A Consistent Gravitationally-Coupled Spin-2 Field Theory

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Abstract Inspired in teleparallel gravity, instead of being represented by a symmetric second-rank tensor, a fundamental spin-2 field is assumed to be represented by a spacetime (world) vector field assuming values in the Lie algebra of the translation group. The flat-space theory naturally emerges in the Fierz formalism, and is both gauge and local Lorentz invariant. The corresponding gravitationally-coupled theory is shown not to present the consistency problems of the usual spin-2 theory constructed on the basis of general relativity.

1 Introduction

It is well known that higher ($s > 1$) spin fields, and in particular a spin-2 field, present consistency problems when coupled to gravitation [1, 2]. The problem is that the divergence identities satisfied by the field equations of a spin-2 field in Minkowski spacetime are no longer valid when it is coupled to gravitation. In addition, the coupled equations are no longer gauge invariant. The basic underlying difficulty is related to the fact that the covariant derivative of general relativity—which defines the gravitational coupling prescription—is non-commutative, and this introduces unphysical constraints on the spacetime curvature.

The tangent bundle is a fundamental spacetime structure that is always present. It is formed by spacetime—the base space—at each point of it there is "attached" a Minkowski tangent-space—the fiber of the bundle. We are going to use the Greek alphabet $\mu, \nu, \rho, \ldots = 0, 1, 2, 3$ to denote indices related to spacetime (world indices), and the first half of the Latin alphabet $a, b, c, \ldots = 0, 1, 2, 3$ to denote algebraic indices related to the tangent spaces, each one a Minkowski spacetime with metric $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$. Denoting a general tetrad field by $h^a{}_\mu$, the spacetime and the tangent space metrics are related by

$$g_{\mu\nu} = h^a{}_\mu h^b{}_\nu \, \eta_{ab}. \tag{1}$$

In the absence of gravitation, $g_{\mu\nu}$ represents the Minkowski metric in a general coordinate system.

Due to the fact that linearized gravity represents a spin-2 field, the dynamics of a fundamental spin-2 field in Minkowski space is expected to coincide with the dynamics of a linear perturbation of the metric around flat spacetime [3]:

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi_{\mu\nu}. \tag{2}$$

For this reason, a fundamental spin-2 field is usually assumed to be described by a rank-two, symmetric tensor $\phi_{\mu\nu} = \phi_{\nu\mu}$. However, conceptually speaking, this is not the most fundamental notion of a spin-2 field. As is well known, although the gravitational interaction of scalar and vector fields can be described in the metric formalism, the gravitational interaction of spinor fields requires a tetrad formalism. The tetrad formalism can then be considered to be more fundamental than the metric formulation in the sense that it is able to describe the gravitational interaction of both tensorial and spinorial fields. Accordingly, the tetrad field can be said to be more fundamental than the metric.

Instead of similar to a linear perturbation of the spacetime metric, therefore, a fundamental spin-2 field should be considered a linear perturbation of the tetrad field. If we denote a general tetrad by $h^a{}_\mu$, and a trivial tetrad representing the Minkowski metric by $e^a{}_\mu$, a fundamental spin-2 field $\phi^a{}_\mu$ should be defined by

$$h^a{}_\mu = e^a{}_\mu + \phi^a{}_\mu. \tag{3}$$

In other words, a fundamental spin-2 field is to be interpreted as a translational-valued vector field:

$$\phi_\mu = \phi^a{}_\mu \, P_a, \tag{4}$$

with $P_a = \partial_a$ the translation generators. Now, in teleparallel gravity the gravitational field is represented by a translational gauge potential $B_\mu = B^a{}_\mu P_a$, which appears as the nontrivial part of the tetrad field [5]. This means essentially that $\phi^a{}_\mu$ plays a role similar to the gauge potential of teleparallel gravity. Accordingly, its dynamics must coincide with the dynamics of linearized teleparallel gravity. It is important to note that, in the usual *metric* formulation of a spin-2 field, the symmetry $\phi_{\mu\nu} = \phi_{\nu\mu}$ of the spin-2 field eliminates six degrees of freedom of the sixteen original degrees of freedom of $\phi_{\mu\nu}$. In the *tetrad* formulation, on the other hand, it is the requirement of local Lorentz invariance in the algebraic index a that eliminates six degrees of freedom of the sixteen original degrees of freedom of $\phi^a{}_\mu$, yielding the same number of independent components of $\phi_{\mu\nu}$. This is analogous to what happens with the metric and the tetrad formulations of gravitation.

In the light of the above considerations, the purpose of this paper is to construct a teleparallel-based field theory for a fundamental spin-2 field. This will be done according to the following scheme. In section 2, for the sake of completeness, we review the basic notions of connections and frames. In section 3, we give a brief description of teleparallel gravity, also known as teleparallel equivalent of general relativity. Then, by using this theory as paradigm, we construct in section 4 the field theory for a fundamental spin-2 field in flat spacetime. Owing to the (abelian) gauge structure of teleparallel gravity, it naturally emerges in the Fierz formulation, being quite similar to electromagnetism. In section 5, by using teleparallel coupling prescription, we obtain the corresponding spin-2 theory in the presence of gravitation. In addition to present duality symmetry, it is gauge and local Lorentz invariant. This means that, even in the presence of gravitation, $\phi^a{}_\mu$ has the correct number of independent components to represent a massless spin-2 field. In section 6, we consider the spin-2 field as source of gravitation, and we obtain the covariant conservation law of the total source of gravitation. Due to the fact that the spin connection of teleparallel gravity represents inertial effects only, not gravitation, the ensuing covariant derivative turns out to be commutative. As a consequence, no constraints on the background geometry show up, yielding a fully consistent gravitationally-coupled spin-2 field theory. Finally, in section 7, we sum up the results obtained.

