Deformation of contour and Hawking temperature

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Abstract

It was found that, in an isotropic coordinate system, the tunneling approach brings a factor of $\frac{1}{2}$ for the Hawking temperature of a Schwarzschild black hole. In this paper, we address this kind of problem by studying the relation between the Hawking temperature and the deformation of integral contour for the scalar and Dirac particles tunneling. We find that correct Hawking temperature can be obtained exactly as long as the integral contour deformed corresponding to the radial coordinate transform if the transformation is a non-regular or zero function at the event horizon.

Keywords: black hole, Hawking temperature, integral contour.

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I. INTRODUCTION

A semi-classical Hamilton-Jacobi method [1]-[19] for controlling Hawking radiation as a tunneling effect has been developed recently. In this method a semiclassical propagator $K(\vec{x}_2, t_2; \vec{x}_1, t_1)$ in a spacetime is described by $N \exp\left[\frac{i}{\hbar}(I(\vec{x}_2, t_2; \vec{x}_1, t_1) + C\right]$ in which the action $I(\vec{x}_2, t_2; \vec{x}_1, t_1)$ acquires a singularity at the event horizon. This singularity can be regularized by specifying a suitable complex contour [1]. After integrating around the pole, we find that the action $I(\vec{x}_2, t_2; \vec{x}_1, t_1)$ is complex. Thus, we know that the probabilities are $\Gamma[\text{emission}] \propto e^{-2\text{Im}[I_+ + C]}$ and $\Gamma[\text{absorption}] \propto e^{-2\text{Im}[I_- + C]} = 1$, and the ratio is

$$\Gamma[\text{emission}] = e^{-2[\text{Im}I_+ - \text{Im}I_-]}\Gamma[\text{absorption}], \tag{1.1}$$

where $I_\pm$ are the square roots of the relativistic Hamilton-Jacobi equation corresponding to outgoing and ingoing particles. In a system with a temperature T_H, the absorption and the emission probabilities are related by $\Gamma[\text{emission}] = e^{-E/T_H}\Gamma[\text{absorption}]$. Then, from the relation

$$e^{-E/T_H} = e^{-2[\text{Im}I_+ - \text{Im}I_-]}, \tag{1.2}$$

we can obtain the Hawking temperature.

It is well known that the Hawking temperature is an attribution of the black hole and is independent of coordinates. This can be seen from its definition: $T_H = \frac{\kappa}{2\pi}$ [20], where κ is the surface gravity of the black hole. However, to calculation the Hawking temperature by tunneling approach, we need to regularize the singularity by specifying a suitable complex contour to bypass the pole. For the Schwarzschild black hole in the standard coordinate representation, we should take the contour to be an infinitesimal semicircle below the pole $r = r_H$ for outgoing particles from inside of the horizon to outside; similarly, the contour is above the pole for the ingoing particles from outside to inside. But, if we use another coordinate representations, we find that the calculation of the Hawking temperature is related to the choice of the integral contour and improper contour would give incorrect result. For example, if a semi-circular contour is still employed in the isotropic coordinate system, the temperature calculated by the Hamilton-Jacobi method is one-half of the standard result (the so-called "factor of $\frac{1}{2}$ problem", see Appendix A); and if we use a semi-circular contour in a general coordinate (2.3), we can prove that the temperature would be $(\alpha + 1)$ times of the standard result (see Appendix B).

The " factor of $\frac{1}{2}$ problem" of the Schwarzschild black hole in the isotropic coordinates is studied by Aknmedov *et al* [21, 22] by deforming the contour, i.e. using a quarter-circular contour instead of the semi-circular contour. How to extend it to a general case? In this manuscript we will study the problem in a general coordinate [5, 6] for a Kerr-Newman black hole via the scalar and Dirac particles tunneling.

This paper is organized as follows. In Sec.II, the different coordinate representations for the Kerr-Newman black hole are presented. In Sec. III, the Hawking temperature of the Kerr-Newman black hole from scalar particles tunneling in a general coordinate is studied. In Sec. IV, the Hawking temperature of the Kerr-Newman black hole from Dirac particles tunneling is studied. The last section is devoted to a summary.

