter epoch). Combining this last equation with the definition of  $\tilde{\Omega}_m$ , (42), we get:

$$\frac{H'}{H} = -\frac{3}{2a}\tilde{\Omega}_m. \quad (48)$$

Finally, using (46) and (48) we can rewrite our third order differential equation (45) in terms of  $\tilde{\Omega}_m$  as:

$$\delta_m''' + \frac{1}{2}\left(16 - 9\tilde{\Omega}_m\right)\frac{\delta_m''}{a} + \frac{3}{2}\left(8 - 11\tilde{\Omega}_m + 3\tilde{\Omega}_m^2 - a\tilde{\Omega}_m'\right)\frac{\delta_m'}{a^2} = 0. \quad (49)$$

Let us stress that this equation is valid for *any* model with variable  $G$  and  $\rho_\Lambda$  in which matter is covariantly conserved, i.e. models of the type iv) in section 2. However, a very particular case where it should give the correct results is for the  $\Lambda$ CDM and CDM models because matter is conserved and the equation (8) is trivially satisfied. For the CDM model, for instance,  $\tilde{\Omega}_m = 1$  and the equation reduces to:

$$\delta_m''' + \frac{7}{2}\frac{\delta_m''}{a} = 0, \quad (50)$$

which has the general solution:

$$\delta_m = C_1 a + C_2 + C_3 a^{-3/2}. \quad (51)$$

Barring the constant and decaying modes, which we can neglect, the relevant solution is the growing mode  $\delta_m \propto a$ , which is linear in the scale factor, as expected. Note that the setting  $\tilde{\Omega}_m = 1$  implies that there is no CC term, and from Eq. (8) we infer that  $G$  must be strictly constant in such case. Finally, Eq. (36) entails that  $\delta G$  is then also constant, and this constant can be set to zero through a redefinition of  $G$ . Let us also remark that, in this limit, one does not expect to recover the perturbative analysis of scenario ii) in section 2, i.e. the results obtained in [28]. The reason is that although in the last reference there are no perturbations on  $G$ , matter and vacuum are exchanging energy. Therefore the underlying physics is completely different and there is in general no simple connection between these two kind of models.

We can use the general equation (49) to study the perturbations in any model with variable  $G$  and  $\rho_\Lambda$  in which matter is conserved. In the next section, we consider a well motivated non-trivial example.

## 4 The class of vacuum power series models in $H$

As we have mentioned in the introduction, there have been many attempts to envisage a dynamical cosmological term  $\rho_\Lambda(t)$  [23]. However, in most cases the approach is purely phenomenological, with no reference to a potential connection with fundamental physics, via QFT or string theory. On the other hand, there are models wherein there is such a possible connection. Obviously, this

represents an additional motivation for their study. Consider the class of models in which the CC evolves as a power series in the Hubble rate:

$$\rho_\Lambda(H) = n_0 + n_1 H + n_2 H^2 + \dots, \quad (52)$$

where  $n_i$  ( $i = 0, 1, 2, \dots$ ) are dimensional coefficients (except  $n_4$ , which is dimensionless). The background solution for this cosmological model up to  $i = 2$  can be found in [18]. The higher order terms in (52), made out of powers of  $H$  larger than 2, are phenomenologically irrelevant at the present time and will not be discussed. Besides, let us stress that from the fundamental point of view of QFT in curved space-time, the general covariance of the effective action can only admit even powers of the expansion rate [15, 19, 37]. So the coefficient  $n_1$  of the linear term in  $H$  cannot be related to the properties of the effective action, but to some phenomenological parameter of the theory (e.g. the viscosity of the DE fluid) which is not part of the fundamental principles. If we dispense with this kind of phenomenological coefficients, we obtain the subclass of models of the form

$$\rho_\Lambda(H) = n_0 + n_2 H^2, \quad (53)$$

which have been advocated as scenarios in which the CC evolution can be linked to the renormalization group (RG) running in QFT [15, 16]. Notice that the coefficient  $n_2$  has dimension of an effective mass squared,  $n_2 = M_{\text{eff}}^2$ . We shall further comment on it below.

The form (53) is indeed crucially different from just considering that the vacuum energy is proportional to  $H^2$ , in the sense that Eq. (53) is an “affine quadratic law” (i.e.  $n_0 \neq 0$ ). While the pure  $H^2$  law is not favored by the experimental data [18], the affine version has been recently tested in a framework where  $G$  is constant, specifically in the framework of the scenario ii) in section 2, and it was found that it can provide a fit to the current observational data of similar quality as the  $\Lambda$ CDM one – see [18] for details.

Consider now the size of the  $n_2$  coefficient in Eq. (53). In its absence, the the CC is strictly constant and  $n_0$  just coincides with the current value  $\rho_\Lambda^0$ . However, in the presence of the  $H^2$  correction, the boundary condition at  $H = H_0$  becomes  $\rho_\Lambda^0 = n_0 + n_2 H_0^2$ . If the additional term is to play a significant role it should not be negligible as compared to  $n_0$ , and as the latter is the leading term it must still be of order  $\sim \rho_\Lambda^0$ . It follows that the effective mass  $M_{\text{eff}}$  associated to the coefficient  $n_2$  cannot be small, but actually very large – specifically, of order of a Grand Unified Theory scale (see below). This also explains why no other even power of  $H$  can play any significant role in the series expansion (52) at any stage of the cosmological history below a typical GUT scale. The upshot is that, in practice, the evolution law (53) is the leading form throughout all relevant cosmic epochs (radiation dominated, matter dominated and late CC dominated epochs).

Likewise, the possibility that the vacuum energy could be evolving linearly with  $H$  – i.e. as if  $n_0 = n_2 = 0$  in Eq. (52) – has also been addressed in the literature and can be motivated through a possible connection of cosmology with the QCD scale of strong interactions [38–40]. However, as we have said, this option is not what one would expect from the the general covariance of the effective action. Actually, the confrontation of the purely linear model  $H$  with the data does not seem to support it [18, 41], and therefore the linear term alone is unfavored. However, it could perhaps enter as a phenomenological term in a general power series vacuum model of the form

(52) – a possibility which is currently under study [42]. For some alternative recent models with variable cosmological parameters, see e.g. [43].

In the following, we focus on the quantum field vacuum model of the form (53). It is particularly convenient to rewrite the coefficients of this equation as follows:  $n_0 = \rho_\Lambda^0 - 3\nu M_P^2 H_0^2/(8\pi)$  and  $n_2 = 3\nu M_P^2/(8\pi)$ . Therefore, the CC evolution law reads

$$\rho_\Lambda(H) = \rho_\Lambda^0 + \frac{3\nu}{8\pi} M_P^2 (H^2 - H_0^2). \quad (54)$$

Here  $\nu$  is a small coefficient ( $|\nu| \ll 1$ ) and  $M_P$  is the Planck mass; it defines the current value of Newton's constant:  $G_0 = 1/M_P^2$ . Clearly, the vacuum energy density (54) is normalized to the present value, i.e.

$$\rho_\Lambda(H_0) = \rho_\Lambda^0 \equiv \frac{3}{8\pi} \Omega_\Lambda^0 H_0^2 M_P^2, \quad (55)$$

where experimentally  $\Omega_\Lambda^0 \simeq 0.7$ . The above parametrization satisfies the aforementioned condition that  $n_0$  is the leading term and is of order  $\rho_\Lambda^0$ , and at the same time the correction term is of order  $M_{\text{eff}}^2 H^2$ , with  $M_{\text{eff}} \sim \sqrt{\nu} M_P$  a large mass, even if  $\nu$  is as small as, say,  $|\nu| \sim 10^{-3}$  or less.

It is now convenient to define a new set of cosmological energy densities normalized with respect to the *current* critical density  $\rho_c^0 = 3H_0^2/(8\pi G_0)$ :

$$\Omega_i(z) \equiv \frac{\rho_i(z)}{\rho_c^0} = \frac{E^2(z)}{g(z)} \tilde{\Omega}_i(z) \quad (i = m, \Lambda), \quad (56)$$

where we have introduced the ratios

$$E(z) \equiv \frac{H(z)}{H_0} = \sqrt{g(z)} [\Omega_m(z) + \Omega_\Lambda(z)]^{1/2}, \quad g(z) \equiv \frac{G(z)}{G_0}, \quad (57)$$

with  $G(z)$  Newton's constant at redshift  $z$ . Notice that, in Eq. (56), we have explicitly related the new parameters  $\Omega_i(z)$  with the old ones  $\tilde{\Omega}_i(z)$ , the latter being defined in Eqs. (42),(43). Obviously, they all coincide at  $z = 0$  with the normalized current densities  $\Omega_i^0$ , i.e.  $\Omega_i(0) = \tilde{\Omega}_i(0) = \Omega_i^0$ .

