Fixed Point Iteration for Estimating The Parameters of Extreme Value Distributions

Tewfik Kernane[1]*and Zohrh A. Raizah[2]
[1]*Department of Mathematics, Faculty of Sciences*
[2]*Scientific Departments, Girls Faculty of Education*
King Khaled University
Abha, Saudi Arabia

February 3, 2009

Abstract

Maximum likelihood estimations for the parameters of extreme value distributions are discussed in this paper using fixed point iteration. The commonly used numerical approach for adressing this problem is the Newton-Raphson approach which requires differentiation unlike the fixed point iteration which is also easier to implement. Graphical approaches are also usualy proposed in the literature. We prove that these reduce in fact to the fixed point solution proposed in this paper.

2000 AMS Classification: 62F10; 62N02.

1   Introduction

Extreme value distributions are largely used in applied engineering and environmental problems (see the book by Coles [4]). Parameter estimation is the first step in the statistical analysis of parametric probability distributions. The most widely used approach is the popular maximum likelihood estimation (MLE), but usually, as for extreme value distributions, the solution is not analytic and must be then approached by numerical techniques. The commonly used one is the Newton-Raphson algorithm to determine the maximum likelihood estimates of the parameters. To employ the algorithm, the second derivatives of the log-likelihood are required. Sometimes the calculations of the derivatives based on the progressively Type II censored samples for example are complicated (see [7]). To avoid such computation, we propose to use the fixed point

---

*Corresponding author: tkernane@gmail.com

iteration algorithm instead. In this paper we prove that generally MLE of extreme value distributions could be expressed in a fixed point iteration form which is easier to implement and does not require differentiation. Graphical techniques were also proposed as alternatives (see Balakrishnan and Kateri [2] and Dodson [5]) which have the disadvantage of using visualization to detect the solution from graphics. We prove that these graphical solutions reduce in fact to fixed points of a suitable iteration forms.

In the following section, we propose fixed point iterations for estimating the parameters of the Gumbel and Weibull distributions from complete data. In section 3, we extend the procedure to censored samples (simple and progressive Type I and Type II) using the Type I least extreme values distribution from which estimations for Gumbel and Weibull distributions can be deduced from suitable transformations. Finally, in section 4 we illustrate the proposed approach using examples quoted from the literature.

2 Maximum Likelihood Estimations for Complete Data

2.1 Type I extreme value distribution

The Type I extreme value distribution which is also called Gumbel distribution function is defined as:

$$F(x) = \exp\left\{ -\exp\left[-\frac{1}{\sigma}(x - \mu) \right] \right\},$$

for $x \in \mathbb{R}$, and it has the probability density

$$f(x) = \frac{1}{\sigma}\exp\left[-\frac{1}{\sigma}(x - \mu) \right]\exp\left\{ -\exp\left[-\frac{1}{\sigma}(x - \mu) \right] \right\},$$

where $\sigma > 0$ and $\mu \in \mathbb{R}$. Let $\underline{x} = (x_1, ..., x_n)'$ be a complete sample from the Type I extreme value distribution.

The maximum likelihood estimator for σ is known to be the solution of the following equation

$$\sigma = \frac{\sum_{i=1}^{n} x_i}{n} - \frac{\sum_{i=1}^{n} x_i \exp(-\frac{x_i}{\sigma})}{\sum_{i=1}^{n} \exp(-\frac{x_i}{\sigma})}. \tag{1}$$

Denote the right hand side of (1) by $g(\sigma; \underline{x})$. Then the equation (1) is in fact in a fixed point form

$$\sigma = g(\sigma; \underline{x}). \tag{2}$$

Unlike the Newton-Raphson method, which is the commonly used nonlinear numerical method for solving MLE of the extreme value distribution (see Cohen [3] p. 143), the fixed point approach does not require differentiation and then is more easier to use. Uniqueness of the solution of (1) can be proved using graphical techniques (see