2 Lorentz connections and frames

A basic ingredient of any relativistic theory is the spin connection A_μ, a connection assuming values in the Lie algebra of the Lorentz group,

$$A_\mu = \tfrac{1}{2}\, A^{ab}{}_\mu\, S_{ab}, \tag{5}$$

with $S_{ab} = -S_{ba}$ a given representation of the Lorentz generators. The corresponding Lorentz covariant derivative is given by the Fock-Ivanenko operator $\mathcal{D}_\mu$ [6], which acting on a Lorentz vector field ϕ^a, for example, assumes the form

$$\mathcal{D}_\mu \phi^a = \partial_\mu \phi^a + A^a{}_{b\mu}\, \phi^b. \tag{6}$$

Analogously to relation (1), to each spin connection $A^a{}_{b\mu}$ there corresponds a linear connection $\Gamma^\rho{}_{\nu\mu}$ given by [7]

$$\Gamma^\rho{}_{\nu\mu} = h_a{}^\rho \partial_\mu h^a{}_\nu + h_a{}^\rho A^a{}_{b\mu} h^b{}_\nu \equiv h_a{}^\rho \mathcal{D}_\mu h^a{}_\nu. \tag{7}$$

The inverse relation is

$$A^a{}_{b\mu} = h^a{}_\nu \partial_\mu h_b{}^\nu + h^a{}_\nu \Gamma^\nu{}_{\rho\mu} h_b{}^\rho \equiv h^a{}_\nu \nabla_\mu h_b{}^\nu, \tag{8}$$

with ∇_μ the covariant derivative in the connection $\Gamma^\nu{}_{\rho\mu}$. Since the last index of the spin connection is a tensorial index, one can write

$$A^a{}_{bc} = A^a{}_{b\mu}\, h_c{}^\mu. \tag{9}$$

The curvature and torsion of $A^a{}_{b\mu}$ are defined respectively by

$$R^a{}_{b\nu\mu} = \partial_\nu A^a{}_{b\mu} - \partial_\mu A^a{}_{b\nu} + A^a{}_{e\nu} A^e{}_{b\mu} - A^a{}_{e\mu} A^e{}_{b\nu} \tag{10}$$

and

$$T^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + A^a{}_{e\nu} h^e{}_\mu - A^a{}_{e\mu} h^e{}_\nu. \tag{11}$$

Using relations (7) and (8), they can be expressed in a purely spacetime form as

$$R^\rho{}_{\lambda\nu\mu} = \partial_\nu \Gamma^\rho{}_{\lambda\mu} - \partial_\mu \Gamma^\rho{}_{\lambda\nu} + \Gamma^\rho{}_{\eta\nu} \Gamma^\eta{}_{\lambda\mu} - \Gamma^\rho{}_{\eta\mu} \Gamma^\eta{}_{\lambda\nu} \tag{12}$$

and

$$T^\rho{}_{\nu\mu} = \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}. \tag{13}$$

According to a theorem by Ricci [8], a general connection $\Gamma^\rho{}_{\mu\nu}$ can always be decomposed in the form[1]

$$\Gamma^\rho{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu} + K^\rho{}_{\mu\nu}, \tag{14}$$

where

$$\overset{\circ}{\Gamma}{}^\sigma{}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) \tag{15}$$

is the Levi-Civita connection, and

$$K^\rho{}_{\mu\nu} = \tfrac{1}{2} \left(T_\nu{}^\rho{}_\mu + T_\mu{}^\rho{}_\nu - T^\rho{}_{\mu\nu} \right) \tag{16}$$

[1] All quantities related to general relativity will be denoted with an over "∘".

is the contortion tensor. Using relation (7), the decomposition (14) can be rewritten as

$$A^a{}_{b\nu} = \overset{\circ}{A}{}^a{}_{b\nu} + K^a{}_{b\nu}, \tag{17}$$

where $\overset{\circ}{A}{}^a{}_{b\nu}$ is the spin connection of general relativity.

The quantities

$$h^a = h^a{}_\mu \, dx^\mu \quad \text{and} \quad h_a = h_a{}^\mu \, \partial_\mu \tag{18}$$

represent Lorentz frames. These frames are classified according to the value of the coefficient of anholonomy $f^c{}_{ab}$, which is defined by the commutation relation [7]

$$[h_a, h_b] = f^c{}_{ab} \; h_c. \tag{19}$$

As a simple calculation shows, it is given by

$$f^a{}_{cd} = h_c{}^\mu \, h_d{}^\nu (\partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu). \tag{20}$$

For example, the class of inertial frames is defined by all frames for which $f^c{}_{ab} = 0$. They are called, for this reason, holonomic frames. Starting from an inertial frame, different classes of frames are obtained by performing *local* (point dependent) Lorentz transformations. Inside each class, different frames are related through *global* (point independent) Lorentz transformations.

