

Homogeneous Einstein Metrics on $SO(n)$

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ABSTRACT

It is well known that every compact simple Lie group G admits an Einstein metric that is invariant under the independent left and right actions of G . In addition to this bi-invariant metric, with $G \times G$ symmetry, it was shown by D'Atri and Ziller that every compact simple Lie group except $SU(2)$ and $SO(3)$ admits at least one further homogeneous Einstein metric, invariant under $G \times H$, where H is some proper subgroup of G . In this paper we consider the Lie groups $G = SO(n)$ for arbitrary n , and provide an explicit construction of $(3k - 4)$ inequivalent homogeneous Einstein metrics on $SO(2k)$, and $(3k - 3)$ inequivalent homogeneous Einstein metrics on $SO(2k + 1)$.

1 Introduction

The Einstein equation $R_{\mu\nu} = \lambda g_{\mu\nu}$ places constraints on a subset of the full Riemann curvature tensor. Since in d dimensions the Riemann tensor has $\frac{1}{12}d^2(d^2 - 1)$ algebraically independent components, while the Ricci tensor has $\frac{1}{2}d(d + 1)$ algebraically independent components, it follows that the Einstein equation impose less and less of a constraint on the curvature as d increases. In $d = 3$ the count of Riemann and Ricci tensor components is the same, and in fact one can express the Riemann tensor algebraically in terms of the Ricci tensor. Dimension $d = 4$ is the lowest in which the Ricci tensor has fewer components than the Riemann tensor.

In consequence of these considerations, one can expect that the richness of solutions of the Einstein equations should increase with increasing dimension. In this paper we consider one aspect of this question, by seeking homogeneous Einstein metrics on certain group manifolds.

It is well known that every compact simple Lie group G admits a bi-invariant Einstein metric, *i.e.* one that is invariant under $G_L \times G_R$, the direct product of independent left-acting and right-acting transitive actions of the group G . If we take T^a to be the generators of the Lie algebra of G , then if g denotes a group element in G we may define left-invariant 1-forms σ_a by

$$g^{-1}dg = \sigma_a T^a. \quad (1)$$

The bi-invariant metric, of the form $\text{tr}(g^{-1}dg)^2$ is, with a suitable choice of basis for T^a , given by

$$ds^2 = c \sigma_a^2, \quad (2)$$

where c is a constant.

It has been shown by D'Atri and Ziller [1] that every simple compact Lie group except $SU(2)$ or $SO(3)$ admits at least one additional homogeneous Einstein metric. These metrics are still invariant under a transitive G action (left or right, according to convention choice; we shall choose the case where the full G_L is preserved). However, the D'Atri and Ziller Einstein metric is invariant only under a proper subgroup of the right-acting G .

The general left-invariant metric can be written as

$$ds^2 = x_{ab} \sigma_a \sigma_b, \quad (3)$$

where x_{ab} is a constant symmetric matrix. For the metric to be Riemannian, the eigenvalues of x_{ab} must all be positive. The D'Atri-Ziller Einstein metric on G falls into the class (3), for some specific choice of $x_{ab} \neq c \delta_{ab}$.

It is known in particular cases that there may exist yet more homogeneous Einstein metrics on a given simple compact group G , over and above the bi-invariant and D'Atri-Ziller examples. (See, for example, [2] for a review.) As far as we are aware, the largest number that have been found explicitly in any example are the six inequivalent homogeneous Einstein metrics on the 14-dimensional exceptional group G_2 , obtained in [3].

In principle, the task of finding left-invariant Einstein metrics of the form (3) is a purely mechanical one. The Ricci tensor can be calculated as an algebraic function of the squashing parameters x_{ab} , and the Einstein equation then reduces to a system of coupled algebraic equations for the x_{ab} . The problem in practice is that if one considers the most general symmetric tensor x_{ab} , the equations become too complicated to be tractable, for essentially any compact simple Lie group larger than $SU(2)$.

In the paper [3], which constructed 4 inequivalent Einstein metrics on $SO(5)$ and 6 on G_2 , the problem was greatly simplified by focusing on restricted metric ansätze, involving only a small number of independent components among the x_{ab} . The choice for these non-vanishing x_{ab} was dictated by requiring invariance under some chosen subgroup of the Lie group G . It turned out that this provided a rather fertile ground in which examples of Einstein metrics could be found. It does not necessarily yield all the homogeneous Einstein metrics on the group, but it does provide a relatively simple way of finding some of them.

