

Spinning binary waveforms via PN expansion: Equal-mass case

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Complete expressions of time-domain gravitational waveforms for spinning binary inspirals via the post-Newtonian (PN) approximation would require determination of the phase, amplitude, inclination angle, precession phase and spin vectors as well as the knowledge of the order coefficients for the PN expansion terms. These quantities are determined by solving simultaneously the spin-precession equations, the evolution equation for the Newtonian angular momentum, and the equation for the orbital frequency. For the spinning binaries with equal masses, determination of these quantities can be done fully analytically, by taking advantage of the equal mass symmetry, therefore by simplifying those equations to solve. We provide the analytical results through 1.5 PN order which includes spin-orbit interactions.

Arun et. al(**author?**) [1] provides the “Ready-to-use” time-domain gravitational waveforms for spinning binary inspirals in Post-Newtonian expansion to 1.5 order. Their formulations, however, have yet to be further specified, depending on the configurations of binaries, such as the mass ratio and spin alignment with the Newtonian angular momentum. In order to make use of their theoretical waveforms in designing our wave templates via the IIR method, it is necessary to solve the equations governing these configurations and to fully specify the waveforms with all known parameters.

The set of equations to solve are the following:

(i) the spin-precession equations

$$\dot{\mathbf{S}}_1 = \boldsymbol{\Omega}_1 \times \mathbf{S}_1, \quad (1)$$

$$\dot{\mathbf{S}}_2 = \boldsymbol{\Omega}_2 \times \mathbf{S}_2, \quad (2)$$

where at 1.5 PN order

$$\boldsymbol{\Omega}_{1,2} = M^{2/3} \omega_{\text{orb}}^{5/3} \left(\frac{3}{4} + \frac{\nu}{2} \mp \frac{3}{4} \delta \right) \hat{\mathbf{L}}_N, \quad (3)$$

with

$$M = M_1 + M_2, \quad (4)$$

$$\nu = \frac{M_1 M_2}{M^2}, \quad (5)$$

$$\delta = \frac{M_1 - M_2}{M}. \quad (6)$$

(ii) the evolution equation for the Newtonian angular momentum

$$\dot{\hat{\mathbf{L}}}_N = -\frac{v}{\nu M^2} (\dot{\mathbf{S}}_1 + \dot{\mathbf{S}}_2), \quad (7)$$

where

$$v \equiv (M \omega_{\text{orb}})^{1/3}. \quad (8)$$

(iii) the equation for the orbital frequency

$$\begin{aligned} \frac{\dot{\omega}_{\text{orb}}}{\omega_{\text{orb}}^2} = & \frac{96}{5} \nu v^5 \left\{ 1 - \left(\frac{743}{336} + \frac{11}{4} \nu \right) v^2 \right. \\ & \left. + \left[\left(\frac{19}{3} \nu - \frac{113}{12} \right) \chi_s \cdot \hat{\mathbf{L}}_N - \frac{113}{12} \delta \chi_a \cdot \hat{\mathbf{L}}_N \right] v^3 + 4\pi v^3 \right\}, \end{aligned} \quad (9)$$

where

$$\chi_s = \frac{1}{2}(\chi_1 + \chi_2), \quad (10)$$

$$\chi_a = \frac{1}{2}(\chi_1 - \chi_2) \quad (11)$$

with the normalized spin vectors

$$\chi_n = \frac{\mathbf{S}_n}{M_n^2}, \quad n = 1, 2, \quad (12)$$

so that $|\chi_n| \leq 1$ for objects that obey the Kerr bound on rotational angular momentum.

I. SPECIFYING THE WAVEFORMS FOR EQUAL-MASS SPINNING BINARIES

A. Configurations for the equal-mass case

One interesting case might be a spinning binary with equal masses, where one can take advantage of the relatively simple spinning configurations due to mass symmetry, thus can analyze the spin effects on the binary such as precession and change in wave frequency more easily. For an equal-mass binary, one has