In special relativity, the anholonomy of the frames is entirely related to the inertial forces present in those frames. The preferred class of inertial frames is characterized by the absence of inertial forces, and consequently represented by holonomic frames. Similarly, in the presence of gravitation there is also a preferred class of frames, denoted h'^a: the class whose anholonomy is related to gravitation only, not with inertial effects. This class of frames, therefore, reduces to the inertial class when gravitation is switched off. Under a local Lorentz transformation

$$h^a = \Lambda^a{}_b(x) \, h'^b, \tag{21}$$

the spin connection $A^a{}_{b\mu}$ changes according to

$$A^a{}_{b\mu} = \Lambda^a{}_c(x) \, A'^c{}_{d\mu} \, \Lambda_b{}^d(x) + \Lambda^a{}_c(x) \, \partial_\mu \Lambda_b{}^c(x). \tag{22}$$

If a connection vanishes in the frame h'^a, therefore, it will be non-vanishing in the Lorentz-rotated frame h^a.

3 Fundamentals of teleparallel gravity

A fundamental property of teleparallel gravity is that, in the class of frames h'_b which reduce to inertial frames in the absence of gravitation, its spin connection[2] vanishes everywhere [5]:

$$\overset{\bullet}{A}{}'^a{}_{b\mu} = 0. \tag{23}$$

In a Lorentz rotated frame $h^a = \Lambda^a{}_b \, h'^b$, it assumes the form

$$\overset{\bullet}{A}{}^a{}_{b\mu} = \Lambda^a{}_e(x) \, \partial_\mu \Lambda_b{}^e(x). \tag{24}$$

[2]All quantities related to teleparallel gravity will be denoted with an over "•".

We see from this expression that the teleparallel spin connection represents purely inertial effects, not gravitation [9].

In the class of frames h'_b, the tetrad of teleparallel gravity is written as

$$h'^a{}_\mu = \partial_\mu x'^a + B'^a{}_\mu, \tag{25}$$

where $B'^a{}_\mu$ is the translational gauge potential. In the general frame h^a, it assumes the form

$$h^a{}_\mu = \overset{\bullet}{\mathcal{D}}_\mu x^a + B^a{}_\mu, \tag{26}$$

where $B^a{}_\mu = \Lambda^a{}_b \, B'^b{}_\mu$, and

$$\overset{\bullet}{\mathcal{D}}_\mu x^a = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu} \, x^b. \tag{27}$$

For a tetrad with a non-trivial gauge potential $B^a{}_\mu$, whereas the curvature of $\overset{\bullet}{A}{}^a{}_{b\mu}$ vanishes,

$$\overset{\bullet}{R}{}^a{}_{b\mu\nu} = 0, \tag{28}$$

the torsion is non-vanishing:

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu h^a{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu h^a{}_\mu \neq 0. \tag{29}$$

In contrast to general relativity, therefore, in which gravitation is represented by curvature, in teleparallel gravity it is represented by torsion. Defining the connection

$$\overset{\bullet}{\Gamma}{}^\lambda{}_{\mu\nu} = h_a{}^\lambda \, \overset{\bullet}{\mathcal{D}}_\nu h^a{}_\mu, \tag{30}$$

usually called Weitzenböck connection, we get

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = h^a{}_\lambda \left(\overset{\bullet}{\Gamma}{}^\lambda{}_{\nu\mu} - \overset{\bullet}{\Gamma}{}^\lambda{}_{\mu\nu} \right). \tag{31}$$

Denoting $h = \det(h^a{}_\mu)$ and $k = 8\pi G/c^4$, the gravitational lagrangian of teleparallel gravity is written as [10]

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{4k} \, \overset{\bullet}{S}{}_a{}^{\rho\sigma} \, \overset{\bullet}{T}{}^a{}_{\rho\sigma}, \tag{32}$$

where

$$\overset{\bullet}{S}{}_a{}^{\rho\sigma} \equiv -\frac{k}{h} \frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial(\partial_\sigma h^a{}_\rho)} = h_a{}^\nu \left(\overset{\bullet}{K}{}^{\rho\sigma}{}_\nu - \delta_\nu{}^\sigma \, \overset{\bullet}{T}{}^{\theta\rho}{}_\theta + \delta_\nu{}^\rho \, \overset{\bullet}{T}{}^{\theta\sigma}{}_\theta \right) \tag{33}$$

is the superpotential, with $\overset{\bullet}{K}{}^{\rho\sigma}{}_\nu$ the contortion tensor written in terms of the teleparallel torsion. The corresponding sourceless gravitational field equation appears naturally in the potential form [11], and is given by

$$\partial_\sigma(h \overset{\bullet}{S}{}_a{}^{\rho\sigma}) - k \, h \, \overset{\bullet}{\jmath}{}_a{}^\rho = 0, \tag{34}$$

where

$$\overset{\bullet}{\jmath}{}_a{}^\rho \equiv -\frac{1}{h} \frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\rho} = \frac{1}{k} \, h_a{}^\lambda \, \overset{\bullet}{S}{}_c{}^{\nu\rho} \, \overset{\bullet}{T}{}^c{}_{\nu\lambda} - \frac{h_a{}^\rho}{h} \, \overset{\bullet}{\mathcal{L}} + \frac{1}{k} \, \overset{\bullet}{A}{}^c{}_{a\sigma} \overset{\bullet}{S}{}_c{}^{\rho\sigma} \tag{35}$$