II. COORDINATE REPRESENTATIONS FOR A KERR-NEWMAN BLACK HOLE

The no-hair theorem postulates that all black hole solutions of the Einstein-Maxwell equations of gravitation and electromagnetism in general relativity can be completely characterized by only three externally observable classical parameters: the mass, the electric charge, and the angular momentum. The final state of a collapsing star is described by the Kerr-Newman black hole. In the Boyer-Lindquist coordinates, its line element reads

$$\begin{aligned} ds^2 = & -\left(1-\frac{2Mr-Q^2}{\rho^2}\right)dt_s^2 - \frac{2(2Mr-Q^2)a\sin^2\theta}{\rho^2}dt_s d\varphi_s + \frac{\rho^2}{\triangle}dr^2 \\ & +\rho^2 d\theta^2 + \left(r^2+a^2+\frac{(2Mr-Q^2)a^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\varphi_s^2, \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} & \rho^2 = r^2 + a^2\cos^2\theta, \qquad \triangle = r^2 - 2Mr + a^2 + Q^2 = (r-r_+)(r-r_-) \\ & r_+ = M + \sqrt{M^2-a^2-Q^2}, \qquad r_- = M - \sqrt{M^2-a^2-Q^2}, \end{aligned}$$

where M, Q and a are the mass, electric charge and angular momentum of the black hole, and r_- and r_+ are the inner and outer horizons. The spacetime has a timelike Killing vector $\tilde{\xi}^\mu_{(t)} = (1,0,0,0)$, and a spacelike Killing vector $\tilde{\xi}^\mu_{(\varphi)} = (0,0,0,1)$.

We note that the Painlevé-type [23], advanced Eddington-Finkelstein [24] and Boyer-Lindquist coordinate representations can be casted into an united form which is given by a general coordinate transform

$$u = \int drF(r), \quad v = \eta t_s + \eta\int(r^2+a^2)G(r)dr, \quad \varphi = \delta\varphi_s + \delta a\int G(r)dr, \tag{2.2}$$

where $(t_s,\ r,\ \theta,\ \varphi_s)$ are the Boyer-Lindquist coordinates; v, u and φ represent the time, radial and angular coordinates respectively, θ remains the same; η and δ are arbitrary nonzero constants which re-scale the time and angle; and G and F are arbitrary functions of r only.

The line element (2.1) in the new coordinate system becomes

$$ds^2 = -\frac{1}{\eta^2}\left(1-\frac{2Mr-Q^2}{\rho^2}\right)\left[dv-\frac{\eta G(r^2+a^2)}{F}du\right]^2$$
$$-\frac{2(2Mr-Q^2)a\sin^2\theta}{\eta\delta\rho^2}\left[dv-\frac{\eta G(r^2+a^2)}{F}du\right]\left[d\varphi-\frac{\delta aG}{F}du\right]$$
$$+\frac{\rho^2}{\triangle F^2}du^2+\rho^2d\theta^2+\left(r^2+a^2+\frac{(2Mr-Q^2)a^2\sin^2\theta}{\rho^2}\right)\frac{\sin^2\theta}{\delta^2}\left[d\varphi-\frac{\delta aG}{F}du\right]^2. \quad (2.3)$$

The timelike and spacelike Killing vectors of the spacetime are

$$\xi^{\mu}_{(t)}=\frac{\partial x^{\mu}}{\partial\tilde{x}^{\nu}}\tilde{\xi}^{\nu}_{(t)}=(\eta,0,0,0), \quad \xi^{\mu}_{(\varphi)}=\frac{\partial x^{\mu}}{\partial\tilde{x}^{\nu}}\tilde{\xi}^{\nu}_{(\varphi)}=(0,0,0,\delta). \quad (2.4)$$

Painlevé-type coordinate representation: In the transformation (2.2), if we take $\eta=\delta=1$, $G(r)=\frac{1}{\triangle}\sqrt{\frac{2Mr-Q^2}{r^2+a^2}}$ and $F(r)=1$, the line element (2.3) becomes the Painlevé-type coordinate representation [23], which has no coordinate singularity at $\triangle(r)=0$.

Advanced Eddington-Finkelstein coordinate representation: In the transformation (2.2), if we let $\eta=1$, $\delta=-1$, $G(r)=\frac{1}{\triangle}$ and $F(r)=1$, the line element (2.3) becomes the advanced Eddington-Finkelstein representation, which has no coordinate singularity just as in the Painlevé-type coordinates [24].

Boyer-Lindquist coordinate representation: In the transformation (2.2), if we let $\eta=\delta=F(r)=1$, $G(r)=0$, the line element (2.3) becomes the Boyer-Lindquist coordinate representation (2.1).

