Noncommutative gauge theory using covariant star product defined between Lie valued differential forms

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Abstract

We develop an internal gauge theory using a covariant star product. The space-time is a symplectic manifold endowed only with torsion but no curvature. It is shown that, in order to assure the restrictions imposed by the associativity property of the star product, the torsion of the space-time has to be covariant constant. An illustrative example is given and it is concluded that in this case the conditions necessary to define a covariant star product on a symplectic manifold completely determine its connection.

1 Introduction

Noncommutative gravity has been intensively studied in the last years. One important motivation is the hope that such a theory could offer the possibility to develop a quantum theory of gravity, or at least to give an idea of how this could be achieved [1, 2, 3, 4, 5, 6]. There are two major candidates to quantum gravity: string theory [7] and loop quantum gravity [8]. Noncommutative geometry and in particular gauge theory of gravity are intimately connected with both these approaches and the overlaps are considerable [2]. String theory is one of the strongest motivations for considering noncommutative space-time geometries and noncommutative gravitation. It has been shown, for example, that in the case when the end points of strings in a theory of open strings are constrained to move on D branes in a constant B-field background and one considers the low-energy limit, then the full dynamics of the theory is described by a gauge theory on a noncommutative space-time [9]. Recently, it has been argued that the dynamics of the noncommutative gravity arising from string theory [10] is much richer than some versions proposed for noncommutative gravity. It is suspected that the reason for this is the noncovariance of the Moyal star product under space-time diffeomorphisms. A geometrical approach to noncommutative gravity, leading to a general theory of noncommutative Riemann surfaces in which the problem of the frame dependence of the star product is also recognized, has been proposed in [11] (for further developments, see [12, 13]).

Now, one important problem is to develop a theory of gravity considering curved noncommutative space-times. The main difficulty is that the noncommutativity parameter $\theta^{\mu\nu}$ is usually taken to be constant, which breaks the Lorentz invariance of the commutation relations between coordinates (see (2.1) below), and implicitly of any noncommutative field theory. One possible way to solve this problem is to consider $\theta^{\mu\nu}$ depending on coordinates and use a covariant star product. In [14] such a product has been defined between differential forms and the property of associativity was verified up to the second order in $\theta^{\mu\nu}$.

In this paper we will adopt the covariant star product defined in [14] and extend the result to case of Lie algebra valued differential forms. We will follow the same procedure as in our previous paper [15]. But, in order to simplify the expression of the covariant product and to give an illustrative example, we will consider the case when the noncommutative space-time is a symplectic manifold M endowed only with torsion (and no curvature). The restrictions imposed by such a covariant star product requires also that the torsion is covariant constant. The motivation for adopting such a manifold is that it allows the construction of noncommutative teleparallel gravity (for the idea of teleparallelism in gravity see [16]). It has been shown that a very difficult problem like the definition of a tensorial expression for the gravitational energy-momentum density can be solved in teleparallel gravity [17, 18]. This density is conserved in covariant sense. Also, it was argued that the possibility of decomposing torsion in irreducible pieces under the global Lorentz group makes of teleparallel gravity a much more convenient theory than general relativity to deal with the quantization problem [19].

The teleparallel gravity is also a very natural candidate for an effective noncommutative field theory of gravitation [20]. In addition it posses many features which makes it particularly well-suited for certain analyses. For instance it enables a pure tensorial proof of the positivity of the energy in general relativity [21], it yields a natural introduction of Ashtekar variables [22], and make it possible to study the torsion at quantum level [23], for example in the gravitational coupling to spinor fields.

On the other hand, we can try to apply the covariant star product to the case when the space-time is a symplectic manifold which has only curvature, but the torsion vanishes. Then, the restriction imposed by the Jacobi identity for the Poisson bracket requires also the vanishing curvature. The corresponding connection is flat symplectic and this reduces drastically the applicability area of the covariant star product. Of course, it is possible to have a manifold with both curvature and torsion.

In Section 2, considering a manifold M endowed only with torsion, we give the definition of the covariant star product between two arbitrary Lie algebra valued differential forms and some of its properties. Then, the star-bracket between such differential forms is introduced and some examples are given.

Section 3 is devoted to the noncommutative internal gauge theory formulated with the new covariant star product. The noncommutative Lie algebra valued gauge potential and the field strength two-form are defined and their gauge transformation laws are established. It is shown that the field strength is gauge covariant and satisfies a deformed Bianchi identity.

An illustrative example is presented in Section 4. It is shown that in our simple example, the conditions necessary to define a covariant star product on a symplectic manifold M completely determine its connection.

Section 5 is devoted to the discussion of the results and to the interpretation of noncommutative gauge theory formulated by using the covariant star product between Lie algebra valued differential forms on symplectic manifolds. Some other possible applications of this covariant star product are also analysed.

The Appendix contains a detailed verification of the associativity property of the covariant star product including only the torsion in its definition.

2 Covariant star product

We consider a noncommutative space-time M endowed with the coordinates x^μ, $\mu = 0, 1, 2, 3$, satisfying the commutation relation

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}(x), \tag{2.1}$$

where $\theta^{\mu\nu}(x) = -\theta^{\nu\mu}(x)$ is a Poisson bivector [14]. The space-time is organized as a Poisson manifold by introducing the Poisson bracket between two functions $f(x)$ and $g(x)$ by

$$\{f, g\} = \theta^{\mu\nu}\partial_\mu f \partial_\nu g. \tag{2.2}$$

In order that the Poisson bracket satisfies the Jacobi identity, the bivector $\theta^{\mu\nu}(x)$ has to obey the condition [24, 25]

$$\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu} = 0. \tag{2.3}$$

If a Poisson bracket is defined on M, then M is called a Poisson manifold (see [24] for mathematical details).

Suppose now that the bivector $\theta^{\mu\nu}(x)$ has an inverse $\omega_{\mu\nu}(x)$, i.e.

$$\theta^{\mu\rho}\omega_{\rho\nu} = \delta^\mu_\nu. \tag{2.4}$$

If $\omega = \frac{1}{2}\omega_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is nondegenerate ($\det\omega_{\mu\nu} \neq 0$) and closed ($\mathrm{d}\omega = 0$), then it is called a symplectic two-form and M — a symplectic manifold. It can be verified that the condition $d\omega = 0$ is equivalent with the equation (2.3) [14, 24, 26]. In this paper we will consider only the case when M is symplectic.