$$M_1 = M_2 = \frac{M}{2}, \quad (13)$$

therefore

$$\nu = \frac{1}{4}, \quad (14)$$

$$\delta = 0, \quad (15)$$

and

$$\chi_n = \frac{4\mathbf{S}_n}{M^2}, \quad n = 1, 2, \quad (16)$$

thus,

$$\chi_s = \frac{2\mathbf{S}}{M^2}, \quad (17)$$

$$\chi_a = \frac{2\bar{\mathbf{S}}}{M^2}, \quad (18)$$

with the definitions

$$\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2, \quad (19)$$

$$\bar{\mathbf{S}} \equiv \mathbf{S}_1 - \mathbf{S}_2. \quad (20)$$

With this simplification, one reduces the set of equations (i), (ii) and (iii) above to the following: (i') the spin-precession equations

$$\dot{\mathbf{S}}_1 = \boldsymbol{\Omega} \times \mathbf{S}_1, \quad (21)$$

$$\dot{\mathbf{S}}_2 = \boldsymbol{\Omega} \times \mathbf{S}_2, \quad (22)$$

with

$$\boldsymbol{\Omega} = \frac{7}{8}M^{2/3}\omega_{\text{orb}}^{5/3}\hat{\mathbf{L}}_N. \quad (23)$$

Further, we may combine Eqs. (21) and (22) via Eq. (19),

$$\dot{\mathbf{S}} = \boldsymbol{\Omega} \times \mathbf{S}, \quad (24)$$

which will be used throughout the rest of the analysis instead of Eqs. (21) and (22).
(ii') the evolution equation for the Newtonian angular momentum

$$\dot{\hat{\mathbf{L}}}_N = -\frac{4v}{M^2} \dot{\mathbf{S}} \quad (25)$$

via Eq. (19).

(iii') the equation for the orbital frequency

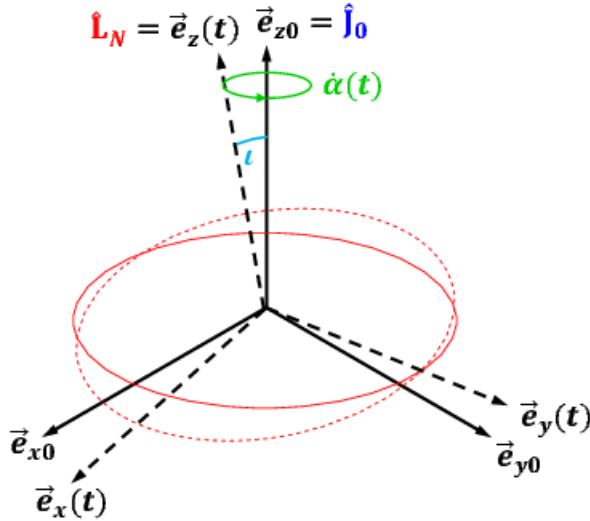
$$\frac{\dot{\omega}_{\text{orb}}}{\omega_{\text{orb}}^2} = \frac{24}{5} v^5 \left[1 - \frac{487}{168} v^2 + \left(4\pi - \frac{47}{3} \frac{\mathbf{S} \cdot \hat{\mathbf{L}}_N}{M^2} \right) v^3 \right]. \quad (26)$$

B. Solving the configuration-equations in the limit $S \ll L$

In order to solve the equations above effectively, one needs to prescribe the time-varying coordinates with respect to the fixed Cartesian coordinates, in which the Newtonian angular momentum becomes the same as that of a non-spinning binary. One such kind of the frame of coordinates may be written as

$$\begin{bmatrix} \vec{e}_x(t) \\ \vec{e}_y(t) \\ \vec{e}_z(t) \end{bmatrix} = \begin{bmatrix} \cos \iota \cos \alpha & \cos \iota \sin \alpha & -\sin \iota \\ -\sin \alpha & \cos \alpha & 0 \\ \sin \iota \cos \alpha & \sin \iota \sin \alpha & \cos \iota \end{bmatrix} \begin{bmatrix} \vec{e}_{x0} \\ \vec{e}_{y0} \\ \vec{e}_{z0} \end{bmatrix}, \quad (27)$$

where ι and α represent the angle of inclination (due to precession) and the phase of precession, respectively [3].



$$\vec{e}_x(t) = \cos \iota \cos \alpha \vec{e}_{x0} + \cos \iota \sin \alpha \vec{e}_{y0} - \sin \iota \vec{e}_{z0}$$

$$\vec{e}_y(t) = -\sin \alpha \vec{e}_{x0} + \cos \alpha \vec{e}_{y0}$$

$$\vec{e}_z(t) = \sin \iota \cos \alpha \vec{e}_{x0} + \sin \iota \sin \alpha \vec{e}_{y0} + \cos \iota \vec{e}_{z0}$$