is the gravitational energy-momentum current. Due to the anti-symmetry of the superpotential in the last two indices, it is conserved in the ordinary sense:

$$\partial_\rho(h \overset{\bullet}{\jmath}_a{}^\rho) = 0. \tag{36}$$

The gravitational energy-momentum current can be decomposed according to

$$\overset{\bullet}{\jmath}_a{}^\rho = \overset{\bullet}{t}_a{}^\rho + \overset{\bullet}{\imath}_a{}^\rho, \tag{37}$$

where [12]

$$\overset{\bullet}{t}_a{}^\rho = \frac{1}{k}\, h_a{}^\lambda\, \overset{\bullet}{S}_c{}^{\nu\rho}\, \overset{\bullet}{T}{}^c{}_{\nu\lambda} - \frac{h_a{}^\rho}{h}\, \overset{\bullet}{\mathcal{L}} \tag{38}$$

is a tensorial current that represents the energy-momentum of gravity alone, and

$$\overset{\bullet}{\imath}_a{}^\rho = \frac{1}{k}\, \overset{\bullet}{A}{}^c{}_{a\sigma} \overset{\bullet}{S}_c{}^{\rho\sigma} \tag{39}$$

is the energy-momentum current of inertia [9]. Of course, by its very nature, it is non-covariant. Using the fact that

$$\partial_\sigma(h \overset{\bullet}{S}_a{}^{\rho\sigma}) - \overset{\bullet}{A}{}^c{}_{a\sigma}(h\, \overset{\bullet}{S}_c{}^{\rho\sigma}) \equiv \overset{\bullet}{\mathcal{D}}_\sigma(h \overset{\bullet}{S}_a{}^{\rho\sigma}), \tag{40}$$

the field equation (34) can be rewritten in the form

$$\overset{\bullet}{\mathcal{D}}_\sigma(h \overset{\bullet}{S}_a{}^{\rho\sigma}) - k\, h\, \overset{\bullet}{t}_a{}^\rho = 0. \tag{41}$$

Then comes the crucial point: since the teleparallel spin connection (24) has vanishing curvature, the corresponding Fock-Ivanenko derivative is commutative:

$$[\overset{\bullet}{\mathcal{D}}_\rho, \overset{\bullet}{\mathcal{D}}_\sigma] = 0. \tag{42}$$

Taking into account the anti-symmetry of the superpotential in the last two indices, it follows from the field equation (41) that the tensorial current (38) is conserved in the covariant sense:

$$\overset{\bullet}{\mathcal{D}}_\rho(h \overset{\bullet}{t}_a{}^\rho) = 0. \tag{43}$$

Of course, as it does not represent the total energy-momentum density—in the sense that the energy-momentum density of inertia is not included—it does not need to be truly conserved.

4 Spin-2 field in flat spacetime

4.1 Gauge transformations

In an inertial frame, the tetrad describing the flat Minkowski spacetime is of the form

$$e'^a{}_\mu = \partial_\mu x'^a. \tag{44}$$

A spin-2 field $\phi'^a{}_\mu$ corresponds to a linear perturbation of this tetrad:

$$h'^a{}_\mu = \partial_\mu x'^a + \phi'^a{}_\mu. \tag{45}$$

In this class of frames, therefore, the vacuum is represented by

$$\phi'^a{}_\mu = \partial_\mu \xi^a(x), \tag{46}$$

with $\xi^a(x)$ an arbitrary function of the spacetime coordinates x^ρ. In fact, such $\phi'^a{}_\mu$ represents simply a gauge transformation

$$x'^a \to x'^a + \xi^a(x) \tag{47}$$

of the tangent space (or fiber) coordinates. This means that the gauge transformation associated to the spin-2 field $\phi^a{}_\mu$ is

$$\phi'^a{}_\mu \to \phi'^a{}_\mu - \partial_\mu \xi^a(x). \tag{48}$$

Of course, $h'^a{}_\mu$ is invariant under such transformations.

4.2 Field strength and Bianchi identity

In the inertial frame e'^a, and following the teleparallel gravity paradigm, the first step towards the construction of a field theory for $\phi'^a{}_\mu$ is to define the analogous of torsion,

$$F'^a{}_{\mu\nu} = \partial_\mu \phi'^a{}_\nu - \partial_\nu \phi'^a{}_\mu. \tag{49}$$

This tensor is actually the spin-2 field-strength. As can be easily verified, $F'^a{}_{\mu\nu}$ is gauge invariant. Furthermore, it satisfies the Bianchi identity

$$\partial_\rho F'^a{}_{\mu\nu} + \partial_\nu F'^a{}_{\rho\mu} + \partial_\mu F'^a{}_{\nu\rho} = 0, \tag{50}$$

which can equivalently be written in the form

$$\partial_\rho(\varepsilon^{\lambda\rho\mu\nu}\, F'^a{}_{\mu\nu}) = 0, \tag{51}$$

with $\varepsilon^{\lambda\rho\mu\nu}$ the totally anti-symmetric, flat spacetime Levi-Civita tensor.