III. TEMPERATURE OF KERR-NEWMAN BLACK HOLE FROM SCALAR TUNNELING IN THE GENERAL COORDINATE SYSTEM

Now we study the scalar tunneling in the general coordinates (2.3). Applying the WKB approximation

$$\phi(v,u,\theta,\varphi)=\exp\left[\frac{i}{\hbar}I(v,u,\theta,\varphi)+I_1(v,u,\theta,\varphi)+\mathcal{O}(\hbar)\right] \quad (3.1)$$

to the charged Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}(\partial_{\bar{\mu}}-\frac{iq}{\hbar}A_{\bar{\mu}})\left[\sqrt{-g}g^{\bar{\mu}\bar{\nu}}(\partial_{\bar{\nu}}-\frac{iq}{\hbar}A_{\bar{\nu}})\phi\right]-\frac{\mu^2}{\hbar^2}\phi=0, \quad (3.2)$$

then, to leading order in $\hbar$, we obtain the relativistic Hamilton-Jacobi equation

$$g^{\bar{\mu}\bar{\nu}}(\partial_{\bar{\mu}}I\partial_{\bar{\nu}}I+q^2A_{\bar{\mu}}A_{\bar{\nu}}-2qA_{\bar{\mu}}\partial_{\bar{\nu}}I)+\mu^2=0, \quad (3.3)$$

where μ is the mass of tunneling particles. From the symmetries of the metric (2.3), we know that there exists a solution of the form (see Appendix C)

$$I = -\frac{1}{\eta}Ev + W(u) + \frac{1}{\delta}m\varphi + J(\theta) + C. \tag{3.4}$$

Substituting the metric (2.3) and Eq. (3.4) into the Hamilton-Jacobi equation (3.3), we obtain

$$\begin{aligned} &\triangle^2\left[FW'(u) - (r^2+a^2)G\Big(E - \frac{qQr}{r^2+a^2} - \frac{ma}{r^2+a^2}\Big)\right]^2 \\ &-\Big(r^2+a^2\Big)^2\left[E - \frac{qQr}{r^2+a^2} - \frac{ma}{r^2+a^2}\right]^2 + \triangle\lambda = 0, \end{aligned} \tag{3.5}$$

with

$$\lambda = \mu^2\rho^2 + J'^2(\theta) + \left(aE\sin\theta - \frac{m}{\sin\theta}\right)^2, \tag{3.6}$$

where $W'(u) = \frac{dW(u)}{du}$, and $J'(\theta) = \frac{dJ(\theta)}{d\theta}$. Then, $W'(u)$ can be expressed as

$$\begin{aligned} W'_{\pm}(u) = &\frac{G}{F}(r^2+a^2)\Big(E - \frac{qQr}{r^2+a^2} - \frac{ma}{r^2+a^2}\Big) \\ &\pm\frac{1}{F\triangle}\sqrt{(r^2+a^2)^2\Big(E - \frac{qQr}{r^2+a^2} - \frac{ma}{r^2+a^2}\Big)^2 - \triangle\lambda}. \end{aligned} \tag{3.7}$$

One solution of Eq. (3.7) corresponds to the scalar particles moving away from the black hole (i.e. "+" outgoing), and the other solution corresponds to particles moving toward the black hole (i.e. "-" incoming). Without loss of generality, the function G can be expressed as $G(r(u)) = \frac{A(r(u))}{\triangle(r(u))} + B(r(u))$, where $A(r(u))$ and $B(r(u))$ are regular functions. Thus, we have

$$\begin{aligned} \mathrm{Im}W_{\pm}(u) = \mathrm{Im}\int du\Bigg[&\Big(\frac{B}{F} + \frac{A}{F\triangle}\Big)(r^2+a^2)\Big(E - \frac{qQr}{r^2+a^2} - \frac{ma}{r^2+a^2}\Big) \\ &\pm\frac{1}{F\triangle}\sqrt{(r^2+a^2)^2\Big(E - \frac{qQr}{r^2+a^2} - \frac{ma}{r^2+a^2}\Big)^2 - \triangle\lambda}\ \Bigg]. \end{aligned} \tag{3.8}$$

Imaginary part of the action can only come from the pole at the horizon. We will work out the integral in two cases: A) F is a regular and non-zero function at the horizon, and B) F is a singular or zero function at the horizon.