Balakrishnan and Kateri [2]). Indeed, the left hand side σ is monotone increasing on σ. We then have to show that $g(\sigma; \underline{x})$ is monotone decreasing on σ.

$$\frac{\partial g(\sigma; \underline{x})}{\partial \sigma} = \frac{\sum_{i=1}^n \frac{x_i}{\sigma^2} \exp(-\frac{x_i}{\sigma}) \sum_{i=1}^n x_i \exp(-\frac{x_i}{\sigma}) - \sum_{i=1}^n \frac{x_i^2}{\sigma^2} \exp(-\frac{x_i}{\sigma}) \left(\sum_{i=1}^n \exp(-\frac{x_i}{\sigma})\right)}{\left(\sum_{i=1}^n \exp(-\frac{x_i}{\sigma})\right)^2}$$

It remains then to prove that

$$g^*(\sigma; \underline{x}) = \sum_{i=1}^n x_i \exp\left(-\frac{x_i}{\sigma}\right) \sum_{i=1}^n x_i \exp\left(-\frac{x_i}{\sigma}\right) - \sum_{i=1}^n x_i^2 \exp\left(-\frac{x_i}{\sigma}\right) \left(\sum_{i=1}^n \exp\left(-\frac{x_i}{\sigma}\right)\right) \le 0.$$

Setting $a_i = x_i \exp\left(-\frac{x_i}{2\sigma}\right)$ and $b_i = \exp\left(-\frac{x_i}{2\sigma}\right)$, $i = 1, ..., n$, $g^*(\sigma; \underline{x})$ becomes

$$g^*(\sigma; \underline{x}) = \left(\sum_{i=1}^n a_i b_i\right)^2 - \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$$

and $g^*(\sigma; \underline{x}) \le 0$ by the Cauchy-Schwarz inequality. The last result can be deduced from the result of Balakrishnan and Kateri [2] from transformation between the Weibull and Type I extreme value distribution. It should be noted that $\lim_{\sigma \to \infty} g(\sigma; \underline{x}) = 0$ and

$$\lim_{\sigma \to 0^+} g(\sigma; \underline{x}) = \frac{\sum_{i=1}^n x_{(i)}}{n} - x_{(1)} = \frac{1}{n} \sum_{i=2}^n \left(x_{(i)} - x_{(1)}\right).$$

We deduce then that $0 \le \sigma \le \frac{\sum_{i=1}^n x_{(i)}}{n} - x_{(1)}$ and $0 \le g(\sigma; \underline{x}) \le \frac{\sum_{i=1}^n x_{(i)}}{n} - x_{(1)}$ which also guarantees the existence of a solution (see Lemma 3.4.1 of [1]). After obtaining a solution for σ we deduce the MLE of μ from

$$\widehat{\mu} = \sigma \left[\ln \frac{n}{\sum_{i=1}^n \exp(-\frac{x_i}{\sigma})}\right]. \tag{3}$$

Example 1 *Consider the data about annual wind-speed maxima in km/h from 1947 to 1984 at Vancouver quoted from the software Xtremes 4.1 [10], provided with the book by Reiss and Thomas [8], stored in the file em-cwind.dat (the source is [6]). By fitting a Gumbel distribution to the data, they provide estimates for σ and μ as $\widehat{\sigma} = 8.3$ and $\widehat{\mu} = 60.3$. The fixed point iteration (2) leads to the solution $\widehat{\sigma} = 8.2891$ and from (3) we obtain $\widehat{\mu} = 60.3504$.*

2.2 Weibull distribution

For the two parameters Weibull distribution $\mathcal{W}(\theta, \beta)$ with pdf

$$f(x; \theta, \beta) = \frac{\beta}{\theta^\beta} x^{\beta-1} \exp\left[-(x/\theta)^\beta\right], \ x > 0, \ \theta, \beta > 0,$$