4.3 Lagrangian and field equation

Considering that the dynamics of a spin-2 field must coincide with the dynamics of linear gravity, its lagrangian will be similar to the lagrangian (32) of teleparallel gravity. One has just to replace the teleparallel torsion $\overset{\bullet}{T}{}^a{}_{\mu\nu}$ by the spin-2 field strength $\sqrt{k}\, F'^a{}_{\mu\nu}$. It is interesting to notice that when we do that, the spin-2 analogous of the teleparallel superpotential $\overset{\bullet}{S}{}^a{}_{\mu\nu}$ is the Fierz tensor [13]

$$\mathcal{F}'^{\ \mu\nu}_a = e'^{\ \rho}_a\, \mathcal{K}'^{\mu\nu}{}_\rho - e'^{\ \nu}_a\, e'^{\ \rho}_b\, F'^{b\mu}{}_\rho + e'^{\ \mu}_a\, e'^{\ \rho}_b\, F'^{b\nu}{}_\rho, \tag{52}$$

with

$$\mathcal{K}'^{\mu\nu}{}_\rho = \tfrac{1}{2}\,(e'^{\ \nu}_a\, F'^{a\mu}{}_\rho + e'^a{}_\rho\, \mathcal{F}'^{\ \mu\nu}_a - e'^{\ \mu}_a\, F'^{a\nu}{}_\rho) \tag{53}$$

the spin-2 analogous of the teleparallel contortion. Considering that in the absence of gravitation $\det(e'^a{}_\mu) = 1$, the lagrangian for a massless spin-2 field is

$$\text{Ł}' = \frac{1}{4}\, \mathcal{F}'^{\ \mu\nu}_a\, F'^a{}_{\mu\nu}. \tag{54}$$

By performing variations in relation to $\phi^a{}_\rho$, we obtain

$$\partial_\mu \mathcal{F}'^{\ \rho\mu}_a = 0. \tag{55}$$

This is the field equation satisfied by a massless spin-2 field in Minkowski spacetime, as seen from the inertial frame $e'^a{}_\mu$. Notice that teleparallel gravity naturally yields the Fierz formulation for a spin-2 field [14], and is gauge invariant.

4.4 Duality symmetry

The spin-2 field can be viewed as an Abelian gauge field with the *internal* index replaced by an *external* Lorentz index. Due to the presence of the tetrad, Lorentz and spacetime indices can be transformed into each other. As a consequence, its Hodge dual will necessarily include additional index contractions in relation to the usual dual. Taking into account al possible contractions, its dual turns out to be given by [15]

$$\star F'^{a}{}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,\mathcal{F}'^{a\rho\sigma}. \tag{56}$$

Substituting the Fierz tensor (52), we find

$$\star F'^{a}{}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left(F'^{a\rho\sigma} - e'^{a\sigma} e'_{b}{}^{\lambda}\, F'^{b\rho}{}_{\lambda} + e'^{a\rho} e'_{b}{}^{\lambda}\, F'^{b\sigma}{}_{\lambda}\right). \tag{57}$$

Let us now consider the Bianchi identity (51). Written for the dual of $F^{a}{}_{\mu\nu}$, it reads

$$\partial_\rho(\varepsilon^{\lambda\rho\mu\nu}\,\star F'^{a}{}_{\mu\nu}) = 0. \tag{58}$$

Substituting $\star F'^{a}{}_{\mu\nu}$ as given by Eq. (56), we get

$$\partial_\rho \mathcal{F}'^{a\mu\rho} = 0, \tag{59}$$

which is the field equation (55). We see in this way that, provided the generalized Hodge dual (56) for soldered bundles is used, the spin-2 field has duality symmetry. This is actually an expected result because the dynamics of a spin-2 field must coincide with the dynamics of linear gravity, which has already been shown to present duality symmetry [16]. We remark finally that, in the usual Fierz formulation of a spin-2 field, identity (51) has to be put by hand in order to get the correct number of equations [17]. In the present formulation it appears naturally as a consequence of the duality symmetry.

4.5 Passage to a general frame

In a Lorentz rotated frame $e^a = \Lambda^a{}_b(x)\, e'^a$, the tetrad assumes the form

$$e^a{}_\mu \equiv \overset{\bullet}{\mathcal{D}}_\mu x^a = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu}\, x^b. \tag{60}$$

In this frame, the vacuum of $\phi^a{}_\mu$ turns out to be represented by

$$\phi^a{}_\mu = \overset{\bullet}{\mathcal{D}}_\mu \xi^a(x), \tag{61}$$

whereas the gauge transformations assume the form

$$\phi^a{}_\mu \to \phi^a{}_\mu - \overset{\bullet}{\mathcal{D}}_\mu \xi^a(x). \tag{62}$$

The field strength (49), on the other hand, becomes

$$F^a{}_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu \phi^a{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu \phi^a{}_\mu. \tag{63}$$

Accordingly, the Bianchi identity reads

$$\overset{\bullet}{\mathcal{D}}_\rho F^a{}_{\mu\nu} + \overset{\bullet}{\mathcal{D}}_\nu F^a{}_{\rho\mu} + \overset{\bullet}{\mathcal{D}}_\mu F^a{}_{\nu\rho} = 0, \tag{64}$$

which is equivalent to

$$\stackrel{\bullet}{\mathcal{D}}_\rho(\varepsilon^{\lambda\rho\mu\nu}\, F^a{}_{\mu\nu}) = 0. \tag{65}$$

Analogously, the Fierz tensor turns out to be

$$\mathcal{F}_a{}^{\mu\nu} = e_a{}^\rho\, \mathcal{K}^{\mu\nu}{}_\rho - e_a{}^\nu\, e_b{}^\rho\, F^{b\mu}{}_\rho + e_a{}^\mu\, e_b{}^\rho\, F^{b\nu}{}_\rho. \tag{66}$$