A. F is a regular and non-zero function at the horizon

If F is a regular and non-zero function at the horizon, using the law of residue we obtain

$$\mathrm{Im}W_{\pm}(u) \ = \ \Big[A(r_+) \pm 1\Big]\frac{r_+^2+a^2}{2(r_+-M)}\left(E - m\Omega_+ - qV_+\right)\pi, \tag{3.9}$$

where $V_+ = \frac{Qr_+}{r_+^2+a^2}$ is the electromagnetic potential, and $\Omega_+ = \frac{a}{r_+^2+a^2}$ is the angular velocity. Then, Eqs. (1.1) and (1.2) show us that the total probability is

$$\Gamma \; = \; \exp\left[-2\pi\frac{r_+^2+a^2}{(r_+-M)}\left(E - m\Omega_+ - qV_+\right)\right], \tag{3.10}$$

and the Hawking temperature is

$$T_H = \frac{r_+ - M}{2\pi(r_+^2+a^2)}, \tag{3.11}$$

which is the same as previous work [1, 2, 23, 25].

B. F is a singular or zero function at the horizon

If F is a non-regular or zero function at the horizon, without loss of generality, we set $F = \triangle^\alpha X(r)$, where α is a non-zero constant and $X(r)$ is a regular and non-zero function. Thus, Eq. (3.8) becomes

$$\begin{aligned}\mathrm{Im}W_\pm(u) = \mathrm{Im}\int du\Bigg[&\left(\frac{B}{\triangle^\alpha X}+\frac{A}{\triangle^{\alpha+1}X}\right)(r^2+a^2)\left(E-\frac{qQr}{r^2+a^2}-\frac{ma}{r^2+a^2}\right)\\ &\pm\frac{1}{\triangle^{\alpha+1}X}\sqrt{\left(r^2+a^2\right)^2\left(E-\frac{qQr}{r^2+a^2}-\frac{ma}{r^2+a^2}\right)^2-\triangle\lambda}\;\Bigg].\end{aligned} \tag{3.12}$$

From which we know

$$\begin{aligned}&\mathrm{Im}[W_+(u)-W_-(u)]\\ &= 2\mathrm{Im}\int du\frac{1}{\triangle^{\alpha+1}X}\sqrt{\left(r^2+a^2\right)^2\left(E-\frac{qQr}{r^2+a^2}-\frac{ma}{r^2+a^2}\right)^2-\triangle\lambda}\,.\end{aligned} \tag{3.13}$$

We now study two cases: 1) $\alpha \neq -1$ and 2) $\alpha = -1$.

1. $\alpha \neq -1$

The Laurent expansion for the factor $\frac{1}{\triangle^{\alpha+1}(r(u))}$ is

$$\frac{1}{\triangle^{\alpha+1}(r(u))} = \frac{X(r(u_+))}{2(\alpha+1)(r_+-M)}\frac{1}{u-u_+}+\sum_{n=0}^{\infty}a_n(u-u_+)^n. \tag{3.14}$$

Then Eq. (3.13) can be written as

$$\begin{aligned}\mathrm{Im}[W_+(u)-W_-(u)] = 2\mathrm{Im}\int du&\left[\frac{1}{\alpha+1}\cdot\frac{1}{2(r_+-M)(u-u_+)}+\frac{1}{X}\sum_{n=0}^{\infty}a_n(u-u_+)^n\right]\\ &\cdot\sqrt{\left(r^2+a^2\right)^2\left(E-\frac{qQr}{r^2+a^2}-\frac{ma}{r^2+a^2}\right)^2-\triangle\lambda}\,.\end{aligned} \tag{3.15}$$

Now, we need to choose a contour to bypass the pole $u = u_+$. We note that, in the Boyer-Lindquist coordinate, the contour can be constructed by taking $r = r_+ + \epsilon e^{i\theta}$, ($\epsilon$ is a positive small real quantity, $\theta \in [0, \pi]$ for outgoing particle, $\theta \in [\pi, 2\pi]$ for ingoing particle). Thus, in the general coordinate (2.3), by substituting $r = r_+ + \epsilon e^{i\theta}$ into $u = \int \triangle^\alpha X(r)dr = \int [(r - r_+)(r - r_-)]^\alpha X(r)dr$, we have

$$\begin{aligned} u &= \int [\epsilon e^{i\theta}(r_+ - r_- + \epsilon e^{i\theta})]^\alpha X(r_+ + \epsilon e^{i\theta}) d\epsilon e^{i\theta} \\ &= u_+ + f(u_+)\epsilon^{\alpha+1} e^{i(\alpha+1)\theta}, \end{aligned} \tag{3.16}$$

where $f(u_+) = \frac{(r_+ - r_-)^\alpha X(r_+)}{\alpha+1}$. Eq. (3.16) indicates that the contour is different from semi-circle now. The integral contours for outgoing particles corresponding to r and u complex plane are showed in figure (1). Using Eqs. (3.15), (3.16) and residue theorem, we have