The total angular momentum

$$\mathbf{J} = \mathbf{J}_0 = \mathbf{L}_N + \mathbf{S} \quad (28)$$

is assumed to be conserved (as radiation reaction is not included throughout our analysis) and is directed along the fixed z -axis in our analysis. We set $\hat{\mathbf{J}}_0 = \vec{e}_{z0}$ and $\hat{\mathbf{L}}_N = \vec{e}_z(t)$, and via the relation of Eq. (27), Eq. (28) may be

rewritten,

$$J_0 \left(-\sin \iota \vec{e}_x(t) + \cos \iota \hat{\mathbf{L}}_N \right) = L_N \hat{\mathbf{L}}_N + S \hat{\mathbf{S}}, \quad (29)$$

where J_0 , L_N and S denote the magnitudes of \mathbf{J}_0 , \mathbf{L}_N and \mathbf{S} , respectively. From this one defines

$$\cos \beta \equiv \hat{\mathbf{S}} \cdot \hat{\mathbf{L}}_N = \frac{J_0 \cos \iota - L_N}{S}, \quad (30)$$

then finds easily

$$\cos \iota = \frac{L_N + S \cos \beta}{J_0}. \quad (31)$$

Also, from Eqs. (28) and (30) we have

$$J_0^2 = L_N^2 + 2L_N S \cos \beta + S^2. \quad (32)$$

1. The angle of inclination

According to Ref.(**author?**) [1], in the limit $S \ll L$ the angle ι can be considered a 0.5 PN correction, and taking advantage of this, we can reduce the lengthy expression for the GW polarizations to a much more compact form. Within this scheme, one finds out of Eqs. (31) and (32)

$$\begin{aligned} \cos \iota &= \frac{1 + \cos \beta \left(\frac{S}{L_N} \right)}{\sqrt{1 + 2 \cos \beta \left(\frac{S}{L_N} \right) + \left(\frac{S}{L_N} \right)^2}} \\ &\xrightarrow{S \ll L_N} 1 - \frac{1}{2} \sin^2 \beta \left(\frac{S}{L_N} \right)^2 + \mathcal{O} \left((S/L_N)^3 \right). \end{aligned} \quad (33)$$

Now, in order to specify the angle ι , we need to determine all the quantities involved in Eq. (33). First, the magnitude of the Newtonian angular momentum can be replaced by the leading order expression in ω_{orb} ,

$$L_N = \nu M^{5/3} \omega_{\text{orb}}^{-1/3} = \frac{1}{4} M^2 v^{-1}, \quad (34)$$

where the latter expression is obtained using Eqs. (8) and (14). Next, from Eq. (24), one finds that the magnitude of the total spin angular momentum S is constant (so is χ_s) for our equal mass binary:

$$\frac{1}{2} \frac{d}{dt} (S^2) = \mathbf{S} \cdot \dot{\mathbf{S}} = \mathbf{S} \cdot (\boldsymbol{\Omega} \times \mathbf{S}) = 0. \quad (35)$$

From this fact and from Eq. (30) together with Eqs. (23), (24) and (25), one also finds that the angle β is constant for the equal mass binary:

$$\frac{d}{dt} (\cos \beta) = \dot{\hat{\mathbf{S}}} \cdot \hat{\mathbf{L}}_N + \hat{\mathbf{S}} \cdot \dot{\hat{\mathbf{L}}}_N = \frac{\dot{\mathbf{S}} \cdot \hat{\mathbf{L}}_N + \mathbf{S} \cdot \dot{\hat{\mathbf{L}}}_N}{S} = -\frac{4v}{M^2} \frac{\mathbf{S} \cdot \dot{\mathbf{S}}}{S} = 0. \quad (36)$$

Then via Eqs. (33), (34), (35) and (36), one can finally specify

$$\cos \iota = 1 - \frac{8S^2 \sin^2 \beta}{M^4} v^2 + \mathcal{O}(v^3) \quad \text{if } S \ll L. \quad (37)$$

From this we may also infer

$$\iota \approx \frac{4S \sin \beta}{M^2} v \quad \text{if } S \ll L. \quad (38)$$

