Rigidity of noncompact complete Bach-flat manifolds

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Abstract

Let (M, g) be a noncompact complete Bach-flat manifold with positive Yamabe constant. We prove that (M, g) is flat if (M, g) has zero scalar curvature and sufficiently small L_2 bound of curvature tensor. When (M, g) has nonconstant scalar curvature, we prove that (M, g) is conformal to the flat space if (M, g) has sufficiently small L_2 bound of curvature tensor and $L_{4/3}$ bound of scalar curvature.

1 Introduction

Let (M, g) be a noncompact complete Riemannian 4-manifold with scalar curvature R, Weyl curvature W, Ricci curvature R_{ij} and curvature tensor *Riem*. A metric is Bach-flat if it is a critical metric of the functional

$$g \longrightarrow \int_M |W|^2 \, dV_g. \tag{1}$$

Bach-flat condition is equivalent to the vanishing of Bach tensor B_{ij}, which is defined by

$$B_{ij} \equiv \nabla^k \nabla^l W_{kijl} + \frac{1}{2} R^{kl} W_{kijl}. \tag{2}$$

(see [1]). Important examples of Bach manifolds are Einstein manifolds, self-dual (anti-self-dual) manifolds, conformally flat manifolds and Käler surfaces with zero scalar curvature (see [2]).

Einstein metrics and Bach-flat metrics share many important properties. When the curvature of a given Einstein metric (M, g) is sufficiently close to that of the constant curvature space, in $L_{\frac{n}{2}}$ sense, it is known that (M, g) is isometric to a quotient of the constant curvature space [3, 4, 5, 6, 7]. In this paper, we study this rigidity phenomena for noncompact complete Bach-flat manifolds. First, we study rigidity of metrics with positive Yamabe constant, zero scalar curvature and small L_2 bound of curvature. Next we look for rigidity of nonconstant scalar curvature spaces with a conformal change of the given metric.

There are known rigidity of Bach-flat metrics. For a compact Bach-flat manifold (M, g) with positive Yamabe constant, Chang, Ji and Yang [8] proved that there is only finite diffeomorphism class with an L_2 bound of Weyl tensor, and (M, g) is conformal to the standard sphere if L_2 norm of Weyl tensor is small enough. For a noncompact complete Bach-flat manifold (M, g) with positive Yamabe constant and zero scalar curvature, Tian and Viaclovsky [9] proved that (M, g) is almost locally Euclidean of order 0 with L_2 bounds of curvature, bounded first Betti number and the uniform volume growth for any geodesic ball.

For rigidity of Bach-flat manifolds, we use an elliptic estimation on the Laplacian of curvature tensor. For this, we introduce the Yamabe constant on (M, g). Let (M, g) be a Riemannian manifold of dimension $n \geq 3$ with scalar curvature R. The Yamabe constant $Q(M, g)$ is defined by

$$Q(M, g) \equiv \inf_{0 \neq u \in C_0^\infty(M)} \frac{\frac{4(n-1)}{(n-2)} \int_M |\nabla u|^2 + R_g u^2 \, dV_g}{\left(\int_M |u|^{2n/(n-2)} dV_g\right)^{(n-2)/n}}.$$

$Q(M, g)$ is conformally invariant and any locally conformally flat manifold and manifolds with zero scalar curvature satisfy $Q(M, g) > 0$ (see [10]).

2 Bach-flat metric with constant scalar curvature

In this section, we study noncompact complete Bach-flat manifolds with nonnegative constant scalar curvature. First, we consider Bach-flat manifolds whose L_2 curvature norm is small. By an elliptic estimation for the Laplacian of curvature tensor, we have:

Theorem 1 *Let (M, g) be a noncompact complete Bach-flat Riemannian 4-manifold with zero scalar curvature and $Q(M, g) > 0$. Then there exists a small number c_0 such that if $\int_M |Riem|^2\, dV_g \le c_0$, then (M, g) is flat, i.e., $Riem = 0$, where $Riem$ is curvature tensor.*