Clearly, the coefficient  $\nu$  in Eq. (54) measures the amount of running of the CC. For any given  $\nu$ , we can compare the value of the dynamical CC term at a cosmic epoch characterized by the expansion rate  $H$  – or by the redshift  $z$  – with the current value (55). The relative correction can be conveniently expressed as follows:

$$\Delta\Omega_\Lambda(z) \equiv \frac{\Omega_\Lambda(z) - \Omega_\Lambda^0}{\Omega_\Lambda^0} = \frac{\nu}{\Omega_\Lambda^0} [E^2(z) - 1], \quad (58)$$

Of course  $G(0) = G_0$ . Moreover, since  $g = 1$  for  $\nu = 0$ , it follows that for small  $\nu$ ,  $g(z)$  deviates little from 1, namely  $g(z) = 1 + \mathcal{O}(\nu)$ . Thus, expanding to order  $\nu$  in the matter epoch, it is easy to show from the previous equations that

$$\Delta\Omega_\Lambda(z) \simeq \nu \frac{\Omega_m^0}{\Omega_\Lambda^0} [(1+z)^3 - 1], \quad (59)$$

where  $g(z) \sim 1$  to this order. If we look back to relatively recent past epochs, e.g. exploring redshifts  $z = \mathcal{O}(1)$  relevant for Type Ia supernovae measurements, we see that  $\Delta\Omega_\Lambda(z)$  can be of order of a few times  $\nu$ . For example, for  $z = 1.5$  and  $z = 2$ , we have  $\Delta\Omega_\Lambda(1.5) \simeq 6\nu$  and

$\Delta\Omega_\Lambda(2) \simeq 11\nu$  respectively, assuming  $\Omega_m^0 = 0.3$ . The correction is thus guaranteed to be small, as desired, but is not necessarily negligible. We shall see, in the next sections, the potential implications on important observables, which will actually put tight bounds on the value of  $\nu$ .

Although the above parametrization of the CC running can be purely phenomenological, let us recall that the dimensionless coefficient  $\nu$  can be interpreted, in more fundamental terms, within the context of QFT in curved-space time; specifically it is proportional to the “ $\beta$ -function” for the RG running of the CC term. The predicted value in this QFT framework is [15, 19, 24, 25]

$$\nu = \frac{\sigma}{12\pi} \frac{M^2}{M_P^2}, \quad (60)$$

where  $M$  is an effective mass parameter, representing the average mass of the heavy particles of the Grand Unified Theory responsible for the CC running through quantum effects (after taking into account the multiplicities of the various species of particles). Obviously,  $M \sim M_{\text{eff}}$ . Since  $\sigma = \pm 1$  (depending on whether bosons or fermions dominate in the loop contributions), the coefficient  $\nu$  can be positive or negative, but  $|\nu|$  is naturally predicted to be smaller than one. For instance, if GUT fields with masses  $M_i$  near  $M_P$  do contribute, then  $|\nu| \lesssim 1/(12\pi) \simeq 2.6 \times 10^{-2}$ , but we expect it to be even smaller in practice because the usual GUT scales are not that close to  $M_P$ . By counting particle multiplicities in a typical GUT, a natural estimate lies in the range  $\nu = 10^{-5} - 10^{-3}$  (see [24, 25] for details).

In the next section, we shall concentrate on a specific running QFT vacuum model of the type (54) where the vacuum energy and the gravitational constant vary simultaneously in accordance to the Bianchi identity (8). We shall also perform a detailed numerical analysis of our results.

## 5 Application: running QFT vacuum model with variable $G$

In this section, we consider a CC running vacuum model in which  $\rho_\Lambda = \rho_\Lambda(H)$  depends on the Hubble rate through the affine quadratic law (54) and in which matter is separately conserved – the time variation of the CC being compensated by that of the Newton’s coupling,  $G = G(H)$ . This setup corresponds to the scenario iv), as defined in section 2 and was first considered at the background level with alternative motivations in Refs. [24, 25]. The novelty here is to consider the detailed treatment of the cosmic perturbations in that scenario. Namely, we apply to it the general treatment of cosmic perturbations with variable  $\rho_\Lambda$  and  $G$  developed in section 3. In this way, we can study the growth of matter density perturbations and obtain a LSS bound on the basic parameter  $\nu$ , the one that controls the variation of  $G$  and  $\rho_\Lambda$ . Actually, the tightly bounded region will appear in the  $(\nu, \Omega_m^0)$  plane.

Specifically, we will constrain the parameter  $\nu$  (60) by requiring that the amount of growth of matter perturbations in our model does not deviate too much from the value deduced from the observations of the number density of local clusters. In fact, let us recall that these observations allow to set the normalization of the matter power spectrum [44, 45], and hence fix its amplitude. In view of the shape independence of the power spectrum for the models under study, which we have discussed in section 3, the truly relevant parameter to constraint our model is the spectral

amplitude [35]. As we shall see, the constraints obtained in this way are in very good agreement with those arising from primordial nucleosynthesis. Throughout this section, we will be using the scale factor and the cosmological redshift  $z = (1 - a)/a$  interchangeably.

The cosmological evolution of our model is determined by the Friedmann equation (3), the Bianchi identity (8) and the RG law for the CC (54). Using the normalized density parameters defined in Eq. (56), the basic cosmological equations can be formulated as

$$E^2(z) = g(z) [\Omega_m(z) + \Omega_\Lambda(z)] , \quad (61)$$

$$(\Omega_m + \Omega_\Lambda)dg + g d\Omega_\Lambda = 0 , \quad (62)$$

$$\Omega_\Lambda(z) = \Omega_\Lambda^0 + \nu [E^2(z) - 1] , \quad (63)$$

$$\Omega_m(z) = \Omega_m^0 (1 + z)^{3(1+\omega_m)} , \quad (64)$$

where the last equation just reflects the covariant conservation of matter, i.e. it is a rephrasing of Eq.(9). Solving the remaining system for the function  $g = g(H)$ , it is easy to arrive at

$$g(H) \equiv \frac{G(H)}{G_0} = \frac{1}{1 + \nu \ln (H^2/H_0^2)} . \quad (65)$$

We confirm that  $g(H)$  depends also on the parameter  $\nu$  and that, to first order, we have  $g = 1 + \mathcal{O}(\nu)$ . Thus, in this model,  $\nu$  plays also the role of the  $\beta$ -function for the RG running of  $G$ . Compared to the quadratic running of the CC with the Hubble rate, indicated in Eq. (63), the running of  $G$  with  $H$  is indeed very slow, it is just logarithmic. The functions  $g(z)$  and  $\Omega_\Lambda(z)$  cannot be determined explicitly in an analytic form. Nevertheless, it is possible to derive an implicit equation for  $g(z)$  by combining (65) with (61) and (63). For the matter-dominated epoch, the final result reads:

$$\frac{1}{g(z)} - 1 + \nu \ln \left[ \frac{1}{g(z)} - \nu \right] = \nu \ln [\Omega_m(z) + \Omega_\Lambda^0 - \nu] , \quad (66)$$

with  $\Omega_m(z) = \Omega_m^0 (1 + z)^3$ . The CC density follows from (63), and is given as a function of  $g(z)$ :

$$\Omega_\Lambda(z) = \frac{\Omega_\Lambda^0 + \nu [\Omega_m(z) g(z) - 1]}{1 - \nu g(z)} . \quad (67)$$

As a simple check, we can see that Eq. (66) is satisfied at  $z = 0$  for any value of  $\nu$ , using  $g(0) = 1$  and the sum rule  $\Omega_m^0 + \Omega_\Lambda^0 = 1$  for flat space. Then from (67) we immediately obtain  $\Omega_\Lambda(0) = \Omega_\Lambda^0$ , as expected.

In Fig. 1, we show the evolution of different background quantities in this model in terms of the redshift  $z$ . For the plots, we have used  $\Omega_m^0 = 0.3$  and both a positive ( $\nu = +4 \times 10^{-3}$ , red solid line) and a negative value ( $\nu = -4 \times 10^{-3}$ , blue dashed line) for the parameter  $\nu$ . As we will see later in this section, according to LSS and nucleosynthesis considerations, we expect  $\nu$  to be (at most) of order  $10^{-3}$ , so these are reasonable values. In Fig. 1a, we plot the evolution of the matter density fraction,  $\tilde{\Omega}_m(z)$  (42), which rapidly approaches unity in the past. This means that  $\tilde{\Omega}_\Lambda(z)$  is negligible for high redshifts – recall the sum rule (43) – so that in this regime our model