2. The orbital frequency

The equation for the orbital frequency, Eq. (26), can be integrated in a straightforward manner. First, we rewrite it using Eqs. (8) and (30),

$$\dot{v} = \frac{8}{5M} v^9 \left[1 - \frac{487}{168} v^2 + \left(4\pi - \frac{47}{3} \frac{S \cos \beta}{M^2} \right) v^3 \right]. \quad (39)$$

Integrating this with respect to v , we obtain

$$\Theta = \frac{1}{(2v)^8} \left[1 + \frac{487}{126} v^2 + \left(-\frac{32\pi}{5} + \frac{376}{15} \frac{S \cos \beta}{M^2} \right) v^3 \right], \quad (40)$$

where

$$\Theta \equiv \frac{t_c - t}{20M}, \quad (41)$$

and t_c denotes the instance of coalescence, at which the frequency tends to infinity (the Post-Newtonian method breaks down well before this point) (author?) [2]. Now, one can invert Eq. (40) and solve it for v ,

$$v = \frac{1}{2} \Theta^{-1/8} \left[1 + \frac{487}{4032} \Theta^{-1/4} + \left(-\frac{\pi}{10} + \frac{47}{120} \frac{S \cos \beta}{M^2} \right) \Theta^{-3/8} + \mathcal{O} \left(\frac{1}{\Theta^{1/2}} \right) \right]. \quad (42)$$

Via Eq. (8), one finds further

$$\omega_{\text{orb}} = \frac{v^3}{M} = \frac{1}{8M} \Theta^{-3/8} \left[1 + \frac{487}{1344} \Theta^{-1/4} + \left(-\frac{3\pi}{10} + \frac{47}{40} \frac{S \cos \beta}{M^2} \right) \Theta^{-3/8} + \mathcal{O} \left(\frac{1}{\Theta^{1/2}} \right) \right]. \quad (43)$$

3. The precession frequency

The precession frequency $\dot{\alpha}$ also needs to be determined. Combining Eqs. (24) and (25), and via Eqs. (23), (29) and (34) we find

$$-\frac{7}{8} J_0 M^{2/3} \omega_{\text{orb}}^{5/3} \sin \iota \vec{e}_y(t) = -\frac{1}{4} M^{5/3} \omega_{\text{orb}}^{-1/3} [i \vec{e}_x(t) + \dot{\alpha} \sin \iota \vec{e}_y(t)]. \quad (44)$$

Here, the $\vec{e}_x(t)$ term being absent from the left-hand side can be justified by comparison of i and $\dot{\alpha}$ on the right-hand side of Eq. (44). To do so, we compare the $\vec{e}_y(t)$ terms on the both sides first to find

$$\dot{\alpha} = \frac{7J_0}{2M} \omega_{\text{orb}}^2 = \frac{7J_0}{2M^3} v^6. \quad (45)$$

Combining this with Eqs. (32) and (34), we have

$$\dot{\alpha} = \frac{7}{8M} v^5 \left[1 + \frac{4S \cos \beta}{M^2} v + \mathcal{O}(v^2) \right], \quad (46)$$

in the limit $S \ll L$. Now, from Eqs. (38) and (39) we find

$$i \approx \frac{32S \sin \beta}{5M^3} v^9. \quad (47)$$

Evidently, this quantity is much smaller than $\dot{\alpha}$ therefore can be ignored in our analysis. By Eq. (42) we specify Eq. (46) further

$$\dot{\alpha} = \frac{7}{256M} \Theta^{-5/8} \left[1 + \frac{2S \cos \beta}{M^2} \Theta^{-1/8} + \mathcal{O} \left(\frac{1}{\Theta^{1/4}} \right) \right]. \quad (48)$$

4. The spin vectors

As shown by Eq. (35) above, S , the magnitude of the total spin \mathbf{S} is constant. So is χ_s due to Eq. (17).