Because the gauge theories involve Lie-valued differential forms such as $A = A^a_\mu(x)T_a\mathrm{d}x^\mu = A_\mu\mathrm{d}x^\mu$, $A_\mu = A^a_\mu(x)T_a$, where T_a are the infinitesimal generators of a symmetry group G, we need to generalize the definition of the Poisson bracket to differential forms and define then an associative star product for such cases. These problems were solved in [14, 24, 26]. In [15] we generalized these results to the case of Lie algebra valued differential forms. This generalization has the effect that the commutator of differential forms can be a commutator or an anticommutator, depending on their degrees.

Assuming that $\theta^{\mu\nu}(x)$ is invertible, we can always write the Poisson bracket $\{x, dx\}$ in the form [14, 24, 26]

$$\{x^\mu, \mathrm{d}x^\nu\} = -\theta^{\mu\rho}\Gamma^\nu_{\rho\sigma}\mathrm{d}x^\sigma, \tag{2.5}$$

where $\Gamma^\nu_{\rho\sigma}$ are some functions of x transforming like a connection under general coordinate transformations. As $\Gamma^\nu_{\rho\sigma}$ is generally not symmetric, one can use the connection one-forms

$$\tilde{\Gamma}^\mu_\nu = \Gamma^\mu_{\nu\rho}\mathrm{d}x^\rho, \quad \Gamma^\mu_\nu = \mathrm{d}x^\rho\Gamma^\mu_{\rho\nu} \tag{2.6}$$

to define two kinds of covariant derivatives $\tilde{\nabla}$ and ∇, respectively. The curvatures for these two connections are

$$\tilde{R}^\nu_{\lambda\rho\sigma} = \partial_\rho\Gamma^\nu_{\lambda\sigma} - \partial_\sigma\Gamma^\nu_{\lambda\rho} + \Gamma^\nu_{\tau\rho}\Gamma^\tau_{\lambda\sigma} - \Gamma^\nu_{\tau\sigma}\Gamma^\tau_{\lambda\rho}, \tag{2.7}$$

$$R^\nu_{\lambda\rho\sigma} = \partial_\rho\Gamma^\nu_{\sigma\lambda} - \partial_\sigma\Gamma^\nu_{\rho\lambda} + \Gamma^\nu_{\rho\tau}\Gamma^\tau_{\sigma\lambda} - \Gamma^\nu_{\sigma\tau}\Gamma^\tau_{\rho\lambda}. \tag{2.8}$$

Because the connection coefficients $\Gamma^\rho_{\mu\nu}$ are not symmetric $\left(\Gamma^\rho_{\mu\nu} \neq \Gamma^\rho_{\nu\mu}\right)$ the symplectic manifold M has also a torsion defined as usual [14]

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}. \tag{2.9}$$

The connection ∇ satisfies the identity [14]

$$[\nabla_\mu, \nabla_\nu]\,\alpha = -R^\sigma_{\rho\mu\nu}\mathrm{d}x^\rho \wedge i_\sigma\alpha - T^\rho_{\mu\nu}\nabla_\rho\alpha, \tag{2.10}$$

and an analogous formula applies for $\tilde{\nabla}$. Here, α is an arbitrary differential k-form

$$\alpha = \frac{1}{k!}\alpha_{\mu_1\cdots\mu_k}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} \tag{2.11}$$

and $i_\sigma\alpha$ denotes the interior product which maps the k-form α into a $(k-1)$-form

$$i_\sigma\alpha = \frac{1}{(k-1)!}\alpha_{\sigma\mu_2\cdots\mu_k}\mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}. \tag{2.12}$$

It has been proven that in order for the Poisson bracket to satisfy the Leibniz rule

$$d\{f, g\} = \{df, g\} + \{f, dg\} \tag{2.13}$$

the bivector $\theta^{\mu\nu}(x)$ has to obey the property [14]

$$\tilde{\nabla}_\rho\theta^{\mu\nu} = \partial_\rho\theta^{\mu\nu} + \Gamma^\mu_{\sigma\rho}\theta^{\sigma\nu} + \Gamma^\nu_{\sigma\rho}\theta^{\mu\sigma} = 0. \tag{2.14}$$

Thus $\theta^{\mu\nu}$ is covariant constant under $\tilde{\nabla}$, and $\tilde{\nabla}$ is an almost symplectic connection. One can use the Leibniz condition (2.14) together with the Jacobi identity for the Poisson bivector $\theta^{\mu\nu}$ to obtain the cyclic relation for torsion

$$\sum_{(\mu,\nu,\rho)} \theta^{\mu\sigma}\theta^{\nu\lambda}T^\rho_{\sigma\lambda} = 0. \tag{2.15}$$

Note that while this relation shows that a torsion-free connection identically satisfies the property (2.15), the Jacobi identity does not require the connection to be torsionless. Also note that (2.14) and the Jacobi identity for the Poisson bivector can be combined to obtain the following cyclic relation

$$\sum_{(\mu,\nu,\rho)} \theta^{\mu\sigma}\nabla_\sigma\theta^{\nu\rho} = 0. \tag{2.16}$$

If in addition to $\tilde{\nabla}_\rho\theta^{\mu\nu} = 0$, one imposes $\nabla_\rho\theta^{\mu\nu} = 0$, the torsion vanishes, $T^\rho_{\mu\nu} = 0$, and there is only one covariant derivative $\nabla = \tilde{\nabla}$. In this paper, we do not require that $\nabla_\rho\theta^{\mu\nu} = 0$.

Using the graded product rule, one arrives at the following general expression of the Poisson bracket between differential forms [14, 26]

$$\{\alpha, \beta\} = \theta^{\mu\nu}\nabla_\mu\alpha \wedge \nabla_\nu\beta + (-1)^{|\alpha|}\tilde{R}^{\mu\nu} \wedge (i_\mu\alpha) \wedge (i_\nu\beta), \tag{2.17}$$

where $|\alpha|$ is the degree of the differential form α, and

$$\tilde{R}^{\mu\nu} = \frac{1}{2}\tilde{R}^{\mu\nu}_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma, \quad \tilde{R}^{\mu\nu}_{\rho\sigma} = \theta^{\mu\lambda}\tilde{R}^{\nu}_{\lambda\rho\sigma} \tag{2.18}$$