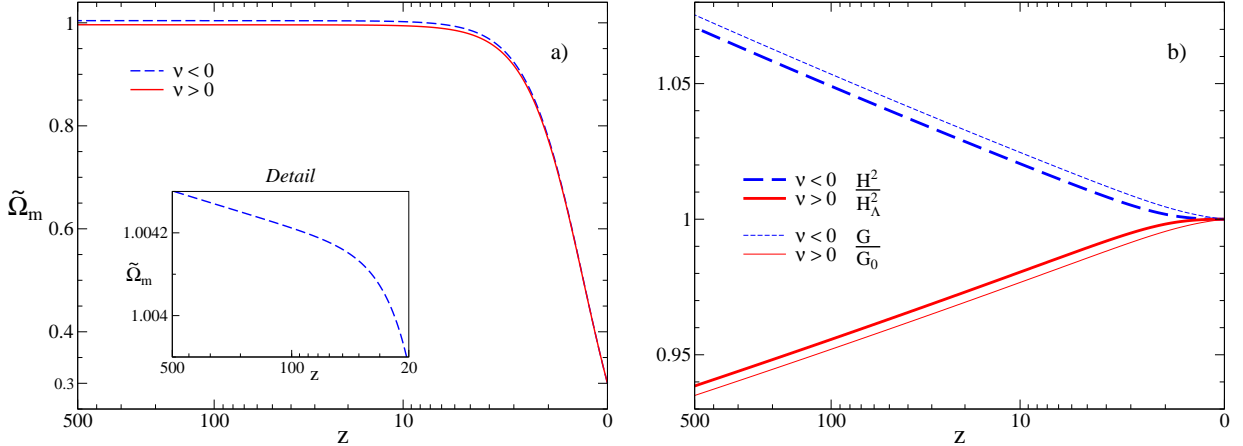


Figure 1: Evolution of different background quantities for the QFT cosmological model with running  $\Lambda$  and  $G$ , in terms of the redshift  $z$ . We take  $\Omega_m^0 = 0.3$  and consider both a positive ( $\nu = +4 \times 10^{-3}$ , red solid lines) and a negative value ( $\nu = -4 \times 10^{-3}$ , blue dashed lines) for the parameter  $\nu$ , which controls the variation of  $G$  and  $\Lambda$ ; a) the matter density fraction,  $\tilde{\Omega}_m(z)$  (42), rapidly approaches unity in the past, meaning that for high redshifts  $\tilde{\Omega}_\Lambda(z)$  is negligible and our model closely resembles a CDM (for which  $\tilde{\Omega}_m(z)$  is *exactly* 1); b) evolution of  $G(z)$  (thin lines) and  $H^2(z)$  (thick lines), normalized to the  $\Lambda$ CDM values, showing that the departure from the standard cosmological model is small. The evolution of both quantities is related by Eq. (71).

closely resembles a CDM model (for which we would have  $\tilde{\Omega}_m(z)$  *exactly* 1.) This could have been anticipated from (63), which far in the past reads:

$$\Omega_\Lambda(z) \simeq \nu E^2(z). \quad (68)$$

Using this expression in (56), we obtain the corresponding asymptotic value of  $\tilde{\Omega}_\Lambda$  in the past:

$$\tilde{\Omega}_\Lambda(z) = \frac{g(z)}{E^2(z)} \Omega_\Lambda(z) \simeq \nu g(z) \sim \mathcal{O}(\nu) \ll 1 \quad (z \gg 1), \quad (69)$$

since indeed  $g(z) \sim \mathcal{O}(1)$ , as can be confirmed from Fig. 1b. Equation (69) tells us that, for any reasonable value of  $\nu$ , the contribution of the running CC in the past will be unimportant. Nevertheless, as seen in the detail frame of Fig. 1a,  $\tilde{\Omega}_m(z)$  in our model is indeed not exactly one, nor really constant. Note that for  $\nu < 0$  the CC density decreases as we go into the past (the opposite is true for  $\nu > 0$ ) until it eventually gets negative. Therefore, our model can accommodate either originally positive or negative values for  $\Lambda$ , thanks to the running nature of this quantity. Notice that the previous results can be viewed as being a consequence of the fact that, for flat space, the tilded normalized densities satisfy the sum rule (43).

In Fig. 1b we show the evolution of  $g(z)$  and  $H^2(z)/H_\Lambda^2(z)$  where  $H_\Lambda$  is the (flat)  $\Lambda$ CDM Hubble function, simply given by:

$$H_\Lambda^2 = H_0^2 [\Omega_m(z) + \Omega_\Lambda^0] = H_0^2 [\Omega_m(1+z)^3 + \Omega_\Lambda^0]. \quad (70)$$

Taking into account (61) and (67) we obtain, to order  $\nu$ ,

$$\frac{H^2(z)}{H_\Lambda^2} = g(z) \frac{\Omega_m(z) + \Omega_\Lambda(z)}{\Omega_m(z) + \Omega_\Lambda^0} \simeq g(z) \left( 1 + \nu \frac{\Omega_m(z) - \Omega_m^0}{\Omega_m(z) + \Omega_\Lambda^0} \right). \quad (71)$$

Therefore the evolutions of  $H^2(z)/H_\Lambda^2$  and  $g(z)$  are expected to be very similar, as indeed shown in Fig. 1b. Furthermore, both quantities stay close to 1, so the deviation from the standard  $\Lambda$ CDM evolution is reasonably small, although it maybe large enough so as to be detected in a future generation of precision cosmology experiments. For instance, for the values of  $\nu$  and  $\Omega_m^0$  that we are considering, the relative deviations of  $\Omega_\Lambda(z)$ ,  $g(z)$  and  $H^2(z)$  at  $z = 2$  with respect to the standard model values are (approximately) 4%, 1% and 0.5%, respectively.

## 5.1 Constraints from large-scale structure

Let us now move to the detailed study of the growth of matter density perturbations in our model, and further elaborate on some important issues raised in section 3. The matter power spectrum for a sufficiently recent value of the scale factor ( $a \gg a_{\text{eq}}$ ) can be written as:

$$P(k, a) \equiv |\delta_m(k, a)|^2 = \mathcal{A} k^n T^2(k) D^2(k, a). \quad (72)$$

Here  $\mathcal{A}$  is a normalization coefficient;  $n$  is the scalar spectral index (which gives us the shape of the primordial spectrum; e.g.  $n = 1$  if we assume a Harrison-Zel'dovich spectrum);  $T(k)$  is the transfer function, which does not depend on the initial conditions but only on the physics of the microscopic constituents; i.e. it encodes the modifications of the primordial spectrum that arise when taking into account the dynamical properties of the particles and their interactions. The  $T(k)$  function receives non-trivial contributions only from the evolution of the different (comoving) wavelengths  $\lambda \sim 1/k$  in the epochs of radiation, horizon crossing and radiation/matter equality, i.e. it reflects the  $k$ -dependent features that occur at early times when  $a$  moves from  $a \ll a_{\text{eq}}$  to  $a \gg a_{\text{eq}}$ . In general, the form of  $T(k)$  depends on the cosmological model under consideration, and determines to a large extent the final shape of the spectrum. However, for the late time evolution ( $a \gg a_{\text{eq}}$ ), there might be more distortions of the spectrum in models that depart significantly from the standard  $\Lambda$ CDM scenario (for which there are no further  $k$ -dependence beyond that already encoded in the transfer function). These additional, model-dependent, effects are reflected in the  $k$ -dependence of the growth factor  $D(k, a)$  in Eq. (72). A dependence of this sort would appear e.g. in models of type ii) in section 2 where the CC decays into matter, as it was previously studied in [28]. However, our claim (formulated in section 3) is that this is not the case for the variable  $CC$  and  $G$  models under consideration, i.e. models of type iv) in section 2 with self-conservation of matter, provided that  $G$  is a slowly varying function of time. Obviously this is so for the model studied in the previous section, where  $G$  varies logarithmically with the expansion of the universe, see Eq. (65). For these models, and of course also for the standard  $\Lambda$ CDM model, the growth factor is independent of  $k$  and can be written as

$$D(a) = \frac{\delta_m(a)}{\delta_{\text{CDM}}(a_0)}, \quad (73)$$

where  $\delta_{\text{CDM}}(a_0)$  is the present matter density contrast in a pure cold dark matter (CDM) scenario, taken as a fiducial model.

### 5.1.1 More on the transfer function for variable $\rho_\Lambda$ and $G$ models.

A few additional comments on the transfer function are now in order. As previously commented, we expect the transfer function in our model to be the same as for the  $\Lambda$ CDM. First of all, let us try to better justify this expectation, which implies that the shape of the matter power spectrum in our model coincides with that of a  $\Lambda$ CDM with the same value of  $\Omega_m^0$ . Then we show that we can still constrain our model by means of the growth factor, obtaining bounds in very good agreement with those arising from primordial nucleosynthesis.