3

which is the Type II extreme value distribution it can be deduced from [2] that

$$\beta = g_w(\beta; \underline{x}),\tag{4}$$

where $g_w(\beta; \underline{x})$ is given by

$$g_w(\beta; \underline{x}) = \left[\frac{\sum_{i=1}^n x_i^\beta \ln x_i}{\sum_{i=1}^n x_i^\beta} - \frac{1}{n} \sum_{i=1}^n \ln x_i \right]^{-1}.$$

It has been proved in [2] that $(g_w(\beta; \underline{x}))^{-1}$ is a monotone increasing function then $g_w(\beta; \underline{x})$ is monotone decreasing which insures existence and uniqueness of the solution of the fixed point iteration (4). The MLE of the parameter θ is deduced from

$$\widehat{\theta} = \left(\frac{1}{n} \sum_{i=1}^n x_i^\beta \right)^{1/\beta}.$$

3 Estimation for Censored Data

3.1 Singly right censored samples

In the case of simple Type-II censored data, let r $(1 < r < n)$ denote the number of observed lifetimes and $x = (x_{(1)}, x_{(2)}, ..., x_{(r)})$ the ordered observed lifetimes from the Type I least extreme value distribution with probability density

$$f(x) = \frac{1}{\sigma} \exp \left[\frac{1}{\sigma} (x - \mu) \right] \exp \left\{ - \exp \left[\frac{1}{\sigma} (x - \mu) \right] \right\}.$$

If X is a random variable from a Type I greatest extreme value distribution with location parameter μ and shape parameter σ then $-X$ follows a Type I least extreme value distribution with location parameter $-\mu$ and shape parameter σ [3]. But if we have a Type-II rigth censored data $x = (x_{(1)}, x_{(2)}, ..., x_{(r)})$ from the Type I least extreme value distribution then $y = (-x_{(r)}, -x_{(r-1)}, ..., -x_{(1)})$ will be a Type-II left censored data from the corresponding Type I greatest extreme value distribution. The MLE of σ is given by the following fixed point iteration expression

$$\sigma = g(\sigma; \underline{x}),\tag{5}$$

where now

$$g(\sigma; \underline{x}) = \frac{\sum_{i=1}^r x_{(i)} \exp\left(-\frac{x_{(i)}}{\sigma}\right) + (n-r)x_{(r)} \exp\left(-\frac{x_{(r)}}{\sigma}\right)}{\sum_{i=1}^r \exp\left(-\frac{x_{(i)}}{\sigma}\right) + (n-r) \exp\left(-\frac{x_{(r)}}{\sigma}\right)} - \frac{\sum_{i=1}^r x_{(i)}}{r}.\tag{6}$$

4

It can be easily proved that $g(\sigma; \underline{x})$ of (5) is a monotone decreasing function in σ exactly along the lines of section 2.1 by taking

$$
\begin{aligned}
a_i &= I\left(1 \le i \le r\right) x_{(i)} \exp\left(-\frac{x_{(i)}}{2\sigma}\right) + I\left(r+1 \le i \le n\right) x_{(r)} \exp\left(-\frac{x_{(r)}}{2\sigma}\right) \\
b_i &= I\left(1 \le i \le r\right) \exp\left(-\frac{x_{(i)}}{2\sigma}\right) + I\left(r+1 \le i \le n\right) \exp\left(-\frac{x_{(r)}}{2\sigma}\right),
\end{aligned}
$$

which guarantees existence and uniqueness of a fixed point for σ. The parameter μ is then obtained from

$$
\widehat{\mu} = \sigma \left[\ln \frac{\sum_{i=1}^{r} \exp\left(-\frac{x_{(i)}}{\sigma}\right) + (n-r) \exp\left(-\frac{x_{(r)}}{\sigma}\right)}{r} \right].
$$