An important issue when looking for Einstein metrics is to be able to recognise whether an ostensibly “new” metric is genuinely new, or whether it is instead just a repetition of a previously-obtained example, possibly disguised by a change of basis. A very useful tool in this regard is provided by calculating some dimensionless invariant quantity built from the metric and the curvature. The simplest example is

$$I_1 = \lambda^{d/2} V, \tag{4}$$

where $R_{ab} = \lambda g_{ab}$, V is the volume of the space, and d is the dimension of the Lie group G . If I_1 takes different values for two Einstein metrics on G then the two metrics are definitely inequivalent. If I_1 is the same for two Einstein metrics then they may be equivalent, and in practice they typically are. The volume form is given by $\sqrt{\det(x_{ab})} \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n$, and we may for convenience just take V to be given by

$$V = \sqrt{\det(x_{ab})}, \tag{5}$$

since the integration over $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n$ will just produce a universal multiplicative constant factor.

Another dimensionless invariant that is sometimes useful is

$$I_2 = R^{abcd} R_{abcd} \lambda^{-2}. \quad (6)$$

In the present paper, we shall look for homogeneous Einstein metrics on the entire class of Lie groups $SO(n)$. Following the methodology of [3], we shall make simple metric ansätze adapted to subgroups of $SO(n)$; specifically, we consider

$$SO(p) \times SO(q) \subset SO(n), \quad p + q = n. \quad (7)$$

A convenient way to formulate the problem is to introduce the left-invariant 1-forms L_{AB} for $SO(n)$, where $L_{AB} = -L_{BA}$ and $1 \leq A \leq n$, etc. These satisfy the Cartan-Maurer equations

$$dL_{AB} = L_{AC} \wedge L_{CB}. \quad (8)$$

If we decompose the fundamental $SO(n)$ index as $A = (i, \tilde{i})$, where $1 \leq i \leq p$ and $p + 1 \leq \tilde{i} \leq n$, then the following metric ansatz is invariant under the right action of $SO(p) \times SO(q)$ (as well as, of course, the left action of $SO(n)$):

$$ds^2 = \frac{1}{2} x_1 L_{ij} L_{ij} + \frac{1}{2} x_2 L_{\tilde{i}\tilde{j}} L_{\tilde{i}\tilde{j}} + x_3 L_{i\tilde{j}} L_{i\tilde{j}}. \quad (9)$$

In the obvious orthonormal frame $e^1 = \sqrt{x_1} L_{12}$, etc., the components of the Ricci tensor in the $SO(p)$, $SO(q)$ and $SO(p+q)/(SO(p) \times SO(q))$ subspaces can easily be shown to be given by

$$\begin{aligned} SO(p) : \text{Ric} &= \left(\frac{p-2}{2x_1} + \frac{qx_1}{2x_3^2} \right) \text{Id}, \\ SO(q) : \text{Ric} &= \left(\frac{q-2}{2x_2} + \frac{px_2}{2x_3^2} \right) \text{Id}, \\ SO(p+q)/(SO(p) \times SO(q)) : \text{Ric} &= \left(\frac{p+q-2}{x_3} - \frac{(p-1)x_1}{2x_3^2} - \frac{(q-1)x_2}{2x_3^2} \right) \text{Id}, \end{aligned} \quad (10)$$

where Id denotes the identity matrix in each subspace. Thus solving the Einstein equation $R_{ab} = \lambda g_{ab}$ amounts to solving the three conditions

$$\begin{aligned} \frac{p-2}{2x_1} + \frac{qx_1}{2x_3^2} &= \lambda, \\ \frac{q-2}{2x_2} + \frac{px_2}{2x_3^2} &= \lambda, \\ \frac{p+q-2}{x_3} - \frac{(p-1)x_1}{2x_3^2} - \frac{(q-1)x_2}{2x_3^2} &= \lambda. \end{aligned} \quad (11)$$

Since the ansatz is symmetrical under the exchange $(p, x_1) \leftrightarrow (q, x_2)$, we may without loss of generality assume that $0 \leq q \leq p$. There are then two special cases that arise, namely when $q = 0$ and $q = 1$, and then all other cases with $q \geq 2$ follow a generic pattern.

The case $q = 0, p = n$:

In this case, only the $SO(p)$ subspace occurs, and the metric is simply the bi-invariant one

$$ds^2 = \frac{1}{2}x_1 L_{ij} L_{ij}. \quad (12)$$

This has

$$\lambda = \frac{(p-2)}{2x_1}. \quad (13)$$

The case $q = 1, p = n - 1$:

In this case, only the $SO(p)$ and the $SO(p+q)/(SO(p) \times SO(1))$ subspaces occur, and the metric takes the form

$$ds^2 = \frac{1}{2}x_1 L_{ij} L_{ij} + x_3 L_{in} L_{in}. \quad (14)$$

The first and third equations in (11) then imply either

$$x_1 = x_3, \quad \lambda = \frac{(p-1)}{2x_3}, \quad (15)$$

or else

$$x_1 = \frac{(p-2)}{p} x_3, \quad \lambda = \frac{(p-1)(p+2)}{2px_3}. \quad (16)$$

The solution (15) is just a repetition of the bi-invariant metric on $SO(n)$ but (16) is an inequivalent Einstein metric; it is invariant under $SO(n) \times SO(n-1)$ but not under $SO(n) \times SO(n)$.

