# Extended Horava gravity and Einstein-aether theory

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Einstein-aether theory is general relativity coupled to a dynamical, unit timelike vector. If this vector is restricted in the action to be hypersurface orthogonal, the theory is identical to the IR limit of the extension of Horava gravity proposed by Blas, Pujolàs and Sibiryakov. Hypersurface orthogonal solutions of Einstein-aether theory are solutions to the IR limit of this theory, hence numerous results already obtained for Einstein-aether theory carry over.

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Much interest has recently been focused on Hořava-Lifshitz gravity [1], which proposes the possibility of a renormalizable, non-Lorentz-invariant UV completion of general relativity. There are so-called projectable and non-projectable versions of this proposal, and both have been shown to suffer from various problems (instabilities, overconstrained evolution, or strong coupling at low energies) related to a badly behaved scalar mode of gravity brought on by the presence of a non-dynamical spatial foliation in the action [2]. A proposal for evading all of these problems, put forth by Blas, Pujolàs and Sibiryakov (BPS) [3], is an "extension" of Hořava gravity. I will call it here BPSH gravity, and below $T$-theory for reasons to become clear. One can view this extension as promoting the fixed foliation to a dynamical one. This extension could still possess strong coupling at low energy [4], but it is also possible that higher derivative terms in the action become important below the strong coupling energy scale and prevent this [5].

It was remarked in Ref. [5] that this extended Horava theory is related to a restricted version of Einstein-aether theory, which is general relativity coupled to a dynamical unit timelike vector field (for recent review see [6]). The restriction amounts to assuming the vector field is hypersurface orthogonal. The purpose of this article is to clarify the relation between these two theories, both at the level of the defining action principles and at the level of solutions to the equations of motion. In particular, the lowest dimension terms (the IR limit) of the BPSH gravity action are equivalent to those of Einstein-aether theory, when the aether vector is assumed to be hypersurface orthogonal. It will be shown that any hypersurface orthogonal solution to Einstein-aether theory is a solution to the IR limit of BPSH gravity, although the converse does not appear to be true. In particular, since all spherically symmetric aether fields are hypersurface orthogonal, the spherically symmetric vacuum, star, black hole, and collapsing star solutions, and FRW cosmological solutions to Einstein-aether theory are all solutions to BPSH gravity. Moreover, some results about the coupling constants and PPN parameters of Einstein-aether theory carry over.

BPSH gravity has been formulated as a theory with a preferred spacetime foliation, defined by a time coordinate $t$ and space coordinates $x^i$. The spacetime metric is given by

$$ds^2 = N^2dt^2 - h_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \qquad (1)$$

where $N$ and $N^i$ are the lapse function and shift vector. The action for the IR sector of BPSH theory is

$$S = \frac{1}{16\pi G_H}\int dtd^3x\, N\sqrt{h}(K_{ij}K^{ij} - \lambda K^2 + \xi\, {}^{(3)}\!R + \alpha a_i a^i). \qquad (2)$$

Here spatial indices are raised and lowered using $h_{ij}$, $K_{ij} = (\dot{h}_{ij} + D_iN_j + D_jN_i)/2N$ is the extrinsic curvature of the spatial surface, dot denotes derivative with respect to $t$, $D_i$ is the spatial covariant derivative, $K = h^{ij}K_{ij}$, ${}^{(3)}R$ is the spatial Ricci curvature scalar, and $a_i = -\partial_i \ln N$. The coefficients $\lambda$, $\xi$, and $\alpha$ are dimensionless constants, and $G_H$ is what BPS denote by $M_P^2/2$.

It is helpful to understand this BPSH gravity theory using a covariant formalism. I shall start from scratch defining a theory motivated by symmetry principles, and then show that the resulting theory is equivalent to BPSH gravity. I focus here purely on the IR limit.

Suppose we wish to write down a generally covariant theory that depends on a spacetime metric, but also on a foliation of spacetime by spacelike surfaces. These surfaces define a notion of "cosmic simultaneity," quite alien to the spirit of general relativity. However, given various difficulties constructing a viable theory of quantum gravity (including non-renormalizability and UV completion, and the problem of time), it is interesting to ask what such a theory would look like and whether it could solve some or all of these problems and be phenomenologically viable.