First, the components of \mathbf{S} in the basis $\{\vec{e}_x(t), \vec{e}_y(t), \vec{e}_z(t)(=\hat{\mathbf{L}}_N)\}$ can be found by solving Eq. (24). To do so, we insert

$$\mathbf{S} = S^{x(t)}\vec{e}_x(t) + S^{y(t)}\vec{e}_y(t) + S^{z(t)}\hat{\mathbf{L}}_N \quad (49)$$

into the equation. We have then,

$$S^{x(t)} = \frac{-S^{z(t)} \sin \iota \dot{\alpha}}{\cos \iota \dot{\alpha} - \frac{7}{8}M^{2/3}\omega_{\text{orb}}^{5/3}}, \quad (50)$$

$$S^{y(t)} = \frac{S^{z(t)} i}{\cos \iota \dot{\alpha} - \frac{7}{8}M^{2/3}\omega_{\text{orb}}^{5/3}}, \quad (51)$$

where the contribution from $S^{x(t)}\vec{e}_x(t) + S^{y(t)}\vec{e}_y(t) + S^{z(t)}\hat{\mathbf{L}}_N$ of $\dot{\mathbf{S}}$ on the left-hand side of Eq. (24) has been disregarded since its magnitude is much smaller than that of $S^{x(t)}\dot{\vec{e}}_x(t) + S^{y(t)}\dot{\vec{e}}_y(t) + S^{z(t)}\dot{\hat{\mathbf{L}}}_N$. From Eqs. (30), (35) and (36) above, we find

$$S^{z(t)} = S \cos \beta = \text{const.} \quad (52)$$

Plugging Eqs. (37), (46), (47) and (52) into Eqs. (50) and (51), and using Eq. (8), we obtain

$$S^{x(t)} \approx -S \sin \beta, \quad (53)$$

$$S^{y(t)} \approx \frac{64}{35}S \sin \beta v^3 \approx 0. \quad (54)$$

Then via Eq. (27) $[S^{x(t)}, S^{y(t)}, S^{z(t)}]$ transforms into $[S^x, S^y, S^z]$ in the basis $\{\vec{e}_{x0}, \vec{e}_{y0}, \vec{e}_{z0}\}$:

$$S^x \approx S(\cos \beta \sin \iota - \sin \beta \cos \iota) \cos \alpha \approx -S \sin \beta \cos \alpha, \quad (55)$$

$$S^y \approx S(\cos \beta \sin \iota - \sin \beta \cos \iota) \sin \alpha \approx -S \sin \beta \sin \alpha, \quad (56)$$

$$S^z \approx S(\sin \beta \sin \iota + \cos \beta \cos \iota) \approx S \cos \beta. \quad (57)$$

Alternatively, we may express

$$\chi_s^x = -\chi_s \sin \beta \cos \alpha, \quad (58)$$

$$\chi_s^y = -\chi_s \sin \beta \sin \alpha, \quad (59)$$

$$\chi_s^z = \chi_s \cos \beta. \quad (60)$$

We can determine the components of $\bar{\mathbf{S}} = \mathbf{S}_1 - \mathbf{S}_2$ in the basis $\{\vec{e}_{x0}, \vec{e}_{y0}, \vec{e}_{z0}\}$ in a similar manner. First, one can show that the \bar{S} , magnitude of $\bar{\mathbf{S}}$ is also constant: so is χ_α due to Eq. (18). Subtracting Eq. (22) from Eq. (21), we have

$$\dot{\bar{\mathbf{S}}} = \boldsymbol{\Omega} \times \bar{\mathbf{S}}. \quad (61)$$

One finds trivially then,

$$\frac{1}{2} \frac{d}{dt} (\bar{S}^2) = \bar{\mathbf{S}} \cdot \dot{\bar{\mathbf{S}}} = \bar{\mathbf{S}} \cdot (\boldsymbol{\Omega} \times \bar{\mathbf{S}}) = 0. \quad (62)$$

In order to determine $[\bar{S}^{x(t)}, \bar{S}^{y(t)}, \bar{S}^{z(t)}]$ in the basis $\{\vec{e}_x(t), \vec{e}_y(t), \hat{\mathbf{L}}_N\}$, similarly we insert

$$\bar{\mathbf{S}} = \bar{S}^{x(t)}\vec{e}_x(t) + \bar{S}^{y(t)}\vec{e}_y(t) + \bar{S}^{z(t)}\hat{\mathbf{L}}_N \quad (63)$$

into Eq. (61) and find

$$\bar{S}^{x(t)} = \frac{-\bar{S}^{z(t)} \sin \iota \dot{\alpha}}{\cos \iota \dot{\alpha} - \frac{7}{8}M^{2/3}\omega_{\text{orb}}^{5/3}}, \quad (64)$$

$$\bar{S}^{y(t)} = \frac{\bar{S}^{z(t)} i}{\cos \iota \dot{\alpha} - \frac{7}{8}M^{2/3}\omega_{\text{orb}}^{5/3}}, \quad (65)$$