In order that (2.17) satisfies the graded Jacobi identity

$$\{\alpha,\{\beta,\gamma\}\} + (-1)^{|\alpha|(|\beta|+|\gamma|)}\{\beta,\{\gamma,\alpha\}\} + (-1)^{|\gamma|(|\alpha|+|\beta|)}\{\gamma,\{\alpha,\beta\}\}, \tag{2.19}$$

the connection $\Gamma^\rho_{\mu\nu}$ must obey the following additional conditions [14]

$$R^{\nu}_{\lambda\rho\sigma} = 0, \tag{2.20}$$

$$\nabla_\lambda \tilde{R}^{\mu\nu}_{\rho\sigma} = 0. \tag{2.21}$$

A covariant star product between arbitrary differential forms has been defined recently in [14] having the general form

$$\alpha \star \beta = \alpha \wedge \beta + \sum_{n=1}^{\infty} \hbar^n C_n(\alpha,\beta), \tag{2.22}$$

where $C_n(\alpha,\beta)$ are bilinear differential operators satisfying the generalized Moyal symmetry [26]

$$C_n(\alpha,\beta) = (-1)^{|\alpha||\beta|+n} C_n(\beta,\alpha). \tag{2.23}$$

The operator C_1 coincides with the Poisson bracket, i.e. $C_1(\alpha,\beta) = \{\alpha,\beta\}$. An expression for $C_2(\alpha,\beta)$ has been obtained also in [14] so that the star product (2.22) satisfies the property of associativity

$$(\alpha \star \beta) \star \gamma = \alpha \star (\beta \star \gamma). \tag{2.24}$$

In this paper we consider the case when the symplectic manifold M has only torsion, i.e. in addition to the necessary constraints (2.14), (2.20) and (2.21) we require

$$\tilde{R}^{\sigma}_{\mu\nu\rho} = 0. \tag{2.25}$$

Since the curvature $R^{\sigma}_{\rho\mu\nu}$ vanishes (2.20), one obtains the following relation between the curvature $\tilde{R}$ and the torsion T [14]

$$\tilde{R}^{\sigma}_{\mu\nu\rho} = \nabla_\mu T^{\sigma}_{\nu\rho}. \tag{2.26}$$

This relation shows that the condition (2.25) requires that the torsion $T^{\sigma}_{\nu\rho}$ is covariant constant, i.e.

$$\nabla_\mu T^{\sigma}_{\nu\rho} = 0. \tag{2.27}$$

Therefore, if the torsion is covariant constant, the symplectic manifold M has only torsion but not curvature.

For such a symplectic manifold, the bilinear differential operators $C_1(\alpha,\beta)$ and $C_2(\alpha,\beta)$ in the star product (2.22) proposed in [14] reduce to the simpler forms

$$C_1(\alpha,\beta) = \{\alpha,\beta\} = \theta^{\mu\nu}\nabla_\mu\alpha \wedge \nabla_\nu\beta, \tag{2.28}$$

$$\begin{aligned} C_2(\alpha,\beta) = {} & \frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu\nabla_\rho\alpha \wedge \nabla_\nu\nabla_\sigma\beta + \frac{1}{3}\left(\theta^{\mu\sigma}\nabla_\sigma\theta^{\nu\rho} + \frac{1}{2}\theta^{\nu\sigma}\theta^{\rho\lambda}T^{\mu}_{\sigma\lambda}\right) \\ & \times \left(\nabla_\mu\nabla_\nu\alpha \wedge \nabla_\rho\beta - \nabla_\nu\alpha \wedge \nabla_\mu\nabla_\rho\beta\right). \end{aligned} \tag{2.29}$$

We can verify that the covariant star product with torsion defined in (2.28)–(2.29) is associative [see Appendix A]. In the next Section we apply this covariant star product in order to develop a noncommutative internal gauge theory.

3 Noncommutative gauge theory

Let us consider the internal symmetry group G and develop a noncommutative gauge theory on the symplectic manifold M endowed with the covariant star product (with torsion) defined above. We proceed as in [15], but considering the star product defined in (2.28)–(2.29). This product differs from that used in [15] by

- The curvature $\tilde{R}^{\sigma}_{\mu\nu\rho}$ is supposed here to vanish as well as $R^{\sigma}_{\mu\nu\rho}$;
- The ordinary derivative $\partial_\sigma \theta^{\nu\rho}$ (see (2.17) in [15]) is replaced with the covariant derivative $\nabla_\sigma \theta^{\nu\rho}$;
- In (2.29) it appears an additional term $\frac{1}{2}\theta^{\nu\sigma}\theta^{\rho\lambda}T^{\mu}_{\sigma\lambda}$ compared with previous version (see [14] for details, and also [26, 25] for other aspects).

The results given in [15] apply with the corresponding changes mentioned above. Before presenting them we make some observations on other possible applications of the covariant star product.

1. It will be interesting to see if the Seiberg-Witten map can be generalized to the case when the ordinary derivatives are replaced with the covariant derivatives and the Moyal star product is replaced by the covariant one.

2. We can consider that the symplectic manifold M is associated to a gauge theory of gravitation with Poincaré group P as local symmetry (see [27, 28, 29] for notations and definitions) and using the covariant star product as in [15]. In this case we introduce the Poincaré gauge fields ${e^a}_\mu$ (tetrads) and ω^{ab}_μ (spin connection) and then we define the covariant derivative as

$$\nabla_\mu = \partial_\mu - \frac{1}{2}\omega^{ab}_\mu \Sigma_{ab}, \tag{3.1}$$

It can be shown that by imposing the tetrad postulate [27]

$$\nabla_\mu {e^a}_\nu - \Gamma^{\rho}_{\mu\nu}{e^a}_\rho = 0 \tag{3.2}$$

one introduces the connection $\Gamma^{\rho}_{\mu\nu}$ in the Poincaré gauge theory and the strength tensors ${F^{ab}}_{\mu\nu}$, ${F^a}_{\mu\nu}$ determine the curvature and torsion of M

$$R^{\rho\sigma}_{\mu\nu} = {F^{ab}}_{\mu\nu}{\bar{e}_a}^{\ \rho}{\bar{e}_b}^{\ \sigma}, \quad T^{\rho}_{\mu\nu} = {F^a}_{\mu\nu}{\bar{e}_a}^{\ \rho} \tag{3.3}$$

where ${\bar{e}_a}^{\ \rho}$ denote the inverse of ${e^a}_\mu$, i.e.

$${\bar{e}_a}^{\ \rho}{e^a}_\sigma = \delta^{\rho}_{\sigma}, \quad {\bar{e}_a}^{\ \rho}{e^b}_\rho = \delta^{b}_{a}. \tag{3.4}$$

Now, let us suppose that we develop an internal gauge theory with the symmetry group G on the symplectic manifold M. It is very important to remark that making the minimal prescription $\partial_\mu \to \nabla_\mu$ the strength tensor $F_{\mu\nu}$ of the internal gauge fields $A_\mu = A_\mu^a T_a$ must be written as