Now, as a simple inspection shows, the lagrangian (54) is invariant under local Lorentz transformations, that is,

$$Ł' \equiv Ł = \frac{1}{4}\, \mathcal{F}_a{}^{\mu\nu}\, F^a{}_{\mu\nu}. \tag{67}$$

The corresponding field equation,

$$\stackrel{\bullet}{\mathcal{D}}_\mu \mathcal{F}_a{}^{\rho\mu} \equiv \partial_\mu \mathcal{F}_a{}^{\rho\mu} - \stackrel{\bullet}{A}{}^b{}_{a\mu}\, \mathcal{F}_b{}^{\rho\mu} = 0, \tag{68}$$

represents the field equation satisfied by a massless spin-2 field in Minkowski spacetime, as seen from the general frame $e^a{}_\mu$. It is clearly invariant under the gauge transformation (62). It is important to observe that the theory has twenty two constraints: sixteen of the invariance under the gauge transformations (48), and six from the invariance of the lagrangian Ł under local Lorentz transformations. The twenty four original components of the Fierz tensor are then reduced to only two, as appropriate for a massless spin-2 field.

5 Spin-2 field in the presence of gravitation

5.1 Gravitational coupling prescription

The presence of gravitation in teleparallel gravity is achieved by replacing the non-gravitational tetrad $e^a{}_\mu = \stackrel{\bullet}{\mathcal{D}}_\mu x^a$ by the teleparallel tetrad $h^a{}_\mu$,

$$e^a{}_\mu \to h^a{}_\mu = e^a{}_\mu + B^a{}_\mu, \tag{69}$$

with $B^a{}_\mu$ the translational gauge potential. As a consequence, the spacetime metric changes from Minkowski $\eta_{\mu\nu}$ to a nontrivial metric $g_{\mu\nu}$:

$$\eta_{\mu\nu} = e^a{}_\mu e^b{}_\nu\, \eta_{ab} \to g_{\mu\nu} = h^a{}_\mu h^b{}_\nu\, \eta_{ab}. \tag{70}$$

Then comes the crucial point: as a (translational-valued) vector field, in the frame h'_a the gravitational coupling prescription of $\phi'_\rho = \phi'^a{}_\rho P'_a$ is achieved by replacing [18]

$$\partial_\mu \phi'_\rho \to \partial_\mu \phi'_\rho - \left(\stackrel{\bullet}{\Gamma}{}^\lambda{}_{\rho\mu} - \stackrel{\bullet}{K}{}^\lambda{}_{\rho\mu}\right) \phi'_\lambda. \tag{71}$$

Of course, because of the identity

$$\stackrel{\bullet}{\Gamma}{}^\lambda{}_{\rho\mu} - \stackrel{\bullet}{K}{}^\lambda{}_{\rho\mu} = \stackrel{\circ}{\Gamma}{}^\lambda{}_{\rho\mu}, \tag{72}$$

with $\stackrel{\circ}{\Gamma}{}^\lambda{}_{\rho\mu}$ the Levi-Civita connection of the metric $g_{\mu\nu}$, the coupling prescription (71) coincides with that of general relativity.

On the other hand, under a local Lorentz transformation, the ordinary derivative of $\phi'^a{}_\rho$ transforms according to

$$\partial_\mu \phi'^a{}_\rho \to \dot{\mathcal{D}}_\mu \phi^a{}_\rho = \partial_\mu \phi^a{}_\rho + \dot{A}^a{}_{b\mu}\, \phi^b{}_\rho. \tag{73}$$

In this general frame, therefore, the gravitational coupling prescription of a fundamental spin-2 field $\phi^a{}_\rho$ is written as

$$\dot{\mathcal{D}}_\mu \phi^a{}_\rho \to \dot{\mathcal{D}}_\mu \phi^a{}_\rho - \left(\dot{\Gamma}^\lambda{}_{\rho\mu} - \dot{K}^\lambda{}_{\rho\mu} \right) \phi^a{}_\lambda. \tag{74}$$

This coupling prescription is an inherent property of teleparallel gravity, and constitutes the main difference of the present approach in relation to the usual one, based on general relativity. It is fundamentally related with fact that teleparallel gravity allows a separation between inertia and gravitation [9]. In fact, observe that whereas the tangent-space index of $\phi^a{}_\rho$ is connected with the inertial effects of the frame, the spacetime index is connected with the coupling to gravitation.

5.2 Field strength and Bianchi identity

Let us now apply the gravitational coupling prescription (74) to the free theory. To begin with we notice that, because the gauge parameter ξ^a has an algebraic index only, the gauge transformation (62) does not change in the presence of gravitation:

$$\phi^a{}_\mu \to \phi^a{}_\mu - \dot{\mathcal{D}}_\mu \xi^a(x). \tag{75}$$

On the other hand, considering that the connection (72) is symmetric in the last two indices, we see that the field strength $F^a{}_{\mu\nu}$ does not change in the presence of gravitation:

$$F^a{}_{\mu\nu} = \dot{\mathcal{D}}_\mu \phi^a{}_\nu - \dot{\mathcal{D}}_\nu \phi^a{}_\mu. \tag{76}$$