Proof. We need to prove that $|Riem| = 0$. The Laplacian of curvature tensor is

$$\begin{aligned}\Delta R_{ijkl} &= 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) + \nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk} \\ &\quad -\nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik} + g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj}). \qquad (3)\end{aligned}$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ (see [11]). Multiplying R_{ijkl} on (3),

$$\begin{aligned}& R_{ijkl}\Delta R_{ijkl} \\ =\ & 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl})R_{ijkl} + (\nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk})R_{ijkl} \\ & +(-\nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik})R_{ijkl} + g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj})R_{ijkl}. \qquad (4)\end{aligned}$$

To simplify notations, we will work in an orthonormal frame. By the Bianchi identity,

$$\nabla^i R_{ijkl} = \nabla_k R_{jl} - \nabla_l R_{jk}. \qquad (5)$$

For a smooth compact supported function ϕ and small $\epsilon > 0$, we integrate the second term in (4)

$$\begin{aligned}& \int_M \phi^2(\nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk})R_{ijkl}\, dV_g \\ =\ & -\int_M \nabla_i \phi^2(\nabla_k R_{jl} - \nabla_l R_{jk})R_{ijkl} + \phi^2(\nabla_k R_{jl} - \nabla_l R_{jk})\nabla_i R_{ijkl}\, dV_g \\ =\ & -\int_M \nabla_i \phi^2 \nabla_t R_{tjkl} R_{ijkl} + \phi^2 |\nabla_i R_{ijkl}|^2\, dV_g \qquad (6) \\ \ge\ & -\int_M \frac{1}{\epsilon}|\nabla\phi|^2 |R_{ijkl}|^2 + (1+\epsilon)\phi^2 |\nabla_i R_{ijkl}|^2\, dV_g. \qquad (7)\end{aligned}$$

Using the same method,

$$\begin{aligned}& \int_M \phi^2(-\nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik})R_{ijkl}\, dV_g \\ \ge\ & -\int_M \frac{1}{\epsilon}|\nabla\phi|^2 |R_{ijkl}|^2 + (1+\epsilon)\phi^2 |\nabla_j R_{ijkl}|^2\, dV_g. \qquad (8)\end{aligned}$$

The first and fourth terms in (4) are contractions of cubic terms of curvature tensor which can be bounded by $c|Riem|^3$ for a constant c. In this paper, we use c and c' to denote some positive constant, which can be varied. By the Kato inequality,

$$|\nabla Riem|^2 \geq |\nabla|Riem||^2$$

and

$$\begin{aligned}
& -\int_M \phi^2 |Riem| \triangle |Riem| \, dV_g & (9)\\
= & -\int_M \phi^2 \left(|\nabla Riem|^2 - |\nabla|Riem||^2 + R_{ijkl} \triangle R_{ijkl} \right) dV_g & (10)\\
\leq & \int_M 2 \left(\frac{1}{\epsilon} |\nabla \phi|^2 |R_{ijkl}|^2 + (1+\epsilon) \phi^2 |\nabla_i R_{ijkl}|^2 \right) + c|Riem|^3 \phi^2 \, dV_g. & (11)
\end{aligned}$$

For a general Riemannian n-manifold, the following hold:

$$\begin{aligned}
(\delta W)_{jkl} &= \nabla^i W_{ijkl} \\
&= \frac{(n-3)}{(n-2)} \left(\nabla_k R_{jl} - \nabla_l R_{jk} - \frac{1}{6} \nabla_k R g_{jl} + \frac{1}{6} \nabla_l R g_{jk} \right) \quad (12)
\end{aligned}$$

and

$$|\nabla^i W_{ijkl}|^2 = (\frac{n-3}{n-2})^2 \left(|\nabla^i R_{ijkl}|^2 - \frac{1}{6} |\nabla R|^2 \right). \quad (13)$$

Let E_{ij} be the traceless Ricci tensor, i.e $E_{ij} = R_{ij} - \frac{1}{4} R g_{ij}$. Multiplying ϕE_{ij} on Bach-flat equation (2) (cf. [12, (3. 24)]),