A spacelike foliation of spacetime can be defined by the level sets of a suitably behaved scalar function $T$. If this function is a dynamical variable in the theory, then general covariance is preserved. A monotonically related function defines the same foliation. If the theory is to not depend on the time labeling, but rather only on the choice of surfaces, the action should depend on $T$ only via the unit (co)vector

$$u_a = WT_{,a}, \quad \text{with} \quad W = (g^{ab}T_{,a}T_{,b})^{-1/2}, \qquad (3)$$

where the subscript "$,a$" denotes the gradient. The convention used here for metric signature is $(+---)$. This vector is *hypersurface orthogonal*, i.e. $u_a v^a = 0$ for any vector $v^a$ tangent to a surface of constant $T$. Conversely, any hypersurface orthogonal vector can be written in the form $WT_{,a}$ for some pair of scalar functions $T$ and $W$.

To formulate a local, generally covariant dynamical theory of the spacetime metric and the $T$ function via the $u_a$ vector, we specify a Lagrangian scalar density. The classification of spacetime scalars that can be written using $g_{ab}$ and $u_a$, with up to two derivatives, was already considered [6, 7] in the context of Einstein-aether theory, or *ae-theory* for short. The only difference between ae-theory and what I will call *T-theory* for short, is that in ae-theory the aether, i.e. the unit vector field, is not required to be hypersurface orthogonal. (Actually the aether has usually been taken to be contravariant rather than covariant, but this changes only appearances.) This means that in ae-theory the aether has three degrees of freedom at each spacetime point, whereas in the present case it has just one, coming from the choice of the time function $T$.

Up to total derivative terms, the most general action for ae-theory (aside from matter couplings) is

$$S = \frac{1}{16\pi G_{æ}} \int \sqrt{-g}\ (-R + L_{æ})\ d^4x \tag{4}$$

where $R$ is the 4d Ricci scalar and

$$L_{\rm ae} = -M^{abmn} \nabla_a u_m \nabla_b u_n, \tag{5}$$

with $M^{abmn}$ is defined as

$$M^{abmn} = c_1 g^{ab} g^{mn} + c_2 g^{am} g^{bn} + c_3 g^{an} g^{bm} + c_4 u^a u^b g_{mn}. \tag{6}$$

The $c_i$ are dimensionless coupling constants, and it is assumed that $u_m$ is constrained to be a unit vector, $g^{mn} u_m u_n = 1$. Note that since the covariant derivative operator $\nabla_a$ involves derivatives of the metric through the connection components, and since the unit vector is nowhere vanishing, the terms quadratic in $\nabla u$ also modify the kinetic terms for the metric.

To pass to $T$-theory we just substitute (3) for $u_m$ in the Lagrangian (5). The equations of motion are the conditions that the action be stationary under variation of the metric and the scalar function $T$. The resulting action looks dangerous because it has two explicit derivatives, and $u_a = WT_{,a}$ already has one implicit derivative. Hence it appears that the equations of motion for $T$ will have fourth derivatives, which doesn't sound healthy. However, since the theory is generally covariant, we may always express the field equations using $T$ itself as one of the spacetime coordinates. Then we have $u_a = \delta_{aT}(g^{TT})^{-1/2}$, which contains no derivatives. Still the variation of $T$ in the action will produce an equation of motion that is third order in derivatives. However, as we shall see it remains second order in time derivatives.

Another issue is the timelike character of $T_{,a}$. It is not clear whether the dynamics of the theory somehow manages to preserve this condition. If not, then where $T_{,a}$ becomes null the unit vector $u_a$ will diverge, and most likely the metric would become singular as well. It seems an important question to determine whether such singularities can arise "unprovoked", and/or more visibly than those hidden by black hole horizons in general relativity.