Due to the fact that the teleparallel Fock-Ivanenko derivative $\dot{\mathcal{D}}_\mu$ is commutative, the Bianchi identity also remains unchanged,

$$\dot{\mathcal{D}}_\rho F^a{}_{\mu\nu} + \dot{\mathcal{D}}_\nu F^a{}_{\rho\mu} + \dot{\mathcal{D}}_\mu F^a{}_{\nu\rho} = 0. \tag{77}$$

Since $\varepsilon^{\lambda\rho\mu\nu}$ is a density of weight $\omega = -1$, in the presence of gravitation the Levi-Civita *tensor* is given by $h\varepsilon^{\lambda\rho\mu\nu}$, and the Bianchi identity can be rewritten in the form

$$\dot{\mathcal{D}}_\rho (h\, \varepsilon^{\lambda\rho\mu\nu}\, F^a{}_{\mu\nu}) = 0. \tag{78}$$

This is similar to what happens to the electromagnetic field in the presence of gravitation.

5.3 Lagrangian and field equation

Analogously to the flat background case, the lagrangian of the spin-2 field in the presence of gravitation can be obtained from the teleparallel lagrangian (32) by replacing the teleparallel torsion $\dot{T}^a{}_{\mu\nu}$ by the spin-2 field strength $\sqrt{k}\, F^a{}_{\mu\nu}$. The result is

$$\mathcal{L} = \frac{h}{4}\, \mathcal{F}_a{}^{\mu\nu}\, F^a{}_{\mu\nu}, \tag{79}$$

where

$$\mathcal{F}_a{}^{\mu\nu} = h_a{}^\rho\, \mathcal{K}^{\mu\nu}{}_\rho - h_a{}^\mu\, h_b{}^\sigma F^{b\nu}{}_\sigma + h_a{}^\nu\, h_b{}^\sigma F^{b\mu}{}_\sigma \tag{80}$$

is the gravitationally-coupled Fierz tensor, with

$$\mathcal{K}^{\mu\nu}{}_\rho = \tfrac{1}{2}\,(h_a{}^\nu\, F^{a\mu}{}_\rho + h^a{}_\rho\, F_a{}^{\mu\nu} - h_a{}^\mu\, F^{a\nu}{}_\rho) \tag{81}$$

the corresponding spin-2 analogous of the coupled contortion. We notice in passing that this lagrangian is invariant under the gauge transformation (62). It is furthermore invariant under local Lorentz transformation $h'^a{}_\mu = \Lambda^a{}_b(x)\, h^a{}_\mu$ of the frames.

Performing variations in relation to $\phi^a{}_\rho$, we get

$$\overset{\bullet}{\mathcal{D}}_\mu \mathcal{F}_a{}^{\rho\mu} + (\overset{\bullet}{\Gamma}{}^\mu{}_{\nu\mu} - \overset{\bullet}{K}{}^\mu{}_{\nu\mu})\, \mathcal{F}_a{}^{\rho\nu} = 0. \tag{82}$$

Using the identity

$$\partial_\mu h = h\, \overset{\circ}{\Gamma}{}^\mu{}_{\lambda\mu} \equiv h\left(\overset{\bullet}{\Gamma}{}^\mu{}_{\lambda\mu} - \overset{\bullet}{K}{}^\mu{}_{\lambda\mu}\right), \tag{83}$$

it can be rewritten in the form

$$\overset{\bullet}{\mathcal{D}}_\mu (h \mathcal{F}_a{}^{\rho\mu}) = 0. \tag{84}$$

This is the equation of motion of a fundamental spin-2 field in the presence of gravitation, as seen from the general frame $h^a{}_\mu$. Observe that it can also be obtained from the free field equation (68) by applying the gravitational coupling prescription (74). Observe also that, like the free theory, the gravitationally-coupled theory has duality symmetry. Finally, it is important to remark that, on account of the commutativity of the covariant derivative $\overset{\bullet}{\mathcal{D}}_\mu$, the gravitationally-coupled spin-2 theory is found to be gauge invariant. Since its lagrangian is invariant under local Lorentz transformations, the spin-2 theory has the correct number of independent components even in the presence of gravitation.

6 Spin-2 field as source of gravitation

Let us consider now the total lagrangian

$$\mathcal{L}_t = \overset{\bullet}{\mathcal{L}} + \mathcal{L}, \tag{85}$$

where $\overset{\bullet}{\mathcal{L}}$ is the teleparallel lagrangian (32), and $\mathcal{L}$ is the lagrangian (79) of a spin-2 field in the presence of gravitation. The corresponding field equation is

$$\partial_\sigma (h \overset{\bullet}{S}_a{}^{\rho\sigma}) - k\, h\, (\overset{\bullet}{t}_a{}^\rho + \overset{\bullet}{\imath}_a{}^\rho) = k\, h\, \theta_a{}^\rho, \tag{86}$$

where $\overset{\bullet}{t}_a{}^\rho$ is the gravitational energy-momentum tensor, $\overset{\bullet}{\imath}_a{}^\rho$ is the energy-momentum pseudotensor of inertia, and

$$\theta_a{}^\rho \equiv -\frac{1}{h}\frac{\delta \mathcal{L}}{\delta h^a{}_\rho} = h_a{}^\nu\, \mathcal{F}_c{}^{\mu\rho}\, F^c{}_{\mu\nu} - \frac{h_a{}^\rho}{h}\, \mathcal{L} \tag{87}$$

is the spin-2 field source energy-momentum tensor. Observe that

$$\theta_\rho{}^\rho \equiv h^a{}_\rho\, \theta_a{}^\rho = 0, \tag{88}$$

as it should be for a massless field. Furthermore, from the invariance of $\mathcal{L}$ under general coordinate transformation, it is found to satisfy the usual (general relativity) covariant conservation law [19]

$$\overset{\circ}{\mathcal{D}}_\rho (h\theta_a{}^\rho) \equiv \partial_\rho (h\theta_a{}^\rho) - \overset{\circ}{A}{}^b{}_{a\rho}\,(h\theta_b{}^\rho) = 0. \tag{89}$$