Also, we define

$$\cos \bar{\beta} \equiv \frac{\bar{\mathbf{S}} \cdot \hat{\mathbf{L}}_N}{\bar{S}}, \quad (66)$$

and have

$$\bar{S}^{z(t)} = \bar{S} \cos \bar{\beta}. \quad (67)$$

Now, substituting Eqs. (37), (46), (47) and (67) into Eqs. (64) and (65), and using Eq. (8), we find

$$\bar{S}^{x(t)} \approx -\frac{\bar{S} \cos \bar{\beta} \sin \beta}{\cos \beta}, \quad (68)$$

$$\bar{S}^{y(t)} \approx \frac{64 S \cos \bar{\beta} \sin \beta}{35 \cos \beta} v^3 \approx 0. \quad (69)$$

However, the magnitude \bar{S} calculated by means of Eqs. (67), (68) and (69) shows

$$\bar{S} \approx \bar{S} \frac{\cos \bar{\beta}}{\cos \beta}, \quad (70)$$

and we easily find

$$\cos \bar{\beta} \approx \cos \beta = \text{const.} \quad (71)$$

Then we may write

$$\bar{S}^{x(t)} \approx -\bar{S} \sin \beta, \quad (72)$$

$$\bar{S}^{y(t)} \approx 0, \quad (73)$$

$$\bar{S}^{z(t)} \approx \bar{S} \cos \beta. \quad (74)$$

The rest of procedure to transform $[\bar{S}^{x(t)}, \bar{S}^{y(t)}, \bar{S}^{z(t)}]$ into $[\bar{S}^x, \bar{S}^y, \bar{S}^z]$ in the basis $\{\vec{e}_{x0}, \vec{e}_{y0}, \vec{e}_{z0}\}$ is the same as the above. We finally have

$$\bar{S}^x \approx -\bar{S} \sin \beta \cos \alpha, \quad (75)$$

$$\bar{S}^y \approx -\bar{S} \sin \beta \sin \alpha, \quad (76)$$

$$\bar{S}^z \approx \bar{S} \cos \beta. \quad (77)$$

Or alternatively,

$$\chi_a^x \approx -\chi_a \sin \beta \cos \alpha, \quad (78)$$

$$\chi_a^y \approx -\chi_a \sin \beta \sin \alpha, \quad (79)$$

$$\chi_a^z \approx \chi_a \cos \beta. \quad (80)$$

C. Determination of the total phase and the amplitude factor

1. The total phase

Arun et. al(**author?**) [1] defines the orbital separation vector $\hat{\mathbf{n}}(t)$, which is set to lie along $\vec{e}_x(t)$ at initial time, i.e., $\hat{\mathbf{n}}(t=0) = \vec{e}_x(t=0)$, and rotates on the plane spanned by $\vec{e}_x(t)$ and $\vec{e}_y(t)$ by the cumulative angle $\Phi(t)$. Then one may write down the following two orthogonal vectors:

$$\hat{\mathbf{n}}(t) = \vec{e}_x(t) \cos \Phi(t) + \vec{e}_y(t) \sin \Phi(t), \quad (81)$$

$$\hat{\lambda}(t) = -\vec{e}_x(t) \sin \Phi(t) + \vec{e}_y(t) \cos \Phi(t). \quad (82)$$

We see that $\Phi(t)$ is the phase measuring how $\hat{\mathbf{n}}(t)$ has rotated relative to the vector $\vec{e}_x(t)$. For our precessing binary, however, $\vec{e}_x(t)$ is itself rotating about $\hat{\mathbf{L}}_N (= \vec{e}_z(t))$, which is associated with the angles ι and α (see Eq. (27)).