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu - i\left[A_\mu, A_\nu\right] + A_\rho T^\rho_{\mu\nu}, \tag{3.5}$$

in order to assure its covariance both under Poincaré group P and internal group G. The gauge invariance under the gauge transformations of G becomes quite clear if we observe that the expression (3.5) is identical with the usual one. Indeed, using the definitions of the covariant derivative ∇_μ and torsion $T^\rho_{\mu\nu}$, we obtain

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - A_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu A_\mu + A_\rho \Gamma^\rho_{\nu\mu} - i\left[A_\mu, A_\nu\right] + A_\rho\left(\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}\right) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right]. \end{aligned} \tag{3.6}$$

Also, because the components of the gauge parameter $\lambda = \lambda^a T_a$ are considered as functions, we can write the gauge transformations as

$$\delta A_\mu = \nabla_\mu \lambda - i\left[A_\mu, \lambda\right], \quad \nabla_\mu \lambda = \partial_\mu \lambda. \tag{3.7}$$

In what follows, we will use these expressions (3.5) and (3.7) in order to show their invariances explicitly.

Suppose now that we have an internal gauge group G whose infinitesimal generators T_a satisfy the algebra

$$\left[T_a, T_b\right] = i f^c_{ab} T_c, \quad a, b, c = 1, 2, \ldots, m, \tag{3.8}$$

with the structure constants $f^c_{ab} = -f^c_{ba}$ and that the Lie algebra valued infinitesimal parameter is

$$\hat{\lambda} = \hat{\lambda}^a T_a. \tag{3.9}$$

We use the hat symbol "ˆ" to denote the noncommutative quantities of our gauge theory. The parameter $\hat{\lambda}$ is a 0-form, i.e. $\hat{\lambda}^a$ are functions of the coordinates x^μ on the symplectic manifold M.

Now, we define the gauge transformation of parameter $\hat{\lambda}$ of the noncommutative Lie valued gauge potential

$$\hat{A} = \hat{A}^a_\mu(x) T_a \mathrm{d}x^\mu = \hat{A}_\mu \mathrm{d}x^\mu, \quad \hat{A}_\mu = \hat{A}^a_\mu(x) T_a, \tag{3.10}$$

by

$$\hat{\delta}\hat{A} = \mathrm{d}\hat{\lambda} - i\left[\hat{A}, \hat{\lambda}\right]_\star. \tag{3.11}$$

Here we consider the following formula for the commutator $[\alpha, \beta]_\star$ of two arbitrary differential forms α and β

$$[\alpha, \beta]_\star = \alpha \star \beta - (-1)^{|\alpha||\beta|} \beta \star \alpha. \tag{3.12}$$

Then, using the definition (2.22) of the star product, we can write (3.11) as

$$\hat{\delta}\hat{A}^a = \mathrm{d}\hat{\lambda}^a + f^a_{bc}\hat{A}^b\hat{\lambda}^c + \frac{\hbar}{2} d^a_{bc} C_1\left(\hat{A}^b, \hat{\lambda}^c\right) - \frac{\hbar^2}{4} f^a_{bc} C_2\left(\hat{A}^b, \hat{\lambda}^c\right) + O\left(\hbar^3\right), \tag{3.13}$$

where we noted $\{T_a, T_b\} = d^c_{ab} T_c$. In fact, this notation is valid if the Lie algebra closes also for anticommutator, as it happens for example in the case of unitary groups. In general, the commutators like $\left[\hat{A}, \hat{\lambda}\right]_\star$ take values in the enveloping algebra [30]. Therefore, the gauge field $\hat{A}$ and the parameter $\hat{\lambda}$ take values in this algebra. Let us write for instance $\hat{A} = \hat{A}^I T_I$ and $\hat{\lambda} = \hat{\lambda}^I T_I$, then

$$\left[\hat{A}, \hat{\lambda}\right]_\star = \frac{1}{2}\left\{\hat{A}^I, \hat{\lambda}^J\right\}_\star [T_I, T_J] + \frac{1}{2}\left[\hat{A}^I, \hat{\lambda}^J\right]_\star \{T_I, T_J\}.$$

Thus, all products of the generators T_I will be necessary in order to close the enveloping algebra. Its structure can be obtained by successively computing the commutators and anticommutators starting from the generators of Lie algebra, until it closes [30],

$$[T_I, T_J] = i f^K_{IJ} T_K, \quad \{T_I, T_J\} = d^K_{IJ} T_K.$$

Therefore, in our above notations and in what follows we understand this structure in general.

The operators C_n of the star product are defined similarly for noncommutative differential forms like $\hat{A}^a$ as for commutative ones. In particular $C_1\left(\hat{A}^b, \hat{\lambda}^c\right)$ and $C_2\left(\hat{A}^b, \hat{\lambda}^c\right)$ are given by (2.28)–(2.29). Here the covariant derivative concerns the space-time manifold M, not the gauge group G, so we use the definition

$$\nabla_\mu \hat{A}^a = \left(\partial_\mu \hat{A}^a_\nu - \Gamma^\rho_{\mu\nu} \hat{A}^a_\rho\right) \mathrm{d}x^\nu \equiv \left(\nabla_\mu \hat{A}^a_\nu\right) \mathrm{d}x^\nu. \tag{3.14}$$

We define also the curvature 2-form $\hat{F}$ of the gauge potentials by

$$\hat{F} = \frac{1}{2} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \hat{F}_{\mu\nu} = \mathrm{d}\hat{A} - \frac{i}{2}\left[\hat{A}, \hat{A}\right]. \tag{3.15}$$

Then, using the definition (2.22) of the star product and the property (2.23) of the operators $C_n\left(\alpha^a, \beta^b\right)$, we obtain from (3.15)

$$\hat{F}^a = \mathrm{d}\hat{A}^a + \frac{1}{2} f^a_{bc} \hat{A}^b \wedge \hat{A}^c + \frac{1}{2}\frac{\hbar}{2} d^a_{bc} C_1\left(\hat{A}^b, \hat{A}^c\right) + \frac{1}{2}\frac{\hbar^2}{4} f^a_{bc} C_2\left(\hat{A}^b, \hat{A}^c\right) + O\left(\hbar^3\right) \tag{3.16}$$