An important observation we can now make is that *the $T$ field equation is implied by the Einstein equation.* One way to think about this is to consider ordinary Einstein gravity coupled to a scalar field. Due to the Bianchi identity $\nabla^a G_{ab} = 0$, the Einstein equation implies conservation of the stress energy tensor. For a scalar field this is enough to imply the matter field equation, unless the scalar field is constant. Another way to see this is to appeal directly to diffeomorphism invariance of the action. Suppose we couple $T$-theory to generic matter fields denoted $\psi$, so the action is schematically $S[g, \psi, T]$. Under a diffeomorphism generated by a vector field $\xi$ the action is invariant, so we have

$$\begin{aligned} \delta_\xi S &= \int (\delta S/\delta g)\mathcal{L}_\xi g + (\delta S/\delta\psi)\mathcal{L}_\xi \psi + (\delta S/\delta T)\mathcal{L}_\xi T \\ &= 0, \end{aligned} \tag{7}$$

where $\mathcal{L}_\xi$ is the Lie derivative. Now suppose the Einstein equation is satisfied, so that $\delta S/\delta g = 0$, and suppose further that the matter field equations are satisfied, so that $\delta S/\delta\psi = 0$. Then at such solutions we have the identity $\int (\delta S/\delta T)\mathcal{L}_\xi T = 0$ for all vector fields $\xi^a$. Unless $T$ is constant — which it cannot be if it is to define a foliation — the Lie derivative $\mathcal{L}_\xi T$ can be freely varied at a point by varying the choice of $\xi^a$, hence it must be that $\delta S/\delta T = 0$, i.e. the $T$ field equation is satisfied.

Since the $T$ field equation need not be imposed explicitly, we can adopt $T$ as one of the spacetime coordinates, and write the theory in this "$T$-gauge" *before* varying the remaining fields, using the 3+1 decomposition of the metric in (1). Then since $u = W dT$ is a unit 1-form, the function $W$ is evidently the same as the lapse $N$.

Decomposing the volume element yields $\sqrt{-g} = N\sqrt{h}$. The covariant derivative of $u_m$ decomposes as

$$\nabla_a u_b = -K_{ab} - u_a a_b, \tag{8}$$

where $K_{ab}$ is the extrinsic curvature of the surfaces normal to $u_a$, and $a_b$ is the acceleration of the normal curves. The extrinsic curvature is symmetric, and it and the acceleration are both spatial ($K_{ab}u^b = 0$ and $a_b u^b = 0$). Using the spatial coordinates $x^i$ the acceleration can be expressed as $a_i = -(\ln N)_{,i}$. The aether Lagrangian (5) can therefore be expressed in the form

$$L_{\rm ae} = -c_{13} K_{ij} K^{ij} - c_2 K^2 + c_{14} a_i a^i, \tag{9}$$

where, $c_{13} = c_1 + c_3$, $c_{14} = c_1 + c_4$, and, as in (2), the spatial indices are raised with $h^{ij}$. The 3+1 decomposition of the $-R$ in the Einstein-Hilbert action adds to the Lagrangian $K_{ij}K^{ij} - K^2 + {}^{(3)}R$.

We conclude that the $T$-theory action (4) and the BPSH action (2) are identical, with the following relations between the various coefficients:

$$G_H/G_{æ} = \xi = (1 - c_{13}), \quad \alpha/\xi = c_{14}, \quad \lambda/\xi = 1 + c_2. \tag{10}$$

In Ref. [3], $\xi$ is fixed to unity by choosing the scale of the $t$ coordinate. Once matter is present, that rescaling is no longer available if matter is to couple minimally to the spacetime metric.