Therefore the total rotation of $\hat{\mathbf{n}}(t)$ about $\hat{\mathbf{L}}_N$ should be a combination of a rotation of $\hat{\mathbf{n}}(t)$ with respect to the comoving basis $\vec{e}_x(t)$ and $\vec{e}_y(t)$, which is parametrized by $\Phi(t)$ and a rotation of the basis due to the precession, which is parametrized by ι and α . Ref.(author?) [1] shows this in the following way. In the basis $\{\hat{\mathbf{n}}, \hat{\lambda}, \hat{\mathbf{L}}_N\}$, we introduce an ‘‘orbital-like’’ frequency ω_{orb} , which is defined as $\omega_{\text{orb}} = (\mathbf{v} \cdot \hat{\lambda})/r$. Then one may write

$$\mathbf{v} = \dot{r}\hat{\mathbf{n}} + r\omega_{\text{orb}}\hat{\lambda}. \quad (83)$$

Now, by means of Eqs. (27), (81) and (82) one can show

$$\dot{\hat{\mathbf{n}}} = \left(\dot{\Phi} + \cos \iota \dot{\alpha} \right) \hat{\lambda} - (i \cos \Phi + \sin \iota \sin \Phi \dot{\alpha}) \hat{\mathbf{L}}_N. \quad (84)$$

However, by imposing $\hat{\mathbf{L}}_N = \hat{\mathbf{n}} \times \mathbf{v} / |\hat{\mathbf{n}} \times \mathbf{v}| = \hat{\mathbf{n}} \times \dot{\hat{\mathbf{n}}} / |\hat{\mathbf{n}} \times \dot{\hat{\mathbf{n}}}|$, one finds that the term proportional to $\hat{\mathbf{L}}_N$ in Eq. (84) must be zero. Thus, we have $\dot{\hat{\mathbf{n}}} \equiv (\mathbf{v} \cdot \hat{\lambda})\hat{\lambda}$, and by identifying this with Eq. (84) via Eq. (83) and , we obtain

$$\omega_{\text{orb}} = \dot{\Phi} + \cos \iota \dot{\alpha}, \quad (85)$$

which may be now interpreted as the angular velocity with which $\hat{\mathbf{n}}$ rotates about $\hat{\mathbf{L}}_N$. The phase $\Phi(t)$ is then the integral

$$\Phi(t) = \int_0^t [\omega_{\text{orb}}(t') - \cos \iota(t')\dot{\alpha}(t')] dt'. \quad (86)$$

By plugging Eqs. (37), (43) and (48) into Eq. (86), and using Eqs. (41) and (42), one finally computes $\Phi(t)$ in the limit $S \ll L$:

$$\Phi(t) = -4\Theta^{5/8} \left[1 + \left(\frac{2435}{4032} - p_S \frac{35}{96} \right) \Theta^{-1/4} + \left(-\frac{3\pi}{4} + p_S \frac{59\chi_s \cos \beta}{64} \right) \Theta^{-3/8} + \mathcal{O} \left(\frac{1}{\Theta^{1/2}} \right) \right], \quad (87)$$

where $p_S = 0$ for the non-spinning case and $p_S = 1$ for the spinning case, and S has been replaced by $\frac{1}{2}M^2\chi_s$ via Eq. (17).

Also, we need a separate expression for the phase of precession. Integrating Eq. (48) with respect to t via Eq. (41), and using Eq. (17), we obtain

$$\alpha(t) = -p_S \frac{35}{24} \Theta^{3/8} \left[1 + \frac{3\chi_s \cos \beta}{2} \Theta^{-1/8} + \mathcal{O} \left(\frac{1}{\Theta^{1/4}} \right) \right], \quad (88)$$

where $p_S = 0$ for the non-spinning case and $p_S = 1$ for the spinning case.

2. The amplitude factor

Ref.(author?) [1] gives the expressions for the waveform polarizations in the following form:

$$h_{+, \times} = \frac{2M\nu v^2}{D_L} \left[H_{+, \times}^{(0)} + H_{+, \times}^{(1/2)} + H_{+, \times}^{(1/2, \text{SO})} + H_{+, \times}^{(1)} + H_{+, \times}^{(1, \text{SO})} + H_{+, \times}^{(3/2)} + H_{+, \times}^{(3/2, \text{SO})} \right]. \quad (89)$$