Due to the anti-symmetry of the superpotential in the last two indices, we see from the field equation (86) that the total energy-momentum density is conserved in the ordinary sense:

$$\partial_\rho [h(\overset{\bullet}{\imath}_a{}^\rho + \overset{\bullet}{t}_a{}^\rho + \theta_a{}^\rho)] = 0. \tag{90}$$

The field equation (86) can be rewritten in the form

$$\overset{\bullet}{\mathcal{D}}_\sigma (h\overset{\bullet}{S}_a{}^{\rho\sigma}) = k\,h\,(\overset{\bullet}{t}_a{}^\rho + \theta_a{}^\rho), \tag{91}$$

where the right-hand side represents the true gravitational field source. We recall that the energy-momentum density of inertia, although entering the total energy-momentum conservation, is not source of gravitation. Now, because $[\overset{\bullet}{\mathcal{D}}_\rho, \overset{\bullet}{\mathcal{D}}_\sigma] = 0$, the true source of gravitation is conserved in the covariant sense:

$$\overset{\bullet}{\mathcal{D}}_\rho [h(\overset{\bullet}{t}_a{}^\rho + \theta_a{}^\rho)] = 0. \tag{92}$$

This property ensures the consistency of the theory in the sense that no constraints on the background spacetime geometry show up.

7 Final remarks

In general relativity the gravitational interaction is described by a geometrization of spacetime. According to this theory, the presence of gravitation produces a curvature in spacetime, and the (universal) gravitational interaction is achieved by assuming that all (spinless) particles follow that curvature. Instead of force equations, therefore, the particles equations of motion are given by geodesics. Due to this geometric structure, general relativity is not a field theory in the usual sense of classical fields. On the other hand, owing to its gauge structure, the teleparallel equivalent of general relativity is much more akin to a usual field theory than general relativity. In fact, in teleparallel gravity the interaction is not geometrized, but described by a gauge force. As a consequence, the particles equations of motion are not geodesics, but force equations.

Now, as is well known, the dynamics of a fundamental spin-2 field in Minkowski spacetime coincides with the dynamics of linear gravity. However, since general relativity is not a true field theory, and considering that teleparallel gravity describes gravitation similarly to all other gauge interactions, in order to construct a field theory for a fundamental spin-2 field one should consider teleparallel gravity, and not general relativity, as paradigm. Accordingly, instead of a symmetric second-rank tensor $\phi_{\mu\nu}$, a spin-2 field turns out to be represented by a spacetime (world) vector field assuming values in the Lie algebra of the translation group. Its components $\phi^a{}_\mu$, like the gauge potential of teleparallel gravity, represent a set of four spacetime vector fields. Due to the gauge structure of teleparallel gravity, the resulting spin-2 field theory in Minkowski spacetime naturally emerges in the Fierz formalism, and turns out to be quite similar to electromagnetism, a gauge theory for the $U(1)$ group. In fact, in addition to satisfying a dynamic field equation, the spin-2 field is found to satisfy also a Bianchi identity, which is related to the dynamic field equation by duality transformation. Furthermore, the gauge and the local Lorentz invariance of the theory provide the correct number of independent components for a massless spin-2 field.

In special relativity, inertial effects are represented by a (vacuum) Lorentz connection [20]. In general relativity, on the other hand, the spin connection $\overset{\circ}{A}_\mu$ represents both inertia and gravitation. This becomes clear if we notice that, in a local frame where inertia compensates gravitation, $\overset{\circ}{A}_\mu$ vanishes identically. Now, differently from general relativity, the spin connection of teleparallel gravity $\overset{\bullet}{A}_\mu$ keeps the special relativity property of representing inertial effects only.[3] As a consequence of this property, the coupling prescription of $\phi^a{}_\mu$ to gravitation is such that all spin-2 field equations and identities (see the field equation (84) and the conservation law (92)) turn out to be written in the terms of the teleparallel covariant derivative

$$\overset{\bullet}{\mathcal{D}}_\mu = \partial_\mu - \overset{\bullet}{A}_\mu, \tag{93}$$

which is commutative due to the inertial nature of the teleparallel spin connection $\overset{\bullet}{A}_\mu$. Owing to this commutativity, the gravitationally-coupled spin-2 theory shows duality symmetry, as well as gauge and local Lorentz invariance. This ensures that, even in the presence of gravitation, the theory has the correct number of components to represent a massless spin-2 particle. Furthermore, the gravitational plus the spin-2 energy-momentum densities—the true source of gravitation—satisfies a conservation law with the same (commutative) covariant derivative. Together, these properties render the (teleparallel-inspired) spin-2 field theory coupled to gravitation fully consistent, and free of the unphysical constraints which appear in the usual approach based on general relativity.[4]

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