The scale dependence of the transfer function for CDM models is different for large scales (small  $k$ 's), where  $T(k) = 1$ , and small scales, where it asymptotes to  $\ln(k)/k^2$  [35]. The turnaround occurs at the value  $k = k_{\text{eq}}$ , corresponding exactly to the scale that enters the Hubble horizon ( $H^{-1}$ ) at the moment of matter-radiation equality, i.e. at the point  $a = a_{\text{eq}}$  which satisfies  $\Omega_m(a_{\text{eq}}) = \Omega_r(a_{\text{eq}})$ , hence

$$a_{\text{eq}} = \frac{\Omega_r^0}{\Omega_m^0}, \quad (74)$$

where  $\Omega_r^0 \sim 10^{-4}$  is the present density of radiation. The modes that enter the horizon at the equality time, with comoving wavelength  $\lambda_{\text{eq}}$ , have a physical wavelength that follows upon multiplication with the scale factor (74), i.e.  $\lambda_{\text{eq}} a_{\text{eq}} = 1/H(a_{\text{eq}})$ , or, equivalently,

$$k_{\text{eq}} = a_{\text{eq}} H(a_{\text{eq}}). \quad (75)$$

Obviously, since  $H$  decreases with time the perturbations with wavelength shorter than  $\lambda_{\text{eq}}$  (i.e. those with  $k > k_{\text{eq}}$ ) will enter the Hubble horizon before the matter-radiation equality, i.e. in the radiation era ( $t < t_{\text{eq}}$ ). From this moment until  $t = t_{\text{eq}}$ , the growth of inhomogeneities in the cold DM component becomes suppressed because the expansion rate during the radiation epoch is faster than the characteristic collapsing rate of the CDM. As a result, the modes in the radiation epoch can grow at most logarithmically with the scale factor. Only after the cold component begins to dominate ( $t > t_{\text{eq}}$ ) the amplitude of the formerly inhibited modes starts growing linearly with the scale factor. Finally, as light can only cross regions smaller than the horizon, the suppression in the radiation epoch does not affect the large-scale perturbations ( $k \ll k_{\text{eq}}$ ), which enter the horizon in the matter epoch. Such different behavior of the perturbations according to their entrance in the horizon before or after the time of equality is the origin of the characteristic shape of the transfer function for CDM-like models in the various available parameterizations [35].

From the previous standard discussion, it is apparent that the shape of the transfer function depends critically on the value of  $k_{\text{eq}}$ , which in turn depends on  $a_{\text{eq}}$  and  $H(a_{\text{eq}})$ . In the models we are studying, dark matter and radiation are separately conserved, and therefore the value of  $a_{\text{eq}}$  will not change with respect to the standard one, Eq. (74). So the remaining issue is to clarify if the change of  $H(a_{\text{eq}})$  in our model as compared to the corresponding  $\Lambda$ CDM value is significant or not. The answer follows easily from Eq. (71). At the high redshift where equality of matter and radiation occurs,  $z = \mathcal{O}(10^3) \gg 1$ , the function that accompanies  $\nu$  on the *r.h.s.* of that equation is virtually equal to one, and we are left with

$$H(a_{\text{eq}}) \simeq \sqrt{g(a_{\text{eq}}) (1 + \nu)} H_\Lambda(a_{\text{eq}}). \quad (76)$$

Here, as before,  $H_\Lambda$  is just the standard  $\Lambda$ CDM Hubble rate. For the maximal values of  $\nu$  that we will be considering ( $|\nu| \sim \mathcal{O}(10^{-3})$ , see the subsequent sections), it is easy to see from (65) that

$$\left| \sqrt{g(a_{\text{eq}})} - 1 \right| \simeq |\nu| \ln \frac{H(a_{\text{eq}})}{H_0} \sim 1\%, \quad (77)$$

where the expansion rate at the time of equality is  $H(a_{\text{eq}}) \sim 10^5 H_0$ . We see that the change of  $H(a_{\text{eq}})$  with respect to  $H_\Lambda(a_{\text{eq}})$  is mainly caused by the variation of  $g$  from 1 at  $t = t_{\text{eq}}$ . However, numerically, the effect is very small. These results allow us to safely conclude that the value of  $k_{\text{eq}}$  expected in our model is essentially the same as that of the  $\Lambda$ CDM model, up to differences of 1% at most. Given that in the  $\Lambda$ CDM model the (comoving) wavenumber at equality is

$$k_{\text{eq}} = a_{\text{eq}} H_\Lambda(a_{\text{eq}}) = \sqrt{\frac{2}{\Omega_R^0}} \Omega_m^0 H_0 \sim 10^{-2} h \text{ Mpc}^{-1}, \quad (78)$$

we see from this expression that a 1% change in  $\sqrt{g(a_{\text{eq}})}$  – hence in  $H(a_{\text{eq}})$  – is equivalent to a 1% change in  $\Omega_m^0$  in the standard scenario (i.e. with  $g = 1$ ). This variation is too small to be within reach of the present observations (see the next section), and can be safely neglected.

The final point is that, as argued throughout this section, neither  $\rho_\Lambda$  nor its perturbations or those of  $G$  are important in the past. Therefore, the evolution of the perturbations both in the radiation and in the matter-dominated eras (and both in the case of sub-Hubble and super-Hubble perturbations) remains essentially unchanged.

All in all, we expect that the transfer function in the model under consideration is very close to that of a  $\Lambda$ CDM with the same value of  $\Omega_m^0$ . From the line of our argumentation that we have used, it is not difficult to convince oneself that this result can be extended to any model with variable  $\Lambda$  and  $G$ , and self-conserved matter, as long as  $G(a)$  is not changing too fast. Adding this property to the fact – cf. section 3 – that the late growth of matter density perturbations in these models does not depend on the wavenumber, the final robust conclusion is that the matter power spectrum for models within the scenario iv) of section 2 will present the same shape as in the standard  $\Lambda$ CDM case. Therefore, the spectrum shape will not be useful to constrain the additional free parameters of the model. This is in sharp contrast to models within scenario ii), in which there is an exchange between dark matter and vacuum energy. For these models there is an explicit dependence on  $k$  beyond that of the transfer function and this produces a late time distortion of the power spectrum with respect to the  $\Lambda$ CDM. As indicated, this was exemplified in Ref. [28, 29] for the case of an evolution law of the type (53).

To summarize, there is a significant difference between the running QFT vacuum model (53) when studied either in scenario ii) or when considered in scenario iv). While in Ref. [28, 29] the shape of the spectrum was used to restrict the parameter  $\nu$  for type ii) models, in the next section we show that for the alternative type iv) models, being them shape-invariant with respect to the  $\Lambda$ CDM model, one can make use of the amplitude of the power spectrum in order to constrain the free parameters.

### 5.1.2 The spectrum amplitude as a way to constrain running G and $\Lambda$ models.

The normalization of the matter power spectrum on scales relevant to large-scale structure can be performed through different methods [46], e.g. from the microwave-background anisotropies or by measuring the local variance of galaxy counts within certain volumes. One of the most robust ways to do it is through the number density of rich galaxy clusters [44, 45], which is very sensitive to the amplitude of the dark matter fluctuations that collapsed to form them. The typical scales for these fluctuations are of order 10 Mpc, which are the smallest ones still in the linear part of the spectrum. The cluster method determines the amplitude of the power spectrum on just that length scale (corresponding to wavenumbers in the upper end of those explored with this method). The normalization is usually phrased in terms of  $\sigma_8$ , the root mean square mass fluctuation in spheres with radius  $8h^{-1}$  Mpc (i.e.  $\sim 10$  Mpc, for  $h \simeq 0.7$ ). By assuming a  $\Lambda$ CDM model, these studies are able to provide constraints in the  $\sigma_8 - \Omega_m^0$  plane. An important feature is that the results are approximately independent of the spectrum shape and galaxy bias [44]. For instance, in this last reference, it is found the relation:

$$\sigma_8 = (0.495_{-0.037}^{+0.034}) (\Omega_m^0)^{-0.60}, \quad (79)$$

valid for spatially flat models with a wide range of shapes (in particular, valid for  $0.2 \leq \Omega_m^0 \leq 0.8$ ). When combined with CMB analyses, cluster studies can determine both  $\sigma_8$  and  $\Omega_m^0$ . In a recent work [45], local cluster counts are used in conjunction with WMAP5 data to find:

$$\Omega_m^0 = 0.30_{-0.02}^{+0.03} \quad (68\%). \quad (80)$$

We will use this result to constrain our  $G$ -variable model. In order to do this, we first define the “amount of growth”, namely the square of the growth factor (73) at present,  $D^2(a_0)$ , which appears directly in the formula of the power spectrum, Eq. (72). The idea is to compare the amount of growth in our model with the amount of growth in the  $\Lambda$ CDM model with  $\Omega_m^0 = 0.30$ . From the standard expression for the growth factor in the  $\Lambda$ CDM model [35] one finds that a  $\sim 10\%$  variation in  $\Omega_m^0$  given by (80) represents a  $\sim 5\%$  change in the amount of growth. As the determination (80) entails only a  $1\sigma$  constraint, we will be conservative and allow up to a  $10\%$  deviation in the amount of growth of our model with respect to the  $\Lambda$ CDM model. We are thus asking our model to pass the following “F-test” [29, 47]:

$$F = \left| 1 - \frac{D^2(a_0, \Omega_m^0, \nu)}{D^2(a_0, 0.3, 0)} \right| \leq 0.1. \quad (81)$$

Notice that  $D^2(a_0, 0.3, 0)$  in the denominator is just the amount of growth in the  $\Lambda$ CDM for the central value of (80), whereas in the numerator we have the amount of growth for the model under consideration at a given non-vanishing value of the relevant parameter  $\nu$ , with  $\Omega_m^0$  left as a free parameter. As a result, the constraint (81) will generate contours in the  $\nu - \Omega_m^0$  plane which will define the allowed region in parameter space.