More explicitly, in terms of components we have

$$\hat{F}^a_{\mu\nu} = \nabla_\mu \hat{A}^a_\nu - \nabla_\nu \hat{A}^a_\mu + f^a_{bc} \hat{A}^b_\mu \hat{A}^c_\nu + \hat{A}^a_\rho T^\rho_{\mu\nu} + \frac{\hbar}{2} d^a_{bc} C_1\left(\hat{A}^b_\mu, \hat{A}^c_\nu\right) + \frac{\hbar^2}{4} f^a_{bc} C_2\left(\hat{A}^b_\mu, \hat{A}^c_\nu\right) + O\left(\hbar^3\right), \tag{3.17}$$

where we used the definition $C_n\left(\hat{A}^b, \hat{A}^c\right) = C_n\left(\hat{A}^b_\mu, \hat{A}^c_\nu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ with

$$C_1\left(\hat{A}^b_\mu, \hat{A}^c_\nu\right) = \left\{\hat{A}^b_\mu, \hat{A}^c_\nu\right\} = \theta^{\rho\sigma} \nabla_\rho \hat{A}^b_\mu \nabla_\sigma \hat{A}^c_\nu, \tag{3.18}$$

$$C_2\left(\hat{A}^b_\mu, \hat{A}^c_\nu\right) = \frac{1}{2} \theta^{\rho\sigma} \theta^{\lambda\tau} \nabla_\rho \nabla_\lambda \hat{A}^b_\mu \nabla_\sigma \nabla_\tau \hat{A}^c_\nu + \frac{1}{3}\left(\theta^{\rho\tau} \nabla_\tau \theta^{\sigma\lambda} + \frac{1}{2} \theta^{\sigma\tau} \theta^{\lambda\phi} T^\rho_{\tau\phi}\right) \times \left(\nabla_\rho \nabla_\sigma \hat{A}^b_\mu \nabla_\lambda \hat{A}^c_\nu - \nabla_\sigma \hat{A}^b_\mu \nabla_\rho \nabla_\lambda \hat{A}^c_\nu\right). \tag{3.19}$$

Under the gauge transformation (3.11) the curvature 2-form $\hat{F}$ transforms as

$$\hat{\delta}\hat{F} = i\left[\hat{\lambda}, \hat{F}\right]_{\star}, \tag{3.20}$$

where we used the Leibniz rule

$$\mathrm{d}\left(\hat{\alpha} \star \hat{\beta}\right) = \mathrm{d}\hat{\alpha} \star \hat{\beta} + (-1)^{|\alpha|}\hat{\alpha} \star \mathrm{d}\hat{\beta} \tag{3.21}$$

which we admit to be valid to all orders in $\hbar$. In terms of the components (3.20) gives

$$\hat{\delta}\hat{F}^a = f^a_{bc}\hat{F}^b\hat{\lambda}^c + \frac{\hbar}{2}d^a_{bc}C_1\left(\hat{F}^b, \hat{\lambda}^c\right) - \frac{\hbar^2}{4}f^a_{bc}C_2\left(\hat{F}^b, \hat{\lambda}^c\right) + O\left(\hbar^3\right), \tag{3.22}$$

In the zeroth order, the formula (3.22) reproduces therefore the result of the commutative gauge theory

$$\delta F^a_{\mu\nu} = f^a_{bc}F^b_{\mu\nu}\lambda^c \;\Leftrightarrow\; \delta F = i[\lambda, F]. \tag{3.23}$$

Using again the Leibniz rule, we obtain the deformed Bianchi identity

$$\mathrm{d}\hat{F} - i\left[\hat{A}, \hat{F}\right]_{\star} = 0. \tag{3.24}$$

If we apply the definition (3.12) of the star commutator, we obtain

$$\mathrm{d}\hat{F} + i\left[\hat{F}, \hat{A}\right] = \left[\frac{\hbar}{2}d^a_{bc}C_1\left(\hat{F}^b, \hat{A}^c\right) - \frac{\hbar^2}{4}f^a_{bc}C_2\left(\hat{F}^b, \hat{A}^c\right)\right]T_a + O\left(\hbar^3\right), \tag{3.25}$$

or in terms of components

$$\mathrm{d}\hat{F}^a - f^a_{bc}\hat{F}^b \wedge \hat{A}^c = \frac{\hbar}{2}d^a_{bc}C_1\left(\hat{F}^b, \hat{A}^c\right) - \frac{\hbar^2}{4}f^a_{bc}C_2\left(\hat{F}^b, \hat{A}^c\right) + O\left(\hbar^3\right). \tag{3.26}$$

We remark that in zeroth order we obtain from (3.25) the usual Bianchi identity

$$\mathrm{d}F + i[F, A] = 0. \tag{3.27}$$

In addition, if the gauge group is $U(1)$, the Bianchi identity (3.24) becomes

$$\mathrm{d}\hat{F} = \hbar C_1\left(\hat{A}, \hat{F}\right) + O\left(\hbar^3\right). \tag{3.28}$$

This result is also in accord with that of [24].

4 An illustrative example

As a very simple example we consider the Poincaré gauge theory to construct the manifold M. Then, suppose that we have the gauge fields ${e^a}_\mu$ and fix the gauge $\omega^{ab}_\mu = 0$ [31]. We define the connection coefficients

$$\Gamma^\rho_{\mu\nu} = {\bar{e}_a}^{\,\rho}\partial_\mu {e^a}_\nu, \tag{4.1}$$

where $\bar{e}_a{}^\rho$ denotes the inverse of $e^a{}_\mu$. Obviously, the connection Γ defined by these coefficients is not symmetric, i.e. $\Gamma^\rho_{\mu\nu} \neq \Gamma^\rho_{\nu\mu}$. Define then the torsion by formula

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}. \tag{4.2}$$

In order to simplify the calculation, we consider the case of spherical symmetry and choose the gauge fields $e^a{}_\mu$ as

$$e^a{}_\mu = \text{diag}\left(A, 1, 1, \frac{1}{A}\right), \quad \bar{e}_a{}^\mu = \text{diag}\left(\frac{1}{A}, 1, 1, A\right), \tag{4.3}$$

where $A = A(r)$ is a function depending only on the radial coordinate r. Then, denoting the spherical coordinates on M by $(x^\mu) = (r, \vartheta, \varphi, t)$, $\mu = 1, 2, 3, 0$, the non-null components of the connection coefficients are

$$\Gamma^0_{10} = -\frac{A'}{A}, \quad \Gamma^1_{11} = \frac{A'}{A}. \tag{4.4}$$

It is easy to see that the only non-null components of the torsion are

$$T^0_{01} = -T^0_{10} = \frac{A'}{A}. \tag{4.5}$$

Also, using the definitions (2.7) and (2.8) of the curvatures, we obtain

$$\tilde{R}^0_{101} = -\tilde{R}^0_{110} = \frac{AA'' - 2A'^2}{A^2}, \quad R^\lambda_{\mu\nu\rho} = 0 \tag{4.6}$$

and all other components of $\tilde{R}^\lambda_{\mu\nu\rho}$ vanish. In these expressions, we denote the first and second derivatives of $A(r)$ by A' and A'' respectively. The vanishing curvature $R^\lambda_{\mu\nu\rho}$ agrees with the constraint (2.20).