The cases $q \geq 2, p = n - q$:

In all these cases, the three subspaces in (10) are all non-empty, and the metric takes the form (9). The three equations (11) then imply

$$\lambda = \frac{qy^2 + p - 2}{2yx_3}, \quad (17)$$

$$x_2 = \frac{[2 - p + 2(p + q - 2)y - (p + q - 1)y^2] x_3}{(q - 1)y}, \quad (18)$$

and then either $y = 1$ or

$$\begin{aligned} 0 &= (p + q - 1)[p^2 + (p + q)(q - 1)]y^3 \\ &\quad - [p(q - 1)(4q - 7) + q(q - 1)(q - 3) + 2p^2(3q - 5) + 3p^3]y^2 \\ &\quad + (p - 2)[p(5q - 7) + 2(q - 1)^2 + 3p^2]y - (p - 2)^2(p + q - 1), \end{aligned} \quad (19)$$

where we have defined

$$y = \frac{x_1}{x_3}. \quad (20)$$

The $y = 1$ solution just gives the bi-invariant metric again. In general, the cubic equation (19) is nontrivial, yielding three inequivalent solutions, and hence three inequivalent Einstein metrics for each choice of p and q . There are two classes of special cases where the cubic factorises over the rationals into a product of a linear and a quadratic polynomial:

(1) $q = 2$. Equation (19) then implies that

$$y = \frac{(p+1)(p-2)}{p^2+p+2}, \quad (21)$$

or else $y = [p \pm \sqrt{p+2}]/(p+1)$. These latter two solutions imply that $x_2 = 0$, and hence the metric is degenerate, but (21) gives a non-trivial, and inequivalent, Einstein metric.

(2) $q = p \geq 3$. Equation (19) has the solutions

$$y = \frac{p-2}{3p-2}, \quad \text{or} \quad y = \frac{2p(p-1) \pm \sqrt{4p^3 - 5p^2 + 2p}}{p(2p-1)}. \quad (22)$$

The two solutions involving the square root give equivalent Einstein metrics (with the roles of x_1 and x_2 exchanged), and so in total we obtain two further inequivalent Einstein metrics in this $q = p$ case.

There are also isolated examples, such as $(p, q) = (6, 3)$ and $(10, 6)$, where the cubic again factorises over the rationals into a product of linear and quadratic polynomials. In these cases, all three roots give inequivalent Einstein metrics.

After including all the inequivalent partitionings of $n = p + q$, we find in summary that the construction described here provides the following numbers of inequivalent Einstein metrics on $SO(n)$:

$$\begin{aligned} SO(2k) &: (3k - 4) \text{ inequivalent Einstein metrics,} & k \geq 2 \\ SO(2k + 1) &: (3k - 3) \text{ inequivalent Einstein metrics,} & k \geq 2. \end{aligned} \quad (23)$$

For example, for $SO(10)$ we have the partitions $(p, q) = (10, 0), (9, 1), (8, 2), (7, 3), (6, 4)$ and $(5, 5)$, giving $1 + 1 + 1 + 3 + 3 + 2 = 11$ inequivalent Einstein metrics. For $SO(11)$ we have the partitions $(p, q) = (11, 0), (10, 1), (9, 2), (8, 3), (7, 4)$ and $(6, 5)$, giving $1 + 1 + 1 + 3 + 3 + 3 = 12$ inequivalent Einstein metrics.

For a given partition of $n = p + q$, the Einstein metrics on $SO(n)$ that we have constructed here have the symmetry $SO(n) \times SO(p) \times SO(q)$, where $SO(n)$ acts transitively on the left, while $SO(p) \times SO(q)$ acts (intransitively) on the right. It is manifest, therefore, that such

metrics on $SO(n)$ with different partitions $n = p + q$ are inequivalent. (This can easily be confirmed, case by case, by comparing the values of the invariants I_1 or I_2 , defined in (4) and (6).) In cases such as $3 \leq q < p$, where three different solutions for the squashing parameters arise for a given p and q , the invariants I_1 or I_2 can be seen to take different values for the three cases, and hence the three Einstein metrics are indeed inequivalent. (The same is true for the $q = p \geq 3$ case, where the two inequivalent solutions can be seen to have different values for I_1 or I_2 .)

It should be emphasised that this construction certainly does not in general exhaust all the possibilities for homogeneous Einstein metrics on $SO(n)$. We have focused only on metric ansätze adapted to the various $SO(p) \times SO(q)$ subgroups in our discussion. Other subgroups can, and in some cases certainly do, give rise to further possibilities.

What we have established, in this paper, is that the number of homogeneous Einstein metrics on compact simple Lie groups can grow without limit as the dimension of the group increases. Specifically, we have exhibited $(3k-4)$ inequivalent homogeneous Einstein metrics on the group manifold $SO(2k)$, and $(3k-3)$ on the group manifold $SO(2k+1)$.

Acknowledgements

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