The admissible values for  $\Omega_m^0$  should, in principle, be those compatible with the shape of the spectrum. However, the shape has been measured by the 2dFGRS and SDSS surveys, existing

a significant difference between their results. On the one hand (assuming a Hubble parameter  $h = 0.72$ ), the SDSS main galaxy analysis [4] favored the result

$$\Omega_m^0 = 0.296 \pm 0.032. \quad (82)$$

Similar values around  $\Omega_m^0 \simeq 0.3$  were found by alternative analyses of the SDSS catalogue [48]. On the other hand, the 2dFGRS collaboration [3] found a much lower matter density

$$\Omega_m^0 = 0.231 \pm 0.021, \quad (83)$$

obtained from measurements of clustering of blue-selected galaxies. The inclusion of Luminous Red Galaxies (LRGs) in the SDSS analysis seems to even increase the discrepancy. For instance, the authors of [49] find  $\Omega_m^0 = 0.32 \pm 0.01$ , although a lower density is recovered when restricting the analysis to large scales ( $\Omega_m^0 = 0.22 \pm 0.04$  for  $0.01 < k < 0.06 h \text{ Mpc}^{-1}$ .) It is widely believed that the differences are due to the scale-dependence of the galaxy bias (which is apparently stronger for the red galaxies that dominate the SDSS catalogue) or even to some kind of systematic effect, but the problem has not been fully settled so far [50]. As an example, the last result by the SDSS team [5], obtained from the analysis of a LRGs sample (in combination with WMAP5 data), yields  $\Omega_m^0 = 0.289 \pm 0.019$ , still much larger than (and incompatible with) the 2dFGRS result, (83), although if we make allowance for a 10% gaussian uncertainty in  $h$  in the 2dFGRS analysis can bring both results within  $1\sigma$ .

At the end of the day, and taking into account the results from 2dFGRs and SDSS, we think that it would be premature to discard any value for  $\Omega_m^0$  in the range (0.21, 0.33) on the grounds of structure formation data – as long as it predicts the right amount of growth, e.g. by satisfying the F-test (81). Therefore, in our analysis, for illustrative purposes, we will consider values of  $\Omega_m^0$  between 0.2 and 0.4.

### 5.1.3 Numerical results.

In this numerical section, and in view of the previous considerations, we wish to determine the set of points in the  $\nu - \Omega_m^0$  plane for which the amount of growth, determined by the value of  $D^2(a_0)$ , in our running  $\rho_\Lambda$  and  $G$  model deviates less than 10% from the central ( $\Omega_m^0 = 0.3$ )  $\Lambda$ CDM value in Eq.(80). To compute the growth factor in our model, we evolve the solution  $\delta_m(a)$  of the differential equation (49) from an initial value  $a = a_i$  up to the present moment ( $a_0 = 1$ ), where  $a_i \ll 1$  is the scale factor at some early time, deep into the matter-dominated era but well after recombination (so that the transfer function regime has already ended). We will take  $a_i = 1/501$  (i.e.  $z_i = 500$ ), although we have checked that the results do not depend significantly on the specific value. As we have seen in Fig. 1a, early on at  $a = a_i$  our model is very similar to the CDM model ( $\tilde{\Omega}_m \simeq 1$ ), for which the matter density perturbations grow linearly with the scale factor, i.e.  $D(a) = a$ . Therefore, we will assume that this is also the case for our running  $G$  and  $\rho_\Lambda$  model, and take  $D(a_i) = a_i$ ,  $D'(a_i) = 1$ ,  $D''(a_i) = 0$  as the initial conditions for the third-order differential equation (49).

The results are shown in Fig. 2a. The shaded areas represent the points allowed by our analysis. Specifically, we compare the case with perturbations in  $\rho_\Lambda$  and  $G$  (green band) with the case in

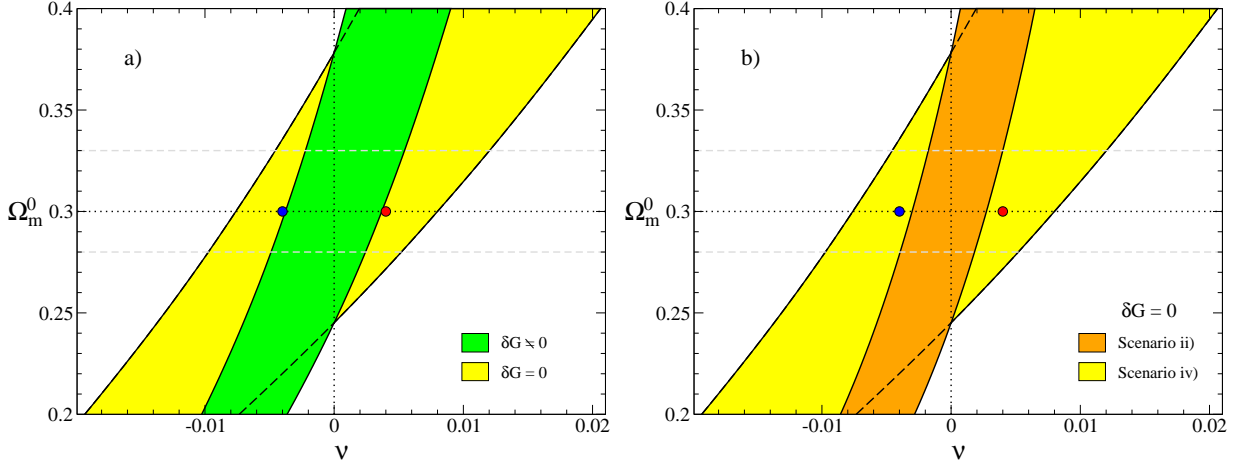


Figure 2: Analysis of the parameter space of the running QFT models. The shaded areas represent the points for which the amount of growth ( $D^2(a_0)$ ) of the model deviates less than 10% from the  $\Lambda$ CDM value. For the latter, we take  $\Omega_m^0 = 0.3$ , Eq.(80), on the basis of data on the matter power spectrum amplitude; a) Scenario iv) section 2 (running  $\Lambda$  and  $G$ , self-conserved matter). The green (resp. yellow) band include (exclude) perturbations in  $\rho_\Lambda$  and  $G$ . With perturbations, the constraint on the parameter space becomes tighter; b) Comparison between two scenarios of section 2 when perturbations in  $\rho_\Lambda$  and  $G$  are neglected: scenario iv) (yellow band) and scenario ii) (orange band). The deviations from the  $\Lambda$ CDM case are larger in scenario ii) owing to the production/decay of matter from the running  $\rho_\Lambda$  at fixed  $G$ , and the constraint is correspondingly tighter. The blue and red points in the plane have coordinates  $(\nu = \pm 4 \times 10^{-3}, \Omega_m^0 = 0.3)$  – used in Figs. 1 and 3. The dashed horizontal lines signal the  $1\sigma$  limits on  $\Omega_m^0$  from Eq. (80).

which these perturbations are neglected (yellow band). In the last situation, the growth factor is obtained by solving Eq. (39), although, as discussed in section 3, it is not possible to neglect them consistently. The most remarkable conclusion that emerges from this numerical analysis is that by considering the effect of the perturbations results in narrower domains, meaning that the deviations with respect to the standard  $\Lambda$ CDM amount of growth tend to be larger – which is why the restriction is accordingly tighter. Therefore, the outcome of the analysis with  $\delta G \neq 0$  is that, for any  $\Omega_m^0$  in the range  $0.2 < \Omega_m^0 < 0.4$ , values of  $|\nu|$  larger than  $10^{-2}$  are ruled out by the data on the number density of local clusters. In particular, this excludes the *canonical* value  $|\nu| = 1/12\pi \simeq 0.026$  obtained from the simplest choice  $M = M_P$  in Eq.(60). Notice that for  $\Omega_m^0$  in the narrower ( $1\sigma$ ) interval (80), the allowed values for  $|\nu|$  are of order  $10^{-3}$  at most. For these values of  $\nu$ , our model is on equal footing with the  $\Lambda$ CDM as far as structure formation is concerned.