Introduce then the noncommutativity parameters $\theta^{\mu\nu}$ and suppose that we choose them so that only the components $\theta^{10} = -\theta^{01}$ are non-null. In addition, if we suppose that these parameters have the expression $\theta^{10} = -\theta^{01} = \frac{1}{A}$, then we have

$$\tilde{\nabla}_1\theta^{01} = -\tilde{\nabla}_1\theta^{10} = 0, \quad \nabla_1\theta^{01} = -\nabla_1\theta^{10} = \frac{A'}{A^2}. \tag{4.7}$$

This agrees with the constraint (2.14) that $\theta^{\mu\nu}$ is covariant constant under $\tilde{\nabla}$.

Finally, if we impose also the condition that the curvature $\tilde{R}^{\mu\nu}_{\rho\sigma} = \theta^{\mu\lambda}\tilde{R}^\nu_{\lambda\rho\sigma}$ vanishes (equivalent with (2.25) due to (2.4)), that implies $\nabla_\lambda\tilde{R}^{\mu\nu}_{\rho\sigma}$ vanishes too (2.21), then from (4.6) we obtain the following differential equation of the second order for the unknown function $A(r)$:

$$AA'' - 2A'^2 = 0. \tag{4.8}$$

The solutions of this equation is

$$A(r) = -\frac{1}{C_1 r + C_2}, \tag{4.9}$$

where C_1 and C_2 are two arbitrary constants of integration. Therefore, in our simple example, the conditions necessary to define a covariant star product on a symplectic manifold M completely determine its connection. In addition, it is very interesting to see that the covariant derivative of the torsion, defined as

$$\nabla_\mu T^\nu_{\rho\sigma} = \partial_\mu T^\nu_{\rho\sigma} + \Gamma^\nu_{\mu\lambda} T^\lambda_{\rho\sigma} - \Gamma^\lambda_{\mu\rho} T^\nu_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma} T^\nu_{\rho\lambda}, \tag{4.10}$$

has the following non-null components

$$\nabla_1 T^0_{01} = -\nabla_1 T^0_{10} = \frac{AA'' - 2A'^2}{A^2}. \tag{4.11}$$

Then, taking into account the equation (4.8), we conclude that the torsion is covariant constant, $\nabla_\mu T^\nu_{\rho\sigma} = 0$, a result which is in concordance with the condition (2.27).

5 Conclusions and discussions

We developed a noncommutative gauge theory by using a star product between differential forms on symplectic manifolds defined as in [14]. We followed the same way as in our recent paper [15], extending the results of the [14] to the case of Lie valued differential forms.

To simplify the calculations, we considered a space-time endowed only with torsion. It has been showed that, in order to satisfy the restrictions imposed by the associativity property of the covariant star product, the torsion of the space-time has to be covariant constant, $\nabla_\mu T^\nu_{\rho\sigma} = 0$. On the other hand, it has been argued that a covariant star product defined in the case when the space-time is a symplectic manifold endowed only with curvature is not possible. This is due to the restrictions imposed by the associativity property of the covariant star product which requires also the vanishing curvature. The corresponding connection is therefore flat symplectic and this reduces the applicability area of the covariant star product.

An illustrative example has been presented starting from the Poincaré gauge theory. Using the gauge fields $e^a{}_\mu$ and fixing the gauge $\omega^{ab}_\mu = 0$ [31] we defined the non-symmetric connection $\Gamma^\rho_{\mu\nu} = \bar{e}_a{}^\rho \partial_\mu e^a{}_\nu$. We deduced that, in this case, the conditions necessary to define a covariant star product on a symplectic manifold M completely determine its connection.

Some other possible applications of this covariant star product have been also analysed. First, it will be very important to generalize the Seiberg-Witten map to the case when the ordinary derivatives are replaced with covariant derivatives and the Moyal star product is replaced by the covariant one. Second, we can try to develop a noncommutative gauge theory of gravity considering the symplectic manifold M as the background space-time. For such a purpose, we have to verify if the non-commutative field equations do not impose too many restrictive conditions on the connection $\Gamma^\rho_{\mu\nu}$, in addition to those required by the existence of the covariant product. However, the problem of which gauge group we can choose remains unsolved. The Poincaré group can not be used because it does not close with respect to the star product. A possibility will be to choose the group $GL(2,\mathbb{C})$, but in this case we obtain a complex theory of gravitation [32, 33]. Another possibility is to consider the universal enveloping of Poincaré group, but this is infinite dimensional and we must find criteria to reduce the number of the degrees of freedom to a finite one. Some possible ideas are given for the case of $SU(N)$ or GUT theories in [34], where it is argued that the infinite number of parameters can in fact all be expressed in terms of right number of classical parameters and fields via the Seiberg-Witten maps.

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A Appendix

We verify the associativity property up to the second order in $\hbar^2$. For simplicity we denote the exterior product by $\alpha \wedge \beta = \alpha\beta$.

Introducing (2.22) into (2.24) we obtain successively

$$\big[\alpha\beta + \hbar C_1(\alpha,\beta) + \hbar^2 C_2(\alpha,\beta) + \cdots\big]\star\gamma = \alpha\star\big[\beta\gamma + \hbar C_1(\beta,\gamma) + \hbar^2 C_2(\beta,\gamma) + \cdots\big] \quad \text{(A.1)}$$

or

$$\begin{aligned} &(\alpha\beta)\gamma + \hbar\left[C_1(\alpha\beta,\gamma) + C_1(\alpha,\beta)\gamma\right] + \hbar^2\left[C_2(\alpha\beta,\gamma) + C_1(C_1(\alpha,\beta),\gamma) + C_2(\alpha,\beta)\gamma\right] + \cdots \\ &= \alpha(\beta\gamma) + \hbar\left[C_1(\alpha,\beta\gamma) + \alpha C_1(\beta,\gamma)\right] + \hbar^2\left[C_2(\alpha,\beta\gamma) + C_1(\alpha,C_1(\beta,\gamma)) + \alpha C_2(\beta,\gamma)\right] + \cdots \end{aligned}$$

Identifying the terms of different orders in $\hbar$ we obtain

$$(\alpha\beta)\gamma = \alpha(\beta\gamma), \quad \text{(A.2)}$$

$$C_1(\alpha\beta,\gamma) + C_1(\alpha,\beta)\gamma = C_1(\alpha,\beta\gamma) + \alpha C_1(\beta,\gamma), \quad \text{(A.3)}$$

$$C_2(\alpha\beta,\gamma) + C_1(C_1(\alpha,\beta),\gamma) + C_2(\alpha,\beta)\gamma = C_2(\alpha,\beta\gamma) + C_1(\alpha,C_1(\beta,\gamma)) + \alpha C_2(\beta,\gamma). \quad \text{(A.4)}$$

(A.2) is verified because the exterior product has this property.