$$\begin{aligned}
0 &= \int_M \phi^2 E_{ij} \left(\nabla_k \nabla_\ell W_{ik\ell j} - \frac{1}{2} W_{ikj\ell} E_{k\ell} \right) dV_g & (14)\\
&= \int_M \phi^2 \left(\left| \delta W \right|^2 - \frac{1}{2} W_{ikj\ell} E_{k\ell} E_{ij} \right) - 2\phi \nabla_k \phi \nabla_l W_{iklj} E_{ij} \, dV_g & \\
&= \int_M \phi^2 \left(\left| \delta W \right|^2 - \frac{1}{2} W_{ikj\ell} E_{k\ell} E_{ij} \right) - 2\phi \nabla_k \phi \nabla_l W_{iklj} E_{ij} \, dV_g & (15)\\
&\geq \int_M (1-\epsilon_2) \phi^2 \left| \delta W \right|^2 - \frac{1}{2} \phi^2 W_{ikj\ell} E_{k\ell} E_{ij} - \frac{1}{\epsilon_2} |\nabla \phi|^2 |E_{ij}|^2 \, dV_g, & (16)
\end{aligned}$$

where $W_{ikj\ell} g_{kl} = 0$ is used in (14). Therefore,

$$\int_M \phi^2 \left| \delta W \right|^2 \leq (1-\epsilon_2)^{-1} \int_M \frac{1}{2} \phi^2 W_{ikj\ell} E_{k\ell} E_{ij} + \frac{1}{\epsilon_2} |\nabla \phi|^2 |E_{ij}|^2 \, dV_g \quad (17)$$

and

$$
\begin{aligned}
& -\int_M \phi^2 |Riem| \triangle |Riem| \, dV_g \\
\leq \; & \int_M 2\Big[\frac{1}{\epsilon}|\nabla\phi|^2|R_{ijkl}|^2 + \frac{4(1+\epsilon)}{1-\epsilon_2}\phi^2 W_{ikj\ell}E_{k\ell}E_{ij} + \frac{8(1+\epsilon)}{(1-\epsilon_2)\epsilon_2}|\nabla\phi|^2|E_{ij}|^2 \\
& + \frac{1}{3}(1+\epsilon)\phi^2|\nabla R|^2\Big] + c|Riem|^3\phi^2 \, dV_g. \qquad (18)
\end{aligned}
$$

Note that second term in (18) is also cubic term of curvature tensor. Now we can bound all cubic terms in the above equation by $c|Riem|^3$, and the first and third terms by $c|\nabla\phi|^2|Riem|^2$ for a suitable constant c. Next we use the fact that scalar curvature is zero. For simplicity of notations, we let $u = |Riem|$. Using the Yamabe constant $\Lambda_0 \equiv Q(M, g)$,

$$
\begin{aligned}
& \Lambda_0 \left(\int_M (\phi u)^4 \, dV_g\right)^{1/2} \\
\leq \; & \int_M |u\nabla\phi + \phi\nabla u|^2 \, dV_g + \frac{1}{6}Ru^2\phi^2 \, dV_g \qquad (19) \\
\leq \; & \int_M u^2|\nabla\phi|^2 + |\nabla u|^2\phi^2 + 2u\phi\nabla\phi\cdot\nabla u + \frac{1}{6}Ru^2\phi^2 \; dV_g \\
\leq \; & \int_M (c+1)|\nabla\phi|^2u^2 + cu^3\phi^2 \, dV_g \\
\leq \; & \int_M (c+1)|\nabla\phi|^2u^2 \, dV_g \\
& + c\left(\int_M (\phi u)^4 \, dV_g\right)^{1/2}\left(\int_M u^2 \, dV_g\right)^{1/2}. \qquad (20)
\end{aligned}
$$

Since $\int_M |Riem|^2 \, dV_g$ is sufficiently small, there exists a constant c' such that

$$
c'(\int_M (\phi u)^4 \, dV_g)^{1/2} \leq \int_M |\nabla\phi|^2 u^2 \, dV_g. \qquad (21)
$$

Now we choose ϕ as

$$
\phi = \begin{cases} 1 & \text{on } B_t \\ 0 & \text{on } M - B_{2t} \\ |\nabla\phi| \leq \frac{2}{t} & \text{on } B_{2t} - B_t \end{cases} \qquad (22)
$$

with $0 \leq \phi \leq 1$ and $B_t = \{x \in M | d(x, x_0) \leq t\}$ for some fixed $x_0 \in M$. From (21)

$$
\begin{aligned}
c' \left(\int_M u^4 \phi^4 \, dV_g \right)^{1/2} &\leq \frac{4}{t^2} \int_{B(2t)-B(t)} u^2 \, dV_g \\
&\leq \frac{4}{t^2} \frac{\sqrt{3}}{2} c_0. \qquad (23)
\end{aligned}
$$

By taking $t \to \infty$, we have $u = 0$. Therefore (M, g) is flat.