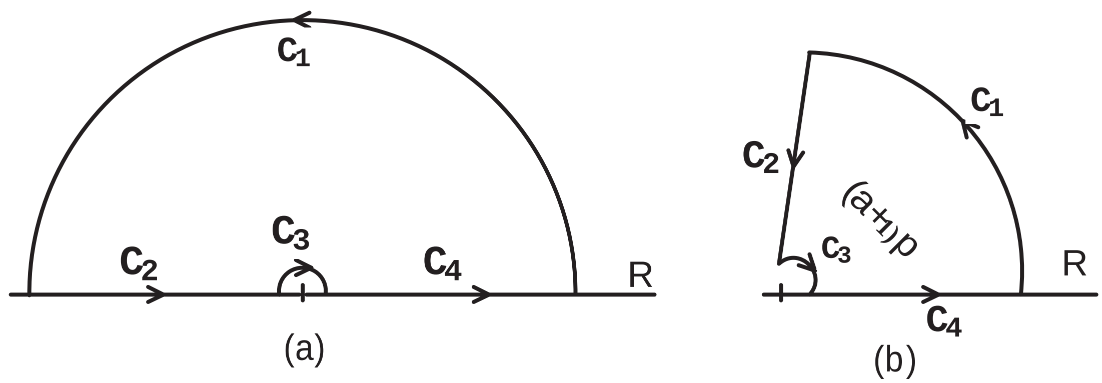

FIG. 1: The figure (a) is the semicircle integrate contour for outgoing particles in the r complex plane; and the figure (b) is the deformation contour in the u complex plane when $\alpha \neq -1$. For $\alpha = -1$, its contour is still a semicircle. The tick mark on the real axis denotes the position of the black hole event horizon.

$$\begin{aligned} &\mathrm{Im}[W_+(u) - W_-(u)] \\ &= -2\mathrm{Im} \lim_{\epsilon \to 0} \int_\pi^0 id\theta \Big[\frac{1}{2(r_+ - M)} + \frac{f(u_+)(\epsilon e^{i\theta})^{\alpha+1}(\alpha+1)}{X} \sum_{n=0}^{\infty} a_n f^n(u_+)(\epsilon e^{i\theta})^{(\alpha+1)n} \Big] \\ &\qquad \cdot \sqrt{\big[(r_+ + \epsilon e^{i\theta})^2 + a^2\big]^2 \Big[E - \frac{qQ(r_+ + \epsilon e^{i\theta}) + ma}{(r_+ + \epsilon e^{i\theta})^2 + a^2} \Big]^2 - \triangle\lambda} \\ &= \pi \frac{r_+^2 + a^2}{(r_+ - M)} \left(E - m\Omega_+ - qV_+ \right), \end{aligned} \tag{3.17}$$

which gives the Hawking temperature (3.11).

2. $\alpha=-1$

It is the tortoise-like coordinate transformation if $\alpha=-1$

$$u=\int X(r)\triangle^{-1}dr. \tag{3.18}$$

By using $r=r_{+}+\epsilon e^{i\theta}$, we know

$$u=u_{+}+i\theta g(u_{+}), \tag{3.19}$$

where $g(u_{+})=\frac{X(r_{+})}{r_{+}-r_{-}}$. Substituting it into Eq. (3.13), we obtain

$$\begin{aligned}
\mathrm{Im}[W_{+}(u)-W_{-}(u)] &= -2\mathrm{Im}\lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{id\theta g(u_{+})}{(r_{+}+\epsilon e^{i\theta})^2+a^2} \\
&\quad\cdot\sqrt{\left[(r_{+}+\epsilon e^{i\theta})^2+a^2\right]^2\left[E-\frac{qQ(r_{+}+\epsilon e^{i\theta})+ma}{(r_{+}+\epsilon e^{i\theta})^2+a^2}\right]^2-\triangle\lambda} \\
&= \pi\frac{r_{+}^2+a^2}{(r_{+}-M)}\left(E-m\Omega_{+}-qV_{+}\right),
\end{aligned} \tag{3.20}$$

which also presents the Hawking temperature (3.11).

Above discussions show us that: i) the integral contour needs to be deformed corresponding to the radial coordinate transformation if this transformations are non-regular or zero at the event horizon; ii) the Hawking temperature is invariant in the general coordinate representation (2.3) for the scalar particle tunneling.