If $y_{(1)}, ..., y_{(r)}$ designate the r smallest observations in a random sample of size n from a two parameter Weibull distribution $\mathcal{W}(\theta, \beta)$ then $x_{(1)}, ..., x_{(r)}$ where $x_{(i)} = \ln y_{(i)}$ will designate equivalent observations in a sample from the Type I distribution of smallest extremes [3]. The MLEs $\widehat{\theta}$ and $\widehat{\beta}$ of the Weibull distribution will then be deduced from the relations $\widehat{\theta} = \exp \widehat{\mu}$ and $\widehat{\beta} = 1/\widehat{\sigma}$. Meanwhile, fixed point iterations hold also in this case for the Weibull distribution from the relation

$$
\beta = g_w(\beta; \underline{x}), \tag{7}
$$

where now

$$
g_w(\beta; \underline{x})^{-1} = \frac{\sum\limits_{i=1}^{r} x_{(i)}^{\beta} \ln x_{(i)} + (n-r) x_{(r)}^{\beta} \ln x_{(r)}}{\sum\limits_{i=1}^{r} x_{(i)}^{\beta} + (n-r) x_{(r)}^{\beta}} - \frac{1}{r} \sum_{i=1}^{r} \ln x_{(i)}. \tag{8}
$$

From [2] it has been proved that $g_w(\beta; \underline{x})^{-1}$ is monotone increasing in β then $g_w(\beta; \underline{x})$ in (7) is monotone decreasing which insures existence and uniqueness of the fixed point $\widehat{\beta}$ of the iteration (7).

The MLE of θ is deduced from

$$
\widehat{\theta} = \left[\frac{1}{r} \left\{ \sum_{i=1}^{r} x_{(i)}^{\beta} + (n-r) x_{(r)}^{\beta} \right\} \right]^{1/\beta}. \tag{9}
$$

For Type-I censoring, it suffices to replace the term $x_{(r)}$ in the $(n-r) x_{(r)} \exp\left(-x_{(r)}/\sigma\right)$ and $(n-r) \exp\left(-x_{(r)}/\sigma\right)$ terms of relation (6) by the pre-specified time of testing T and in the terms $(n-r) x_{(r)}^{\beta} \ln x_{(r)}$ and $(n-r) x_{(r)}^{\beta}$ in the relation (8).

Example 2 *In this example we quote a Type-II censored data from [5]. These data are about testing twenty identical grinders, with the test ending at time 152.7. In this*

period, twelve grinders failed. The observed failure times are presented in the following table

Table 1

Type-II censored failure data from Dodson (2006)

| 12.5 | 24.4 | 58.2 | 68.0 | 69.1 | 95.5 | 96.6 | 97.0 | 114.2 | 123.2 | 125.6 | 152.7 |

By assuming a Weibull distribution the graphical approach of [2] leads to solutions $\widehat{\beta} = 1.647$ and $\widehat{\theta} = 162.223$. Using the fixed point iteration (7) we obtain the fixed point solution $\widehat{\beta} = 1.6467$ and from (9) $\widehat{\theta} = 162.223$. This shows that the graphical approach proposed in [2] reduces in fact to a fixed point iteration.

3.2 Progressively censored samples

Consider a progressively Type II censored sample from a Type I least extreme values distribution with R_j, $j = 1, ..., r$ the number of censored items at failure time j. We have the following fixed point iteration for the MLE of σ

$$\sigma = g(\sigma; \underline{x}), \tag{10}$$

where

$$g(\sigma; \underline{x}) = \frac{\sum_{i=1}^{r} (R_i + 1) \, x_{(i)} \exp\left(\frac{x_{(i)}}{\sigma}\right)}{\sum_{i=1}^{r} (R_i + 1) \exp\left(\frac{x_{(i)}}{\sigma}\right)} - \frac{1}{r} \sum_{i=1}^{r} x_{(i)}.$$