In our equal-mass case ($\nu = 1/4$), the amplitude factor for each PN group will then be determined by means of Eq. (42)

$$F^{(0)} = \frac{Mv^2}{2D_L} = \frac{M}{8D_L} \Theta^{-1/4}, \quad (90)$$

$$F^{(1/2)} = \frac{Mv^3}{2D_L} = \frac{M}{16D_L} \Theta^{-3/8}, \quad (91)$$

$$F^{(1)} = \frac{Mv^4}{2D_L} = \frac{M}{32D_L} \Theta^{-1/2} \left(1 + \frac{487}{1008} \Theta^{-1/4} \right), \quad (92)$$

$$F^{(3/2)} = \frac{Mv^5}{2D_L} = \frac{M}{64D_L} \Theta^{-5/8} \left[1 + \frac{2435}{4032} \Theta^{-1/4} + \left(-\frac{\pi}{2} + p_S \frac{47\chi_s \cos \beta}{48} \right) \Theta^{-3/8} \right], \quad (93)$$

where $p_S = 0$ for the non-spinning case and $p_S = 1$ for the spinning case, and in the last equation S has been replaced by $\frac{1}{2}M^2\chi_s$ via Eq. (17).

SUMMARY

1. The total phase

$$\Phi(t) = -4\Theta^{5/8} \left[1 + \left(\frac{2435}{4032} - p_S \frac{35}{96} \right) \Theta^{-1/4} + \left(-\frac{3\pi}{4} + p_S \frac{59\chi_s \cos \beta}{64} \right) \Theta^{-3/8} + \mathcal{O} \left(\frac{1}{\Theta^{1/2}} \right) \right]$$

2. The precession phase

$$\alpha(t) = -p_S \frac{35}{24} \Theta^{3/8} \left[1 + \frac{3\chi_s \cos \beta}{2} \Theta^{-1/8} + \mathcal{O} \left(\frac{1}{\Theta^{1/4}} \right) \right]$$

3. Inclination angle

$$\iota \approx 2\chi_s \sin \beta v \quad \text{if } S \ll L$$

4. The spin vectors

$$\begin{bmatrix} \chi_s^x \\ \chi_s^y \\ \chi_s^z \end{bmatrix} = \begin{bmatrix} -\chi_s \sin \beta \cos \alpha \\ -\chi_s \sin \beta \sin \alpha \\ \chi_s \cos \beta \end{bmatrix}$$

$$\begin{bmatrix} \chi_a^x \\ \chi_a^y \\ \chi_a^z \end{bmatrix} = \begin{bmatrix} -\chi_a \sin \beta \cos \alpha \\ -\chi_a \sin \beta \sin \alpha \\ \chi_a \cos \beta \end{bmatrix}$$

5. The amplitude factors

$$\begin{aligned} F^{(0)} &= \frac{Mv^2}{2D_L} = \frac{M}{8D_L} \Theta^{-1/4} \\ F^{(1/2)} &= \frac{Mv^3}{2D_L} = \frac{M}{16D_L} \Theta^{-3/8} \\ F^{(1)} &= \frac{Mv^4}{2D_L} = \frac{M}{32D_L} \Theta^{-1/2} \left(1 + \frac{487}{1008} \Theta^{-1/4} \right) \\ F^{(3/2)} &= \frac{Mv^5}{2D_L} = \frac{M}{64D_L} \Theta^{-5/8} \left[1 + \frac{2435}{4032} \Theta^{-1/4} + \left(-\frac{\pi}{2} + p_S \frac{47\chi_s \cos \beta}{48} \right) \Theta^{-3/8} \right] \end{aligned}$$

Above, $p_S = 0$ for the non-spinning case and $p_S = 1$ for the spinning case.

$\chi_s \equiv 2S/M^2 = \text{constant}$ and $\chi_a \equiv 2\bar{S}/M^2 = \text{constant}$; $\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2$, $\bar{\mathbf{S}} \equiv \mathbf{S}_1 - \mathbf{S}_2$, $M \equiv M_1 + M_2$.

$\beta \equiv \cos^{-1} \left(\frac{\mathbf{L}_N \cdot \mathbf{S}}{L_N S} \right) = \cos^{-1} \left(\frac{\mathbf{L}_N \cdot \bar{\mathbf{S}}}{L_N \bar{S}} \right) = \text{constant}$.

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- [1] K. G. Arun, Alessandra Buonanno, Guillaume Faye, and Evan Ochsner, Phys. Rev. D **79**, 104023 (2009).
 [2] L. Blanchet, Living Reviews in Relativity, "Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries" (2006).
 [3] Ref. **(author?)** [1] has a slightly different prescription for the time-varying coordinates.