Note that athough we begin with four coefficients $c_{1,2,3,4}$ in the most general $T$-theory action, when expressed in 3+1 form in the $T$-gauge only three independent combinations of these parameters enter (9). From the covariant point of view this happens because, when $u_a$ is hypersurface orthogonal, there is a relation between three of the terms in the Lagrangian (5). This can be seen by considering the twist 3-form, $\omega = u \wedge du$, which vanishes identically when $u = NdT$. In terms of the dual vector $\omega^a = \epsilon^{abcd}u_b\nabla_c u_d$ we have the identity

$$\omega_a\omega^a = -2(\nabla_a u_b)(\nabla^{[a}u^{b]}) + (u^b\nabla_b u_a)(u^c\nabla_c u^a), \tag{11}$$

where the square brackets denote index antisymmetrization. When $u_a$ is hypersurface orthogonal $\omega^a$ vanishes, so the $c_1$, the $c_3$ or the $c_4$ term in the Lagrangian can be written in terms of the other two. Alternatively, one can use this identity to remove the square of the antisymmetric part $(\nabla_{[a}u_{b]})(\nabla^{[a}u^{b]})$, replacing both the $c_1$ and $c_3$ terms by a single term of the form $(\nabla_{(a}u_{b)})(\nabla^{(a}u^{b)})$ (where the round brackets denote index symmetrization).

One more term may be eliminated from the action (4) by making a field redefinition of the metric [8, 9]

$$g'_{ab} = g_{ab} + (\zeta - 1)u_a u_b, \tag{12}$$

which "stretches" the metric tensor in the aether direction by a positive factor $\zeta$. (A negative factor would return a Euclidean signature metric [19].) This does not change the function $T$, but the unit vector does change, to $u'_a = \sqrt{\zeta}u_a$. If matter is coupled minimally to the metric $g_{ab}$, then it is not coupled minimally to $g'_{ab}$ but rather to $g'_{ab} - (1 - 1/\zeta)u'_a u'_b$. However, for considerations not involving matter, such a field redefinition can simplify the theory. The action (4) for $(g'_{ab},\, u'_a = \sqrt{\zeta}u_a)$ takes the same form as that for $(g_{ab},\, u_a)$, with new coefficients $c'_i$. The relation between the $c'_i$ and $c_i$ was worked out in Ref. [9], and is conveniently given in terms of certain combinations with simple scaling behavior:

$$\begin{aligned} c'_{14} &= c_{14} \\ c'_{123} &= \zeta c_{123} \\ c'_+ - 1 &= \zeta(c_+ - 1) \\ c'_- - 1 &= \zeta^{-1}(c_- - 1), \end{aligned} \tag{13}$$

where $c_{123} = c_1 + c_2 + c_3$ and $c_\pm = c_1 \pm c_3$. For example, one can arrange for $c'_+ = 0$ by choosing $\zeta = 1/(1 - c_+)$ (provided $c_+ < 1$). Thus by a metric redefinition one can eliminate the symmetrized term $(\nabla_{(a}u_{b)})(\nabla^{(a}u^{b)})$ in the Lagrangian. Then, using the twist identity (11), one can either replace the remaining, antisymmetrized term by an acceleration squared term, or vice versa. The first way yields the Lagrangian

$$L_{\text{ae}} = \gamma_2(\nabla_a u^a)^2 + \gamma_4(u^b\nabla_b u_a)(u^c\nabla_c u^a), \tag{14}$$

where $\gamma_{2,4}$ depend on the original values $c_{1,2,3,4}$. The second way yields the equivalent Lagrangian

$$L_{\text{ae}} = \gamma_2(\nabla_a u^a)^2 + \gamma_-\nabla_{[a}u_{m]}\nabla^{[a}u^{m]}. \tag{15}$$

The equations of motion of $T$-theory are closely related to those of ae-theory. Let $E^n$ denote the variation of the action with respect to $u_n$,

$$E^n = 16\pi G\,\frac{\delta S}{\delta u_n} = \nabla_b(M^{abmn}\nabla_a u_m). \tag{16}$$

The variations of $u_n$ induced by variations of $T$ and of the inverse metric $g^{ab}$ are

$$\begin{aligned} \delta_T u_n &= N(\delta^m_n - u_n u^m)\nabla_m \delta T & (17) \\ \delta_g u_n &= -\tfrac{1}{2}u_n u_a u_b \delta g^{ab}. & (18) \end{aligned}$$

The $T$ equation of motion is thus

$$\nabla_m\Big(N(\delta^m_n - u^m u_n)E^n\Big) = 0. \tag{19}$$

As mentioned earlier, if $T$ is used as one of the coordinates, then $E^n$ has just two derivatives in it, and then the $T$ field equation (19) is of third order in derivatives. However, due to the presence of the spatial projection $\delta^m_n - u^m u_n$, the derivative in the outer $\nabla_m$ operator is purely spatial, so the equation is of only second order in derivatives with respect to $T$.