It can be proved using similar arguments of the preceding section that $g(\sigma; \underline{x})$ is a decreasing function on σ which insures existence and uniqueness of a fixed point for σ. The MLE of μ is then deduced from

$$\mu = \sigma \ln \left[\frac{1}{r} \sum_{i=1}^{r} (R_i + 1) \exp\left(\frac{x_{(i)}}{\sigma}\right) \right]. \tag{11}$$

For the case of a Weibull distribution with progressively Type II censored sample, we obtain from [2] that there exists a unique fixed point solution $\widehat{\beta}$ of the following equation

$$\beta = g_w(\beta; \underline{x}),$$

where

$$g_w(\beta; \underline{x}) = \left[\frac{\sum_{i=1}^{r} (R_i + 1) \, x_{(i)}^{\beta} \ln x_{(i)}}{\sum_{i=1}^{r} (R_i + 1) \, x_{(i)}^{\beta}} - \frac{1}{r} \sum_{i=1}^{r} \ln x_{(i)} \right]^{-1}. \tag{12}$$

We have for the MLE of θ

$$\widehat{\theta} = \left(\frac{1}{r} \sum_{i=1}^{r} (R_i + 1) \, x_{(i)}^{\beta} \right)^{1/\beta}. \tag{13}$$

6

Example 3 *Consider the progressive Type-II censored data analysed by Viveros and Balakrishnan [9] and given in the following table*

Table 2

Progressive Type-II censored failure data from Viveros and Balakrishnan (1994)

i	1	2	3	4	5	6	7	8
$x_{(i)}$	-1.6608	-0.2485	-0.0409	0.2700	1.0224	1.5789	1.8718	1.9947
R_i	0	0	3	0	3	0	0	5

Fitting a Type-I least extreme values distribution to these data and using the Newton-Raphson numerical method they obtain $\widehat{\sigma} = 1.026$ and $\widehat{\mu} = 2.222$. Using fixed point iteration (10) we obtain the solution $\widehat{\sigma} = 1.0264$ and using (11) we obtain $\widehat{\mu} = 2.222$. In order to compare the convergence rate of the fixed point iteration with that of the Newton–Raphson method and the EM algorithm used in [7], same initial value $\sigma_0 = 0.7912$ (and $\mu_0 = 1.4127$ for the Newton–Raphson and EM algorithm methods) were used and the level of accuracy was fixed at 5×10^{-5}. The Newton–Raphson method used in [9] took 37 iterations and the EM algorithm took 151 while the fixed point iteration took 12 iterations to converge to the same values.

References

[1] Atkinson, K., and Han, W., *Elementary Numerical Analysis,* John Wiley & Sons, 3rd ed. 2004.

[2] Balakrishnan, N., Kateri, M., On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data. *Statistics and Probability Letters* (2008).

[3] Cohen, A.C., *Truncated and Censored Samples: Theory and Applications.* Dekker, New York 1991.

[4] Coles, S., *An Introduction to Statistical Modeling of Extreme Values*, Springer-Verlag, 2001.

[5] Dodson, B., *The Weibull Analysis Handbook*, 2nd ed. ASQ Quality Press, Milwaukee, 2006.

[6] Gentlman, J.F. and Whitmore G.A. (eds.) (1995). *Case Studies in Data Analysis.* Lect. Notes Statist. 94, Springer-Verlag, New York.

[7] Ng, H.K.T.; Chan, P.S.; Balakrishnan, N., Estimation of parameters from progressively censored data using EM algorithm, *Computational Statistics & Data Analysis* **39** (2002) 371 – 386.

[8] Reiss, R.D. and Thomas, M., *Statistical Analysis of Extreme Values: with Applications to Insurance, Finance, Hydrology and Other Fields*. 3rd ed., 2007, Birkhauser.

[9] Viveros, R., Balakrishnan, N., 1994. Interval estimation of parameters of life from progressively censored data. *Technometrics* 36, 84–91.

[10] XTREMES, Version 4.1, May 9, 2007, 1991-2007 Xtremes Group, Siegen.