In Fig. 2b we compare the results for two of the scenarios of section 2: scenario iv), represented by the yellow band (this is the scenario we have been analyzing so far, with variable  $\rho_\Lambda$  and  $G$  and self-conserved matter; and scenario ii), indicated by the orange band; for the latter,  $G$  is constant



and the CC exchanges energy with matter. In both cases, we are neglecting the perturbations<sup>5</sup> in  $\rho_\Lambda$ , which in scenario iv) implies that  $\delta G = 0$  as well, cf. Eq. (36). The effective equation for the matter density contrast in scenario ii) is the following:

$$\ddot{\delta}_m + \left(2H + \frac{\Psi}{\rho_m}\right) \dot{\delta}_m - \left[4\pi G\rho_m - 2H\frac{\Psi}{\rho_m} - \frac{d}{dt}\left(\frac{\Psi}{\rho_m}\right)\right] \delta_m = 0, \quad (84)$$

$$\Psi \equiv \dot{\rho}_m + 3H\rho_m = -\dot{\rho}_\Lambda. \quad (85)$$

This equation, whose primary derivation was performed within the Newtonian formalism [51], can also be derived from the general relativistic treatment of perturbations [21], as explained in [18]. It is convenient to express it in terms of the scale factor  $a$ :

$$\delta_m'' + \left(\frac{3}{a} + \frac{H'}{H} + \frac{\Psi}{\rho_m H a}\right) \delta_m' - \left[\frac{3}{2}\tilde{\Omega}_m - \frac{2}{H}\frac{\Psi}{\rho_m} - \frac{a}{H}\left(\frac{\Psi}{\rho_m}\right)'\right] \frac{\delta_m}{a^2} = 0. \quad (86)$$

For  $\Psi = 0$ , we recover equation (39) (with  $\delta G = 0$ ), as expected. In Fig. 2b, we see that the differences in the amount of growth with respect to the  $\Lambda$ CDM case are larger for scenario ii); the natural interpretation is that this is caused by the production/decay of matter associated to the time evolution of  $\rho_\Lambda$ .

The blue and red points in Figs. 2a,b correspond to the values analyzed in Fig. 1, i.e.  $\Omega_m^0 = 0.3$  and  $\nu = \pm 4 \times 10^{-3}$ . We will use again these values to exemplify the evolution of the matter and Newton's coupling perturbations. Fig. 3a shows the growth factor as a function of the redshift, both when allowing for perturbations in  $G$  and  $\rho_\Lambda$  and when they are neglected. For  $\nu < 0$ , our model predicts more growth than the  $\Lambda$ CDM model. When  $\delta G = 0$ , this is just due to the fact that  $\rho_\Lambda$  is decreasing when we go into the past, so its repulsive effect diminishes. When considering the perturbations in  $G$  and  $\Lambda$ , the deviation with respect to the standard case increases even more, as already commented in Fig. 2a. The opposite situation occurs for  $\nu > 0$ ; here the suppression of growth is attributed to the enhanced repulsion of matter associated to a positive  $\rho_\Lambda$ , which is increasing with  $z$ . Such suppression is larger when we include the  $G$  and  $\Lambda$  perturbations. A detailed study of these effects for model ii) of section 2 was performed in Ref. [29], in an effective approach with no perturbations in the CC.

In Fig. 3b, the evolution of  $\delta G/G$ , as obtained from Eq. (37), is depicted. This equation only determines  $\delta G'$ , so we need to give an initial condition for  $\delta G$  at  $a = a_i$ . In order to do so, we note that the fact that  $\tilde{\Omega}_m(z) \simeq 1$  in the past (reflected in Fig. 1a) ensures that the perturbations in  $G$  are not playing any important role, according to our discussion at the end of section 3. Therefore, the most natural condition seems to be  $\delta G(a_i) = 0$ , and, besides, we expect  $\delta G/G$  to remain negligible until very recent times ( $z \lesssim 10$ ), when  $\tilde{\Omega}_m$  begins to depart from 1. This is indeed what can be seen in Fig. 3b, giving additional support to our assumption. Let us remind from (36) that  $\delta_\Lambda = -\delta G/G$ . Thus, for  $\nu < 0$  we have  $\delta G > 0$  and hence  $\delta_\Lambda < 0$ , which explains the further enhancement in the growth of matter perturbations with respect to the case with  $\delta G = \delta\rho_\Lambda = 0$ .

The main conclusion of this section is that for  $\nu$  of order  $10^{-3}$  or smaller, our model is in perfect agreement with recent data on the normalization of the power spectrum. For the time

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<sup>5</sup>Remember that within scenario ii), including the perturbations in  $\Lambda$  causes the growth factor to become scale-dependent [28],  $D = D(k, a)$ , so the kind of analysis we are performing here would be no longer possible.

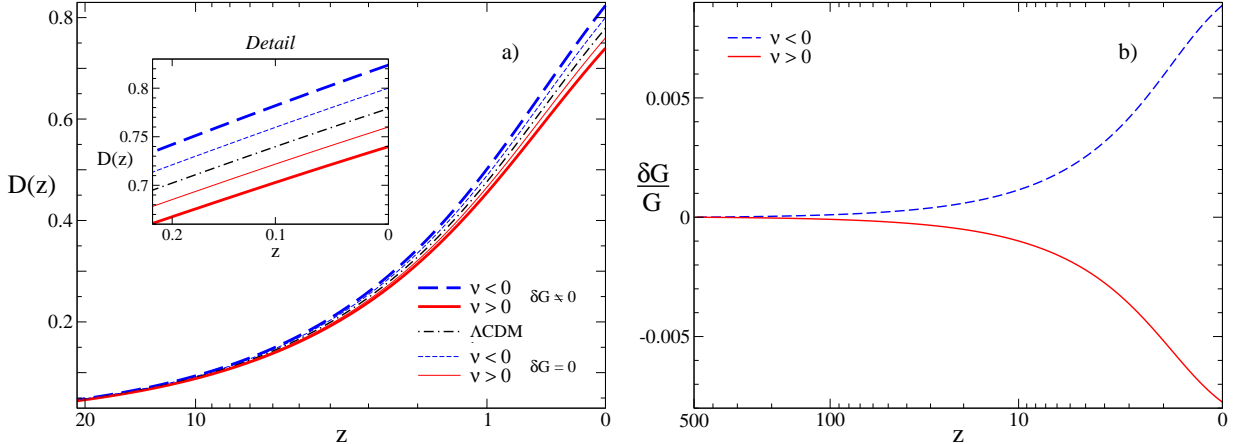


Figure 3: Evolution of the perturbations for the QFT model with running  $\Lambda$  and  $G$ , using the same values of  $\Omega_m^0$  and  $\nu$  as in Fig. 1; a) The growth factor  $D(z)$ , showing an enhancement (suppression) of the growth with respect to the  $\Lambda$ CDM case for negative (positive)  $\nu$ . The effect is present even when we neglect the perturbations in  $G$  and  $\rho_\Lambda$ , but it is larger if we consider them; b) Evolution of the perturbations in  $G$ , under the natural assumption that they are negligible at  $z_i = 500$ . The perturbations remain unimportant until very recent times, when  $\tilde{\Omega}_m$  begins to depart significantly from 1 (cf. Fig. 1). Remembering that  $\delta_\Lambda = -\delta G/G$ , we can explain the enhancement (suppression) of matter perturbations observed in a) on account of the opposite sign of  $\delta_\Lambda = \delta\rho_\Lambda/\rho_\Lambda$ , see Eq. (39).

being, there is still some controversy on the value of  $\Omega_m^0$  resulting from the power spectrum shape measured by the two main galaxy surveys. For a given value of  $\Omega_m^0$ , the power spectrum in our model presents, in very good approximation, the same shape as in the  $\Lambda$ CDM model. However, a non-vanishing  $\nu$  could become manifest through a difference in the amount of growth. Therefore, a future simultaneous precision cosmology measurement of the shape and amplitude (normalization) of the matter power spectrum should have the potential to discriminate between a model of variable  $\rho_\Lambda$  and  $G$ , and the  $\Lambda$ CDM.

In the following section, we show that  $\nu \sim \mathcal{O}(10^{-3})$  is also the maximum allowed value that we would expect from arguments related to primordial nucleosynthesis, fact that reinforces the validity of this bound.

## 5.2 Constraints from primordial nucleosynthesis

Big Bang nucleosynthesis (BBN) can also provide limits on the possible variation of Newton's coupling  $G$ . The BBN predictions for the light element abundances are sensitive to a number of parameters, such as the baryon-to-photon ratio  $\eta = n_B/n_\gamma$  (in view of the fact that the nuclear reaction rates depend on the nucleon density) or the value of the Hubble function at the nucleosynthesis time,  $H_N \equiv H(z = z_N)$  (given that the expansion rate competes against the reaction rates). Since  $\eta \propto \Omega_B$  can be accurately determined through CMB measurements, one can use the observed abundances to constrain the expansion rate, and since  $H \propto \sqrt{G}$  these constraints can be

directly translated<sup>6</sup> into bounds on  $g_N \equiv g(z_N)$ , where  $g(z)$  was defined in Eq. (57).