Using (2.29), the (A.3) becomes

$$\theta^{\mu\nu}\left[\nabla_\mu(\alpha\beta)(\nabla_\nu\gamma) + (\nabla_\mu\alpha)(\nabla_\nu\beta)\gamma - (\nabla_\mu\alpha)\nabla_\nu(\beta\gamma) + \alpha(\nabla_\mu\beta)(\nabla_\nu\gamma)\right] = 0$$

that is satisfied due to the Leibniz rule $\nabla_\mu(\alpha\beta) = (\nabla_\mu\alpha)\beta + \alpha(\nabla_\mu\beta)$.

In order to verify (A.4) we write it as

$$\begin{aligned} \delta C_2(\alpha,\beta,\gamma) &\equiv C_2(\alpha,\beta\gamma) - C_2(\alpha,\beta)\gamma - C_2(\alpha\beta,\gamma) + \alpha C_2(\beta,\gamma) \\ &= \{\{\alpha,\beta\},\gamma\} - \{\alpha,\{\beta,\gamma\}\} \end{aligned} \quad \text{(A.5)}$$

We calculate the right-hand side of (A.5) first

$$\begin{aligned} &\{\{\alpha,\beta\},\gamma\} - \{\alpha,\{\beta,\gamma\}\} = (-1)^{|\alpha|(|\beta|+|\gamma|)}\{\beta,\{\gamma,\alpha\}\} \\ &= (-1)^{|\alpha|(|\beta|+|\gamma|)}\theta^{\mu\nu}(\nabla_\mu\beta)\nabla_\nu\left(\theta^{\rho\sigma}(\nabla_\rho\gamma)(\nabla_\sigma\alpha)\right) = -\theta^{\mu\nu}(\nabla_\nu\theta^{\rho\sigma})(\nabla_\rho\alpha)(\nabla_\mu\beta)(\nabla_\sigma\gamma) \\ &\quad + \theta^{\mu\nu}\theta^{\rho\sigma}\left[(\nabla_\mu\nabla_\rho\alpha)(\nabla_\nu\beta)(\nabla_\sigma\gamma) - (\nabla_\mu\alpha)(\nabla_\rho\beta)(\nabla_\sigma\nabla_\nu\gamma)\right], \end{aligned} \quad \text{(A.6)}$$

where the graded symmetry property (2.23) (for $n = 1$) and the graded Jacobi identity (2.19) of the Poisson bracket are used in the first equality, the expression (2.28) for the Poisson bracket in the second step, and the symmetry properties of $\theta^{\mu\nu}$ and the exterior product, $\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha$, in the last equality. Then we introduce the decomposition of the second covariant derivative of an arbitrary differential form as

$$\nabla_\mu \nabla_\rho \alpha = \frac{1}{2}\left\{\nabla_\mu, \nabla_\rho\right\}\alpha - \frac{1}{2}T^\lambda_{\mu\rho}\nabla_\lambda, \tag{A.7}$$

implied by (2.10) and (2.20), into (A.6). Finally, using the cyclic property (2.15) for the torsion, we obtain

$$\begin{aligned}\{\{\alpha,\beta\},\gamma\} - \{\alpha,\{\beta,\gamma\}\} = &-\left(\theta^{\mu\sigma}\nabla_\sigma\theta^{\nu\rho} + \frac{1}{2}\theta^{\nu\sigma}\theta^{\rho\lambda}T^\mu_{\sigma\lambda}\right)(\nabla_\nu\alpha)(\nabla_\mu\beta)(\nabla_\rho\gamma)\\ &+\frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}\left[\left\{\nabla_\mu,\nabla_\rho\right\}\alpha(\nabla_\nu\beta)(\nabla_\sigma\gamma) - (\nabla_\mu\alpha)(\nabla_\rho\beta)\left\{\nabla_\nu,\nabla_\sigma\right\}\gamma\right],\end{aligned} \tag{A.8}$$

Next, we calculate the left-hand side of (A.5), the Hochschild coboundary of C_2 (see [14] for details). First we calculate $C_2(\alpha,\beta\gamma) - C_2(\alpha,\beta)\gamma$ and $C_2(\alpha\beta,\gamma) - \alpha C_2(\beta,\gamma)$, then substracting them yields

$$\begin{aligned}\delta C_2(\alpha,\beta,\gamma) = &\frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}\left[(\nabla_\mu\nabla_\rho\alpha)2(\nabla_{(\nu}\beta\nabla_{\sigma)}\gamma) - 2(\nabla_{(\mu}\alpha\nabla_{\rho)}\beta)(\nabla_\nu\nabla_\sigma\gamma)\right]\\ &+\frac{1}{3}\left(\theta^{\mu\sigma}\nabla_\sigma\theta^{\nu\rho} + \frac{1}{2}\theta^{\nu\sigma}\theta^{\rho\lambda}T^\mu_{\sigma\lambda}\right)\left((\nabla_\rho\alpha)2(\nabla_{(\mu}\beta\nabla_{\nu)}\gamma) - 2(\nabla_{(\mu}\alpha\nabla_{\nu)}\beta)(\nabla_\rho\gamma)\right),\end{aligned} \tag{A.9}$$

where we denote $\nabla_{(\mu}\alpha\nabla_{\rho)}\beta = \frac{1}{2}\left[(\nabla_\mu\alpha)(\nabla_\rho\beta) + (\nabla_\rho\alpha)(\nabla_\mu\beta)\right]$. By using the symmetries of the factor $\theta^{\mu\nu}\theta^{\rho\sigma}$ and the cyclic relation implied by (2.16) and (2.15),