Next we consider complete Bach-flat metric with positive constant scalar curvature. Using an elliptic estimation for traceless Ricci tensor, we prove

Theorem 2 *Let (M, g) be a noncompact complete Riemannian 4-manifold with nonnegative constant scalar curvature R, Weyl curvature W and traceless Ricci curvature E_{ij}. Assume that (M, g) is Bach-flat and $Q(M, g) > 0$. Then there exists a small number c_0 such that if $\int_M |W|^2 + |E_{ij}|^2 \, dV_g \leq c_0$, then (M, g) is an Einstein manifold.*

There is no noncompact complete Einstein manifold of positive scalar curvature. By Theorem 2, we have an obstruction for the existence of a noncompact complete Bach-flat manifold.

Theorem 3 *Let (M, g) be a noncompact complete Riemannian 4-manifold with positive constant scalar curvature R, Weyl curvature W and traceless Ricci curvature E_{ij}. Assume that (M, g) is Bach-flat and $Q(M, g) > 0$. Then, there exists a positive number c_1 such that $\int_M |W|^2 + |E_{ij}|^2 \, dV_g \geq c_1$.*

Proof of Theorem 2.

Let E_{ij} be the traceless Ricci tensor and $|E| = |E_{ij}|$. Using Bianchi identity, Bach tensor can be expressed in the following way (cf. [13, (1.18)]),

$$
\begin{aligned}
B_{ij} &= -\frac{1}{2} \triangle E_{ij} + \frac{1}{6} \nabla_i \nabla_j R - \frac{1}{24} \triangle R g_{ij} - E^{kl} W_{ikjl} + E_i^k E_{jk} \\
&\quad -\frac{1}{4} |E|^2 g_{ij} + \frac{1}{6} R E_{ij} \qquad (24)
\end{aligned}
$$

By the Kato inequality, $|\nabla E|^2 \geq |\nabla |E||^2$ and $\mathrm{tr} E^3 \leq \frac{1}{\sqrt{3}} |E|^3$, there exists a positive constant c satisfying the following equation for a Bach-flat metric

$$
\begin{aligned}
|E|\triangle|E| &= |\nabla E|^2 - |\nabla|E||^2 - 2E^{kl}W_{ikjl}E^{ij} + 2\mathrm{tr}E^3 + \frac{1}{3}R|E|^2 \quad (25)\\
&\geq -2E^{kl}W_{ikjl}E^{ij} + 2\mathrm{tr}E^3 + \frac{1}{3}R|E|^2 \quad (26)\\
&\geq -c|W||E|^2 - \frac{2}{\sqrt{3}}|E|^3 + \frac{1}{3}R|E|^2. \quad (27)
\end{aligned}
$$

Let $u = |E|$. Multiplying a smooth compact supported function ϕ to (27) and integrating on M,

$$
\int_M \phi^2|\nabla u|^2 + 2\phi u \nabla\phi \cdot \nabla u \, dV_g \leq \int_M c|W|u^2\phi^2 + \frac{2}{\sqrt{3}}u^3\phi^2 - \frac{1}{3}Ru^2\phi^2 \, dV_g.
$$

Using the Yamabe constant,

$$
\begin{aligned}
&\Lambda_0 \left(\int_M (\phi u)^4 dv_g\right)^{1/2}\\
\leq &\int_M |u\nabla\phi + \phi\nabla \ u|^2 dV_g + \frac{1}{6}Ru^2\phi^2\\
\leq &\int_M u^2|\nabla\phi|^2 + c|W|u^2\phi^2 + \frac{2}{\sqrt{3}}u^3\phi^2 - \frac{1}{6}Ru^2\phi^2 dV_g. \quad (28)
\end{aligned}
$$

Note that the second term of (28) is bounded by

$$
c\left(\int_M |W|^2 dV_g\right)^{1/2} \left(\int_M u^4\phi^4 \ dV_g\right)^{1/2}
$$

and the third term is bounded by

$$
\frac{2}{\sqrt{3}}\left(\int_M u^4\phi^4 dV_g\right)^{1/2} \left(\int_M u^2 dV_g\right)^{1/2}.
$$