IV. TEMPERATURE OF KERR-NEWMAN BLACK HOLE FROM DIRAC PARTICLE TUNNELING

In this section, we study the Dirac particle tunneling of the Kerr-Newman black hole in the coordinates (2.3). The Dirac equation is [26]

$$\left[\gamma^{\alpha}e_{\alpha}^{\bar{\mu}}(\partial_{\bar{\mu}}+\Gamma_{\bar{\mu}}-\frac{iq}{\hbar}A_{\bar{\mu}})+\frac{\mu}{\hbar}\right]\psi=0, \tag{4.1}$$

with

$$\Gamma_{\bar{\mu}}=\frac{1}{8}[\gamma^{a},\gamma^{b}]e_{a}^{\bar{\nu}}e_{b\bar{\nu};\bar{\mu}},$$

where γ^{a} is the Dirac matrix, and $e_{a}^{\bar{\mu}}$ is the inverse tetrad defined by $\{e_{a}^{\bar{\mu}}\gamma^{a},\quad e_{b}^{\bar{\nu}}\gamma^{b}\}=2g^{\bar{\mu}\bar{\nu}}\times 1$. For the Kerr-Newman metric in the general coordinate system (2.3), the tetrad $e_{a}^{\bar{\mu}}$ can be taken as

$$
\begin{aligned}
e_a^v &= \left(\tfrac{\sqrt{\chi - \triangle^2\eta^2(r^2+a^2)^2G^2}}{\rho\sqrt{\triangle}},\ 0,\ 0,\ 0 \right), \\
e_a^u &= \left(-\tfrac{1}{\rho\sqrt{\triangle}}\tfrac{\triangle^2 FG\eta(r^2+a^2)}{\sqrt{\chi-\triangle^2\eta^2(r^2+a^2)^2G^2}},\ \tfrac{1}{\rho\sqrt{\triangle}}\tfrac{\triangle F\sqrt{\chi}}{\sqrt{\chi-\triangle^2\eta^2(r^2+a^2)^2G^2}},\ 0,\ 0 \right), \\
e_a^\theta &= \left(0,\ 0,\ \tfrac{1}{\rho},\ 0 \right), \\
e_a^\varphi &= \left(\tfrac{a\eta\delta}{\rho\sqrt{\triangle}}\tfrac{(2Mr-Q^2)-\triangle^2(r^2+a^2)G^2}{\sqrt{\chi-\triangle^2\eta^2(r^2+a^2)^2G^2}},\ \tfrac{a\delta G\sqrt{\triangle}}{\rho}\tfrac{\chi-(2Mr-Q^2)\eta^2(r^2+a^2)}{\sqrt{\chi(\chi-\triangle^2\eta^2(r^2+a^2)^2G^2)}},\ 0,\ \tfrac{\eta\delta\rho}{\sin\theta\sqrt{\chi}} \right), \\
\chi &= \eta^2\Big[\big(r^2+a^2\big)^2 - \triangle a^2\sin^2\theta\Big].
\end{aligned}
\tag{4.2}
$$

Without loss of generality, we can choose the following ansatz for spin up and spin down Dirac particles according to [27],

$$
\begin{aligned}
\psi_\uparrow &= \begin{pmatrix} A(v,u,\theta,\varphi)\xi_\uparrow \\ B(v,u,\theta,\varphi)\xi_\uparrow \end{pmatrix} \exp\big(\frac{i}{\hbar}I_\uparrow(v,u,\theta,\varphi)\big) = \begin{pmatrix} A(v,u,\theta,\varphi) \\ 0 \\ B(v,u,\theta,\varphi) \\ 0 \end{pmatrix} \exp\big(\frac{i}{\hbar}I_\uparrow(v,u,\theta,\varphi)\big), \\
\psi_\downarrow &= \begin{pmatrix} C(v,u,\theta,\varphi)\xi_\downarrow \\ D(v,u,\theta,\varphi))\xi_\downarrow \end{pmatrix} \exp\big(\frac{i}{\hbar}I_\downarrow(v,u,\theta,\varphi)\big) = \begin{pmatrix} 0 \\ C(v,u,\theta,\varphi) \\ 0 \\ D(v,u,\theta,\varphi) \end{pmatrix} \exp\big(\frac{i}{\hbar}I_\downarrow(v,u,\theta,\varphi)\big),
\end{aligned}
\tag{4.3}
$$

where "↑" and "↓" represent the spin up and spin down cases, and $\xi_\uparrow$ and $\xi_\downarrow$ are the eigenvectors of σ^3. Inserting Eqs. (4.2) and (4.3) into Eq. (4.1), and employing the ansatz