$$\sum_{(\mu,\nu,\rho)}\left(\theta^{\mu\sigma}\nabla_\sigma\theta^{\nu\rho} + \frac{1}{2}\theta^{\nu\sigma}\theta^{\rho\lambda}T^\mu_{\sigma\lambda}\right) = 0, \tag{A.10}$$

we find

$$\begin{aligned}\delta C_2(\alpha,\beta,\gamma) = &\frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}\left[\left\{\nabla_\mu,\nabla_\rho\right\}\alpha(\nabla_\nu\beta)(\nabla_\sigma\gamma) - (\nabla_\mu\alpha)(\nabla_\rho\beta)\left\{\nabla_\nu,\nabla_\sigma\right\}\gamma\right]\\ &-\left(\theta^{\mu\sigma}\nabla_\sigma\theta^{\nu\rho} + \frac{1}{2}\theta^{\nu\sigma}\theta^{\rho\lambda}T^\mu_{\sigma\lambda}\right)(\nabla_\nu\alpha)(\nabla_\mu\beta)(\nabla_\rho\gamma),\end{aligned} \tag{A.11}$$

This is the same result as in (A.8) and therefore we have verified the associativity of our star product to the second order in $\hbar^2$.

References

[1] R. J. Szabo, Quantum gravity, field theory and signatures of noncommutative spacetime, arXiv:0906.2913 [hep-th].

[2] F. Lizzi, The structure of space-time and noncommutative geometry, arXiv:0811.0268 [hep-th].

[3] V. O. Rivelles, Noncommutative field theories and gravity, Phys. Lett. B558 (2003) 191, hep-th/0212262

[4] D. V. Vassilevich, Diffeomorfism covariant star products and noncommutative gravity, Class. Quant. Grav. 26 (2009) 145010, arXiv:0904.3079 [hep-th].

[5] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, J. Wess, A gravity theory on noncommutative spaces, Class. Quant. Grav. 22 (2005) 3511, hep-th/0504183.

[6] P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess, Noncommutative geometry and gravity, Class. Quant. Grav. 23 (2006) 1883, hep-th/0510059.

[7] J. Polchinschi, String theory, Cambridge University Press, 1998.

[8] C. Rovelli, Quantum gravity, Cambridge University Press, 2004.

[9] N. Seiberg, E. Witten, String theory and noncommutative geometry, JHEP 09 (1999) 032, hep-th/9908142.

[10] L. Álvarez-Gaumé, F. Meyer, M. A. Vázquez-Mozo, Comments on noncommutative gravity, Nucl. Phys. B753 (2006) 92, hep-th/0605113.

[11] M. Chaichian, A. Tureanu, R. B. Zhang, X. Zhang, Riemannian geometry of noncommutative surfaces, J. Math. Phys. 49 (2008) 07351, hep-th/0612128.

[12] D. Wang, R. B. Zhang, X. Zhang, Quantum deformations of the Schwarzschild and Schwarzschild-de Sitter spacetimes, Class. Quant. Grav. 26 (2009) 085014, arXiv:0809.0614 [hep-th].

[13] R. B. Zhang, X. Zhang, Projective module description of embedded noncommutative spaces, arXiv:0810.2357 [hep-th].

[14] S. McCurdy, B. Zumino, Covariant star product for exterior differential forms on symplectic manifolds, arXiv:0910.0459 [hep-th].

[15] M. Chaichian, A. Tureanu, G. Zet, Gauge field theories with covariant star product, JHEP 07 (2009) 084, arXiv:0905.0608 [hep-th].

[16] A. Einstein, Math. Ann. 102 (1930) 685.

[17] T. G. Lucas, Y. N. Obukhov, J. G. Pereira, Regularizing role of teleparallelism, arXiv:0909.2418 [hep-th].

[18] R. Aldrovandi, T. G. Lucas, J. G. Pereira, Inertia and gravitation in teleparallel gravity, arXiv:0812.0034 [hep-th].

[19] V. C. Andrade, A. L. Barbosa, J. G. Pereira, Gravity and duality symmetry, Int. J. Mod. Phys. D14 (2005) 1635, gr-qc/0501037.

[20] E. Langmann, R. J. Szabo, Teleparallel gravity and dimensional reduction of noncommutative gauge theory, Phys. Rev. D64 (2001) 104019, hep-th/0105094.

[21] J. M. Nester, Positive energy via teleparallel Hamiltonian, Int. J. Mod. Phys. A4 (1989) 1755.

[22] E. W. Mielke, Generating function for Ashtekar's complex variables in general relativity, Ann. Phys. 219 (1992) 78.

[23] S. Okubo, BRST operator for space with zero curvature but with non-zero toersion tensor, Gen. Rel. Grav. 23 (1991) 599.

[24] C.-S Chu, P.-M. Ho, Poisson algebra of differential forms, Int. J. Mod. Phys. 12 (1997) 5573, q-alg/9612031.

[25] S. McCurdy, A. Tagliaferro, B. Zumino, The star product for differential forms on symplectic manifolds, arXiv:0809.4717v1 [hep-th].

[26] A. Tagliaferro, A star product for differential forms on symplectic manifolds, arXiv:0809.4717v2 [hep-th].

[27] M. Blagojević, Gravitation and gauge symmetries, Institute of Physics Publishing, Bristol and Philadelphia, 2002.

[28] M. Chaichian, A. Tureanu, G. Zet, Corrections to Schwarzschild solution in noncommutative gauge theory of gravity, Phys. Lett. B660 (2008) 573, arXiv:0710.2075 [hep-th].

[29] M. Chaichian, A. Tureanu, M. Setare, G. Zet, On black holes and cosmological constant in noncommutative gauge theory of gravitation, JHEP 0804 (2008) 064, arXiv:07011.4546 [hep-th].

[30] H. García-Compeán, O. Obregón, C. Ramírez, M. Sabido, Noncommutative self-dual gravity, Phys. Rev. D68 (2003) 044015, hep-th/0302180.

[31] G. Zet, Schwarzschild solution on a space-time with torsion, gr-qc/0308078.

[32] A. Stern, Particle classification and dynamics in $GL(2,C)$ gravity, Phys. Rev. D79 (2009) 105017, arXiv0903.0882 [hep-th].

[33] A. Chamseddine, $SL(2,C)$ gravity with complex vierbein and its noncommutative extension, Phys. Rev. D69 (2004) 024015, hep-th/0309166.

[34] X. Calmet, B. Jurco, P. Schupp, J. Wess, M. Wohlgenannt, The standard model on non-commutative space-time, Eur. Phys. J. C23 (2002) 363, hep-th/0111115.