The $g^{ab}$ field equation, including matter fields, is

$$G_{ab} = T^{æ}_{ab} + 8\pi G T^{\text{matter}} \tag{20}$$

where $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ is the usual Einstein tensor and $T^{æ}_{ab}$ denotes the aether stress tensor, which includes the contributions from the variations in the aether action (4) of the explicit metrics in $\sqrt{-g}$ and $M^{abmn}$, the metrics in the covariant derivative operators, and those buried in the definition of $u_m$. In particular, according to (18) the latter contribute

$$-\tfrac{1}{2}(E^n u_n)u_a u_b \tag{21}$$

to $T^{æ}_{ab}$, which has the form of the stress tensor of a pressureless dust.

Let us now compare the form of the field equations (19) and (20) to those of ae-theory. In ae-theory, the fundamental field $u_m$ is constrained to satisfy $g^{mn}u_m u_n = 1$.

This constraint has usually been implemented with a Lagrange multiplier term in the action, but it can also be taken into account by restricting the variations to satisfy $\delta g^{mn}u_m u_n + 2u^n \delta u_n = 0$. Thus the part of the aether variation orthogonal to $u^n$ is unconstrained, while the part parallel to $u_n$ is given by $\delta_{\|}u_n = -\frac{1}{2}u_n u_a u_b \delta g^{ab}$, just as in (18). Thus the metric variation, accompanied by this parallel aether variation, yield the Einstein equation (20) as before, with the contribution (21) in the aether stress tensor. The orthogonal aether variation yields

$$(\delta^m_n - u^m u_n)E^n = 0. \tag{22}$$

This implies the $T$-theory field equation (19), although the converse is not true. We conclude that *any hypersurface orthogonal solution to Einstein-aether theory is also a solution to $T$-theory, i.e. to BPSH gravity.*

In particular, since any spherically symmetric aether field is hypersurface orthogonal, any such ae-theory solution is a $T$-theory solution. An example is the vacuum solution to $T$-theory found recently by Kiritsis [11] which is identical to the one found previously for ae-theory [12]. A second, non asymptotically flat solution of $T$-theory was found in Ref. [11]. This is also an ae-theory solution: it corresponds to the interior part of the wormhole in the global solution found in Ref. [12]. In fact, for an aether parallel to the time translation Killing field in spherical symmetry, the aether equation is automatically satisfied [12], hence *all* such $T$-theory solutions are ae-theory solutions. More generally, however, it appears that not all spherically symmetric $T$-theory solutions need be ae-theory solutions.

The conclusion that spherical ae-theory solutions are $T$-theory solutions applies also to the black hole solutions in Ref. [13], the neutron star solutions in Ref. [14], and even the time-dependent collapse solutions of Ref. [15]. It also applies to the homogeneous isotropic cosmological solutions discussed in Refs. [17, 18]. Moreover, we can infer that the relation [18] between Newton's constant $G_N$ and the coefficient $G_{æ}$ in the action (4) is the same as in ae-theory, as is the relation [18] between the cosmological gravitational constant $G_{\rm cosmo}$ that appears in the Friedmann equation, because these relations can be obtained with hypersurface orthogonal solutions. This agrees with the relations reported in Ref. [3]. Similarly, the PPN parameters $\beta$ and $\gamma$ must have the same values as in ae-theory, in particular they are the same (unity) as in general relativity [16]. The preferred frame PPN parameters $\alpha_{1,2}$ may well be different in the two theories however, since they characterize solutions for sources in motion with respect to the aether and therefore are sensitive to solutions beyond spherical symmetry.

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