$$
I_\uparrow = -\frac{1}{\eta}Ev + W(u) + \frac{1}{\delta}m\varphi + J(\theta) + C, \tag{4.4}
$$

to the lowest order in $\hbar$, we obtain

$$\left[-e_0^v\frac{1}{\eta}\big(E-\frac{qQr}{\rho^2}\big)+e_0^u W'(u)+e_0^\varphi\frac{1}{\delta}\big(m-\frac{qQr}{\rho^2}a\sin^2\theta\big)+\mu\right]A$$

$$+B\left[e_1^u W'(u)+e_1^\varphi\frac{1}{\delta}\big(m-\frac{qQr}{\rho^2}a\sin^2\theta\big)\right]=0 \quad , \tag{4.5}$$

$$B\left[e_2^\theta J'(\theta)+ie_3^\varphi\frac{1}{\delta}\big(m-\frac{qQr}{\rho^2}a\sin^2\theta\big)\right]=0 \quad , \tag{4.6}$$

$$-\left[-e_0^v\frac{1}{\eta}\big(E-\frac{qQr}{\rho^2}\big)+e_0^u W'(u)+e_0^\varphi\frac{1}{\delta}\big(m-\frac{qQr}{\rho^2}a\sin^2\theta\big)-\mu\right]B$$

$$-A\left[e_1^u W'(u)+e_1^\varphi\frac{1}{\delta}\big(m-\frac{qQr}{\rho^2}a\sin^2\theta\big)\right]=0 \quad , \tag{4.7}$$

$$-A\left[e_2^\theta J'(\theta)+ie_3^\varphi\frac{1}{\delta}\big(m-\frac{qQr}{\rho^2}a\sin^2\theta\big)\right]=0 \quad . \tag{4.8}$$

Eqs. (4.6) and (4.8) both yield $\left[e_2^\theta J'(\theta)+ie_3^\varphi\frac{1}{\delta}(m-\frac{qQr}{\rho^2}a\sin^2\theta)\right]=0$, regardless of A or B. Then substituting tetrad elements (4.2) into (4.5)–(4.8), after tedious calculating, we obtain

$$\begin{aligned}&\triangle^2\left[FW'(u)-G(r^2+a^2)\Big(E-\frac{qQr}{r^2+a^2}-\frac{ma}{r^2+a^2}\Big)\right]^2\\&-\Big(r^2+a^2\Big)^2\left[E-\frac{qQr}{r^2+a^2}-\frac{ma}{r^2+a^2}\right]^2\\&+\triangle\left[\mu^2\rho^2+J'^2(\theta)+\Big(aE\sin\theta-\frac{m}{\sin\theta}\Big)^2\right]=0,\end{aligned} \tag{4.9}$$

which is the same as Eq. (3.5). Therefore, it is easy to find the Hawking temperature (3.11).

The spin-down calculation is similar to the spin-up case discussed above, and the result is the same.

V. SUMMARY

We firstly cast three well-known coordinate representations, i.e. the Painlevé-type, advanced Eddington-Finkelstein and Boyer-Lindquist coordinate representations for the Kerr-Newman black hole, into an united and general coordinate representation (2.3). Then, based on this coordinate representation, we study the relation between the Hawking temperature and the deformation of integral contour for the scalar and Dirac particle tunneling. We find that correct Hawking temperature can be obtained exactly as long as the integral contour deformed corresponding to the radial coordinate transform if the transformation is a non-regular or zero function at the event horizon.

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Appendix A: Improper choice of contour in isotropic coordinates for Schwarzschild black hole

Improper choice of integral contour can led to incorrect temperature. In this section, we review the following process mentioned in [14, 15]. By taking an isotropic coordinate transformation

$$t \to t, \quad r \to \rho, \quad \ln \rho = \int \frac{dr}{r\sqrt{1-\frac{2M}{r}}}, \tag{A1}$$

the line element of the Schwarzschild black hole becomes

$$ds^2 = -\left(\frac{2\rho - M}{2\rho + M}\right)^2 dt^2 + \left(\frac{2\rho + M}{2\rho}\right)^4 d\rho^2 + \frac{(2\rho + M)^4}{16\rho^2} d\Omega^2. \tag{A2}$$