The current constraints on  $g_N$  available in the literature usually make use either of the deuterium abundance ( $D/H$ ) or the  ${}^4\text{He}$  mass fraction ( $Y_p$ ). Deuterium has the advantage that it is not produced in significant quantities after nucleosynthesis; its primordial abundance can be determined in a quite precise manner by studying the spectrum of the light from distant quasars, which exhibits an absorption line corresponding to the deuterium present in (high-redshift) intervening neutral hydrogen systems. This turns deuterium into an excellent probe of the universe at the time of BBN. However, its predicted abundance depends much more strongly on  $\eta$  than on  $H_N$ . As a consequence, the constraints on  $g_N$  based on deuterium measurements are not very stringent. For instance, in [52] it is found  $g_N = 1.01_{-0.16}^{+0.20}$  at the 68.3% confidence level or:

$$g_N = 1.01_{-0.30}^{+0.42} \quad (95\%). \quad (87)$$

The abundance of  ${}^4\text{He}$  is much more sensitive to the expansion rate, but accurate observational values are difficult to obtain, since there are many potential sources of systematic uncertainties [53]. As a result, a wide range of values for  $Y_p$  can be found in the literature, see e.g. Table 12 in [54]. In this reference, the authors used  $Y_p = 0.250 \pm 0.003$  (in fact, they use  $Y_p$  in combination with  $D/H$ , although the dominant effect is that of helium [54]) to obtain (see section 8.2.5 of the latter):

$$0.964 < g_N < 1.086 \quad (95\%), \quad (88)$$

whereas in [55] a more conservative range for the  ${}^4\text{He}$  mass fraction is adopted ( $Y_p = 0.2495 \pm 0.0092$ ), leading to:

$$0.9 < g_N < 1.13 \quad (68\%). \quad (89)$$

Applying (87)-(89) to our model, we can effectively constrain our parameter  $\nu$ . In order to do so, we compute  $g_N$  from (65):

$$g_N = \frac{1}{1 + \nu \ln (H_N^2/H_0^2)} \simeq \frac{1}{1 + \nu \ln [\Omega_r^0(1 + z_N)^4]}, \quad (90)$$

where  $\Omega_r^0$  is the radiation energy density fraction at present, we have neglected both the matter and CC contributions to the expansion rate at  $z = z_N$ , and is evident that  $g_N$  can be neglected as well in the logarithm. Taking  $\Omega_r^0 \sim 10^{-4}$  (which includes photons and three light neutrino species) together with  $z_N \sim 10^9$ , and comparing the last expression to (87)-(89), we find:

$$\begin{aligned} \text{From (87)} &\quad \longrightarrow \quad -4.1 \times 10^{-3} \lesssim \nu \lesssim 5.5 \times 10^{-3} \\ \text{From (88)} &\quad \longrightarrow \quad -1.1 \times 10^{-3} \lesssim \nu \lesssim 5.1 \times 10^{-4} \\ \text{From (89)} &\quad \longrightarrow \quad -1.6 \times 10^{-3} \lesssim \nu \lesssim 1.5 \times 10^{-3} \end{aligned}$$

Let us remind that (88) was derived from a value for  $Y_p$  with a (possibly unrealistic) very small error and that (89) is a 68% value (whereas the other two limits are given at the 95% confidence level.) In any case, the conclusion arising from this nucleosynthesis analysis seems to be that the

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<sup>6</sup>Provided that the total energy density at  $z = z_N$  is approximately the same as in the standard case. Eq. (69) shows that this condition is satisfied in our model.

parameter  $\nu$  can be, at most, of order  $10^{-3}$ . This is in complete agreement with the constraint on  $\nu$  obtained in section 5.1 from structure formation considerations. Therefore, we have arrived to the same result by using two very different methods, which gives additional credit to our conclusions.

## 6 Conclusions

In this paper, we have derived the general set of cosmological perturbation equations for FLRW models with variable cosmological parameters  $\rho_\Lambda$  and  $G$  in which matter is covariantly conserved. To our knowledge, this is the first time that a complete set of coupled differential equations for  $\delta\rho_\Lambda$  and  $\delta G$  is presented in the literature. We have shown that the linear growth of matter perturbations of this model,  $D(a) \propto \delta_m(a)$ , is independent of the wave number  $k$ . Adding this property to the fact that we generally expect these models to present the same transfer function as the  $\Lambda$ CDM, the scaling dependence and hence the shape of the power spectrum will not change with the late time evolution and it will coincide with that of the  $\Lambda$ CDM model. This fact is remarkable as it is in contrast to the situation, more frequently studied in the literature, in which the time-evolving vacuum models exchange energy with matter at fixed Newton's coupling  $G$  [18].

We have exemplified the difference in the power spectrum of running vacuum models, with and without matter conservation, by considering the class of cosmological models characterized by the running CC law (53), which we call quantum field vacuum models because the evolution of the CC in them is of the form that one would naturally expect from QFT, and more specifically from the renormalization group evolution of the cosmological parameters. Such quantum field vacuum models depend on a single parameter  $\nu$ , which acts as the  $\beta$ -function for the running of the vacuum energy density  $\rho_\Lambda = \rho_\Lambda(H)$  and the running gravitational coupling  $G = G(H)$ . It is interesting to note that while the running of the vacuum energy  $\rho_\Lambda(H)$  is quadratical in the expansion rate, the running of  $G(H)$  is logarithmic in  $H$ , Eq. (65), and therefore much milder. For  $\nu > 0$ , the running of  $G$  is asymptotically free, hence  $G$  decreases (resp. increases) logarithmically with the redshift (resp. with the expansion), whereas for  $\nu < 0$  it decreases with the expansion. The background evolution of these models was studied in [24, 25].

In the present paper, we have concentrated on the study of the cosmic perturbation equations of the running  $\rho_\Lambda(H)$  and  $G(H)$  models. After confronting the predicted matter growth with the various cosmological data, we have found that the final region of the parameter space is a naturalness region in which  $\nu$  could be, in absolute value, of order  $10^{-3}$  at most. We have also checked that this bound is compatible with the restriction on  $G$  variation that follows from primordial nucleosynthesis. Remarkably enough, the results emerging from the perturbations analysis are derived from the amplitude of the power spectrum rather than from its shape. Numerically, the obtained upper bound on  $\nu$  is perfectly compatible with the quantum field theoretical definition of this parameter, see Eq. (60), and it implies that the mass scales that would enter the quantum running of the cosmological parameters could be one order of magnitude below the Planck scale, at most. The result  $|\nu| < \mathcal{O}(10^{-3})$  is also compatible with previous analyses, using various independent procedures, of models with running  $\rho_\Lambda$  in which the vacuum can decay into matter [18, 28–30]. This decay feature, however, is impossible for the models with running  $\rho_\Lambda(H)$  and  $G(H)$  that we

have studied here, and this is the basic reason why they can exhibit the same power spectrum profile as the  $\Lambda$ CDM. We cannot exclude that this property could have been responsible for the fact that we have observationally missed this fundamental possibility up to now. As we have emphasized, we expect that these models should eventually be testable in the next generation of precision cosmology observations from the analysis of the spectrum amplitude, rather than of the spectral shape.

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## 7 Appendix: cosmic perturbations in the Newtonian gauge

In this appendix, we sketch the derivation of the perturbation equations in the Newtonian or longitudinal gauge [56]. Our aim is to explicitly check that the same physical consequences that we have derived in section 3 for the synchronous gauge are recovered in another gauge. In this way, we confirm in our particular context the general expectation that calculations of cosmic perturbations at deep sub-horizon scales should not present significant gauge dependence.

The most general perturbation of the spatially flat FLRW metric can be conveniently written as follows [56]

$$ds^2 = a^2(\eta)[(1 + 2\psi)d\eta^2 - \omega_i d\eta dx^i - ((1 - 2\phi)\delta_{ij} + \chi_{ij}) dx^i dx^j], \quad (91)$$

where  $\eta$  is the conformal time, defined through  $d\eta = dt/a$ . The above metric is expressed in the notation of [21] and consists of the 10 degrees of freedom associated to the two scalar functions  $\psi$ ,  $\phi$ , the three components of the vector function  $\omega_i$  ( $i = 1, 2, 3$ ), and the five components of the traceless second-rank tensor  $\chi_{ij}$ . Clearly, the synchronous gauge that we used in section 3 is obtained by setting  $\psi = 0$ ,  $\omega_i = 0$  and absorbing the function  $\phi$  into the trace of  $\chi_{ij}$ , which then contributes six degrees of freedom. In this gauge, the metric part of the analysis of cosmic perturbations is tracked by the nonvanishing trace of the metric disturbance:  $h \equiv g^{\mu\nu} h_{\mu\nu} = g^{ij} h_{ij} = -h_{ii}/a^2$ . The corresponding equations have been presented in detail in section 3 after defining the ‘‘hat variable’’:  $\hat{h} = -\partial h/\partial t = \partial (h_{ii}/a^2)/\partial t$ .