Assume that

$$
c\left(\int_M |W|^2 dV_g\right)^{1/2} + \frac{2}{\sqrt{3}}\left(\int_M u^2 dV_g\right)^{1/2} \leq \Lambda_0. \quad (29)
$$

Then, three terms in the right hand side of (28) can be absorbed in the left hand side. Therefore, there exists a constant $c' > 0$ such that

$$
c'\left(\int_M u^4\phi^4 dV_g\right)^{1/2} \leq \int_M u^2|\nabla\phi|^2 dV_g \quad (30)
$$

Now we choose ϕ as

$$\phi = \begin{cases} 1 & \text{on } B_t \\ 0 & \text{on } M - B_{2t} \\ |\nabla\phi| \le \frac{2}{t} & \text{on } B_{2t} - B_t \end{cases} \tag{31}$$

with $0 \le \phi \le 1$. From (29) and (30)

$$\begin{aligned} c' \left(\int_M u^4 \phi^4 dV_g \right)^{1/2} &\le \frac{4}{t^2} \int_{B(2t)-B(t)} u^2 dV_g \\ &\le \frac{4}{t^2} \frac{\sqrt{3}}{2} \Lambda_0. \end{aligned} \tag{32}$$

By taking $t \to \infty$, we have $u = 0$. Therefore (M, g) is Einstein.

3 Bach-flat metric with nonconstant scalar curvature

In this section, we study noncompact complete Bach-flat metric with non-constant scalar curvature. We apply a result of the Yamabe problem on noncompact manifold to study rigidity. For a given manifold (M, g), we find a conformal metric $\overline{g} = u^{4/(n-2)} g$ whose scalar curvature is zero. This is equivalent to find a solution for the following partial differential equation

$$-\Delta_g u + \frac{1}{6} R_g u = 0. \tag{33}$$

The following existence of a conformal metric with zero scalar curvature was proved by Kim [14].

Theorem 4 *Let (M, g) be a noncompact complete Riemannian manifold of dimension $n \ge 3$ with scalar curvature R. Assume that $Q(M, g) > 0$ and $\int_M |R|^{2n/(n+2)} + |R|^{n/2} \, dV_g < \infty$. Then, there exists a conformal metric $\overline{g} = u^{4/(n-2)} g$ whose scalar curvature is zero. Moreover, u satisfies the following:*

$$\int_M |\nabla(u-1)|^2 + |u-1|^{2n/(n-2)} \, dV_g < \infty \tag{34}$$

and

$$\int_M |\nabla(u-1)|^2 + |u-1|^{2n/(n-2)}\, dV_g \to 0 \ \ as$$
$$\int_M |R|^{2n/(n+2)} + |R|^{n/2}\, dV_g \to 0. \tag{35}$$

By Theorem 4 and an elliptic estimation for solutions of (33), new metric $(M, \overline{g})$ in Theorem 4 is also complete (cf. [15, ch. 8]). By a standard elliptic estimation, $C^{2,\alpha}$ norm of $u-1$ is bounded by $L_{n/2}$ and $L_{2n/(n+2)}$ norm of R. Therefore, in dimension 4, if (M, g) has sufficiently small L_2 bound of $|Riem|$ and $L_{4/3}$ bound of R, there is a conformal metric $\overline{g}$ with zero scalar curvature and small L_2 norm of $|Riem_{\overline{g}}|$ with respect to metric $\overline{g}$. Applying Theorem 1 to new metric $(M, \overline{g})$, we have:

Theorem 5 *Let (M, g) be a noncompact complete Bach-flat Riemannian 4-manifold with scalar curvature R and $Q(M, g) > 0$. Then there exists a small number c_0 such that if $\int_M |Riem|^2 + |R|^{4/3}\, dV_g \le c_0$, then (M, g) is conformal to a flat space.*

Remark 1 *The constant c_0 in Theorem 1, 2, 5 and c_1 in Theorem 3 depend on $Q(M, g)$.*

Acknowledgments

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) KRF-2005-202-C00036. This work was initiated when the author visited Princeton University. The author thanks P. Yang for his help and comments on this work and Princeton University for hospitality.

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