The horizon now is $\rho_H = M/2$. Substituting it and $\phi = e^{i[-Et+W(\rho)+J(\theta,\varphi)]}$ into Hamilton-Jacobi equation (3.3), we obtain

$$\mathrm{Im}W_\pm(\rho) \;=\; \pm\mathrm{Im}\left[\int \frac{(2\rho+M)^3 d\rho}{4\rho^2(2\rho - M)}\sqrt{E^2 - (\frac{2\rho - M}{2\rho + M})^2(m^2 + g^{ij}J_iJ_j)}\,\right]. \tag{A3}$$

Because the imaginary part of above integration comes from the pole $\rho = M/2$, we only consider the integral around the pole. If we still set a semi-circular contour bypass the pole, as we do in the Schwarzschild coordinates, i.e. setting $\rho = M/2 + \epsilon e^{i\theta}, \theta \in [0, \pi]$, then Eq. (A3) becomes

$$\begin{aligned}\mathrm{Im}W_\pm(\rho) \;&=\; \mp\mathrm{Im}\lim_{\epsilon\to 0}\left[\int_\pi^0 \frac{4(M+\epsilon e^{i\theta})^3 i d\theta}{(M+2\epsilon e^{i\theta})^2}\sqrt{E^2 - (\frac{\epsilon e^{i\theta}}{M+\epsilon e^{i\theta}})^2(m^2 + g^{ij}J_iJ_j)}\,\right] \\ &=\; \pm 4\pi M E.\end{aligned} \tag{A4}$$

By using Eqs. (1.1) and (1.2), the probability is

$$\Gamma = \frac{\Gamma[\text{emission}]}{\Gamma[\text{absorption}]} = \exp\Big[-4\mathrm{Im}W_+\Big] = \exp\Big[-16\pi M E\Big] = \exp\Big[-\frac{E}{T}\Big], \tag{A5}$$

and incorrect temperature is

$$T = \frac{1}{16\pi M} = \frac{1}{2} T_H, \tag{A6}$$

where T_H is the Hawking temperature of the Schwarzschild black hole. This is the so-called "factor of $\frac{1}{2}$ problem".

Appendix B: improper choice of integral contour gives incorrect temperature in the general coordinates (2.3)

In Eq. (3.15), if we still set a semi-circular contour to bypass the pole $u = u_+$, i.e. $u = u_+ + \epsilon e^{i\theta}$, we obtain

$$\mathrm{Im}[W_+ - W_-] \;=\; \frac{\pi}{\alpha+1}\left[\frac{r_+^2 + a^2}{(r_+ - M)}\left(E - m\Omega_+ - qV_+\right)\right], \tag{B1}$$

and the temperature would be

$$T = (\alpha+1) T_H. \tag{B2}$$

Appendix C: definition of radiating particle energy and angular momentum

It is well known that, there are conservational quantities as long as the spacetime possesses some certain symmetries. In the Borer-Lindquist coordinate system, the line element (2.1) obviously has temporal-translational invariance and φ_s-translational one, so we can define the particle energy as $E = -\partial_{t_s} I$, and the particle angular momentum as $m = \partial_{\varphi_s} I$. Thus, the action can be written as $I = -Et_s + W(r) + m\varphi_s + J(\theta)$, which is essentially related to the time-like Killing vector $\tilde{\xi}^\mu_{(t)} = (1, 0, 0, 0)$ and the space-like Killing one $\tilde{\xi}^\mu_{(\varphi)} = (0, 0, 0, 1)$.

As mentioned in ref. [28], the scalar product between time-like Killing vector and particle four-momentum $p^\mu = m dx^\mu / d\lambda$ is a constant for the particle moving along geodesic, i.e.

$$\xi_\mu p^\mu = \text{constant}. \tag{C1}$$

Furthermore, this scalar product is also a constant in different coordinate systems. Hence, these quantities can be used to define particle energy [7] and angular momentum in different coordinate systems, i.e.

$$E = -\xi^\mu_{(t)} p_\mu, \quad m = \xi^\mu_{(\varphi)} p_\mu. \tag{C2}$$

In the general coordinates (2.3), the energy and angular momentum of test particle are

$$\begin{aligned} E &= -\xi^{\mu}_{(t)} p_{\mu} = -\xi^{\mu}_{(t)} \partial_{\mu} I = -\eta \partial_{v} I, \\ m &= \xi^{\mu}_{(\varphi)} p_{\mu} = \xi^{\mu}_{(\varphi)} \partial_{\mu} I = \delta \partial_{\varphi} I. \end{aligned} \tag{C3}$$

Thus, the expression of the action can be taken as

$$I = -\frac{1}{\eta} E v + W(u) + J(\theta) + \frac{1}{\delta} m \varphi. \tag{C4}$$

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