An alternative possibility is to use the (conformal) Newtonian gauge [56, 57]. In this case, we set  $\omega_i = \chi_{ij} = 0$  ( $\forall i, j$ ) in Eq. (91). A useful feature of this gauge is that the metric is diagonal and another is that, in the Newtonian limit,  $\psi$  plays the role of the gravitational potential. It is now straightforward to repeat the calculation of the perturbation equations in a similar way as in section 3. In the absence of anisotropic shear stress (which is always the case with non-relativistic particles, such as baryons and dark matter), the perturbed Einstein’s equations for  $i \neq j$  give  $\psi = \phi$  [57], see also [58]. Indicating by a prime the derivatives with respect to the scale factor

( $f' \equiv df/da$ ), the final perturbation equations in that gauge read as follows:

$$\begin{aligned}
k^2\phi &= -4\pi a^2 \left\{ G \left[ \delta\rho_m + \delta\rho_\Lambda + 3H\rho_m \frac{a^2}{k^2}\theta \right] + (\rho_m + \rho_\Lambda)\delta G \right\}, \\
\delta\rho'_m + \rho_m \left( \frac{\theta}{aH} - 3\phi' \right) + \frac{3}{a}\delta\rho_m &= 0, \\
\theta' + \frac{2}{a}\theta &= \frac{k^2}{Ha^3}\phi, \\
\delta G'(\rho_m + \rho_\Lambda) + \delta G\rho'_\Lambda + G'(\delta\rho_m + \delta\rho_\Lambda) + G\delta\rho'_\Lambda &= 0, \\
G\delta\rho_\Lambda + \rho_\Lambda\delta G + \left( \frac{Ha^3}{k^2} \right) \rho_m G'\theta &= 0.
\end{aligned} \tag{92}$$

Notice that the role of  $\hat{h}$  in the synchronous gauge is taken up here by the variable  $\phi = \psi$ , which turns out to satisfy an algebraic rather than a differential equation. Worth noticing is also the fact that the last two equations of (92) are identical to equations (32) and (33) respectively, except that in the present case we used derivatives with respect to the scale factor rather than with respect to the cosmic time (as in this way length scales can be better compared with the horizon).

We are now ready to show that for deep sub-Hubble (sub-horizon) perturbations, i.e. those satisfying  $k/(aH) \gg 1$ , these equations lead to the same second and third-order differential equations for the normalized matter overdensity  $\delta_m = \delta\rho_m/\rho_m$ , which we have previously found within the synchronous gauge in section 3. To start with, the first equation of (92) can be cast as:

$$\phi = -\frac{3}{2} \frac{H^2 a^2}{k^2} \left\{ \tilde{\Omega}_m \delta_m + \tilde{\Omega}_\Lambda \delta_\Lambda + \frac{\delta G}{G} \right\}, \tag{93}$$

where  $\delta_\Lambda = \delta\rho_\Lambda/\rho_\Lambda$ , and of course we use the same notation as in section 3 concerning the parameters  $\tilde{\Omega}_m$  and  $\tilde{\Omega}_\Lambda$ . In the previous equation, we have dropped the term  $3H\rho_m a^2 \theta/k^2$  since it is negligible for sub-Hubble perturbations. In this same regime of perturbations, Eq.(93) obviously implies that  $\phi \ll \delta_m$ . Next, let us note that the second equation of (92) can be simplified by using the conservation of matter, giving:

$$\delta'_m + \frac{\theta}{aH} = 3\phi'. \tag{94}$$

Differentiating this equation and using the third one in (92) to substitute for  $\theta'$ , we find:

$$\delta''_m - \frac{\theta}{aH} \left( \frac{3}{a} + \frac{H'}{H} \right) = 3\phi'' - \left( \frac{k}{Ha} \right)^2 \frac{\phi}{a^2}. \tag{95}$$

Furthermore, if we eliminate  $\theta$  from this equation by means of (94) and also  $\phi$  using (93), we get:

$$\delta''_m - 3\phi'' + (\delta'_m - 3\phi') \left( \frac{3}{a} + \frac{H'}{H} \right) = \frac{3}{2a^2} \left\{ \tilde{\Omega}_m \delta_m + \tilde{\Omega}_\Lambda \delta_\Lambda + \frac{\delta G}{G} \right\}. \tag{96}$$

Again we can neglect the terms that are small at sub-Hubble scales; for example, in the previous equation we can neglect  $\phi'$  ( $\phi''$ ) as compared to  $\delta'_m$  ( $\delta''_m$ ) on account of the previously discussed relation  $\phi \ll \delta_m$ . In this way, we arrive at:

$$\delta''_m + \left( \frac{3}{a} + \frac{H'}{H} \right) \delta'_m = \frac{3}{2a^2} \left\{ \tilde{\Omega}_m \delta_m + \tilde{\Omega}_\Lambda \delta_\Lambda + \frac{\delta G}{G} \right\}. \tag{97}$$

The almost final step is to substitute  $\delta_\Lambda$  by means of the last equation of (92). Once more, we can argue (as we did in section 3) that apart from the damping factor  $Ha/k \ll 1$  for the sub-Hubble modes, there are additional suppression effects, such as the presence of  $G'$  which is small as compared to  $G$ , and hence  $(G'/G)\theta$  can be considered almost second-order. Therefore, the last equation of (92) boils down in practice to  $G\delta\rho_\Lambda + \rho_\Lambda\delta G = 0$ , which simply renders  $\delta_\Lambda = -\delta G/G$ . Using also the sum rule (43), the relation (97) finally takes on the simpler form

$$\delta_m'' + \left(\frac{3}{a} + \frac{H'}{H}\right)\delta_m' = \frac{3\tilde{\Omega}_m}{2a^2} \left(\delta_m + \frac{\delta G}{G}\right). \quad (98)$$

This equation exactly coincides with Eq.(39), showing indeed that our result was not gauge-dependent. Obtaining the third-order equation for  $\delta_m$  is now straightforward. From the last two equations of (92) one can show that

$$\delta_m = -\frac{(\delta G)'}{G'}. \quad (99)$$

From here we proceed as in section 3 for the synchronous gauge, i.e. we differentiate Eq.(98) with respect to the scale factor and we reach once more the third order differential equation (49). (q.e.d.)

Some discussion is now in order. Let us first note that the variable cosmological constant scenario ii) of section 2, whose perturbation equations were originally analyzed in [28] in the synchronous gauge, has been reanalyzed in the Newtonian gauge in [32] and the same conclusions have been attained. We should also mention a very recent paper [59] where a nice discussion is made on comparing the synchronous and Newtonian gauges in the context of a model with self-conserved DE and without DE perturbations. As it is well-known, in that case the two gauges give the same evolution for the cosmological perturbations for scales well below the horizon – see also [57, 60]. This feature was expected in our model too, even if it is a bit more complicated than the one considered in [59] (our DE is also self-conserved, but we do have both DE and  $G$  perturbations), and in fact the physical discussion about the gauge differences is very similar. In the same reference, it is also pointed out that the  $k$ -dependent effects are negligible for large values of  $k$ , although they could start to be appreciable for  $k < 0.01h$  (or, equivalently, for length scales larger than  $\ell \sim 1/k \simeq 100h^{-1}$  Mpc.).

The issues about gauge dependence are of course important in general, because scale-dependent effects may change from gauge to gauge. For instance, let us note that there is no counterpart (within the synchronous gauge analysis considered in section 3) of the scale-dependent effects introduced by the first equation (92). At the same time, there may be scale dependent features which are tied to the particular cosmological model under consideration, and they need not show up in another model, even if we compare them within the same gauge. Indeed, take once more the model discussed in Ref. [59], where the DE is self-conserved and there are no perturbations in this variable. In such situation, the cosmological perturbations of matter in the synchronous gauge turn out to be exactly scale-independent (without any approximation). In our case, in contrast, there is some residual scale dependence in the synchronous gauge, although it can be shown to be negligible – as we have extensively discussed in section 3. Gauge differences, however, can be significant in certain cases, specially for calculations involving very large scales, showing that in general it is



non-trivial to relate the perturbation variables to observable quantities in different gauges [57,60]. In that case, one must keep the appropriate  $k$ -dependence (as e.g. in the Newtonian gauge [59]) or resort to a gauge-invariant formalism [61,62]. However, for the range of wavenumbers usually explored in the analysis of the matter power spectrum (viz.  $0.01h \text{ Mpc}^{-1} < k < 0.2h \text{ Mpc}^{-1}$  [3]), the gauge differences, and in particular the  $k$ -dependence, should be small enough so as to be licitly neglected. In other words, for such scales, the sub-Hubble approximation that we have taken in this Appendix holds good and the calculations in the two gauges are found to be exactly coincident, as we expected.

Finally, let us note that the reason why the physical discussion can be more transparent in the Newtonian gauge is because the time slicing in this gauge respects the isotropic expansion of the background. The synchronous gauge, instead, corresponds to free falling observers at all points, entailing that its predictions are relevant only to length scales significantly smaller than the horizon, but at these scales it renders the same physics as the Newtonian gauge [57,59,60].

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