

# Consistency relation for the Lorentz invariant single-field inflation

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(Dated: February 15, 2010)

In this paper we compute the sizes of equilateral and orthogonal shape bispectrum for the general Lorentz invariant single-field inflation. The stability of field theory implies a non-negative square of sound speed which leads to a consistency relation between the sizes of orthogonal and equilateral shape bispectrum, namely  $f_{NL}^{orth.} \lesssim -0.054 f_{NL}^{equil.}$ . In particular, for the single-field Dirac-Born-Infeld (DBI) inflation, the consistency relation becomes  $f_{NL}^{orth.} \simeq 0.070 f_{NL}^{equil.} \lesssim 0$ . These consistency relations are also valid in the mixed scenario where the quantum fluctuations of some other light scalar fields contribute to a part of total curvature perturbation on the super-horizon scale and may generate a local form bispectrum. A distinguishing prediction of the mixed scenario is  $\tau_{NL}^{loc.} > (\frac{6}{5} f_{NL}^{loc.})^2$ . Comparing these consistency relations to WMAP 7yr data, there is still a big room for the Lorentz invariant inflation, but DBI inflation has been disfavored at more than 68% CL.

PACS numbers: 98.80.Cq, 98.80.Es

The quantum fluctuation during inflation seeds the temperature anisotropies in the cosmic microwave background radiation (CMBR) and formation of large-scale structure of galaxies in our universe. The primordial cosmological perturbations are so tiny that the generation and evolution of fluctuations has been investigated within linear perturbation theory. Within this approach, the primordial perturbation is Gaussian; or equivalently, its Fourier components are uncorrelated and have random phases. The simplest version of inflation predicts such a nearly Gaussian distribution [1]. A non-vanishing three-point correlation function of the curvature perturbation, or its Fourier transform, the bispectrum, is an important indicator of a non-Gaussian feature in the cosmological perturbations because it represents the lowest order statistics able to distinguish non-Gaussian from Gaussian perturbations. Recently the non-Gaussianity emerges as a more and more important observable.

In general the bispectrum depends on configuration in momentum space [2, 3]. A large non-local shape, such as equilateral and orthogonal shape, bispectrum is obtained in the single-field inflation model where the higher derivative terms are involved [4–8]. In the multi-field inflation the perturbations along the entropy directions which are transverse to the motion direction (adiabatic direction) can be converted into the adiabatic perturbation on the super-horizon scale and generate a local form bispectrum, for example the curvature perturbation generated at the end of multi-field inflation due to the curved geometry of the hyper-surface for inflation to end [9–11] and the curvaton model [12–16].

In this paper we mainly focus on equilateral, orthogonal, and local form bispectrum whose sizes are measured by  $f_{NL}^{equil.}$ ,  $f_{NL}^{orth.}$  and  $f_{NL}^{loc.}$  respectively. WMAP 7yr data [17] indicates  $f_{NL}^{loc.} = 32 \pm 21$ ,  $f_{NL}^{equil.} = 26 \pm 140$  and

$f_{NL}^{orth.} = -202 \pm 104$  at 68% CL; and  $-10 < f_{NL}^{loc.} < 74$ ,  $-214 < f_{NL}^{equil.} < 266$ ,  $-410 < f_{NL}^{orth.} < 6$  at 95% CL. Up to now the data implies that the distribution of the primordial curvature perturbation deviates from an exact Gaussian distribution at  $1\sigma$  level, but a Gaussian distribution is still consistent with the data at  $2\sigma$  level.

In last decades a lot of efforts have been put for constructing a realistic inflation model from string theory. Many of these models fall into a class called brane inflation [18]. The dynamics of brane is governed by the DBI action. In this paper we will figure out the consistency relation between the sizes of equilateral and orthogonal shape bispectrum for the general Lorentz invariant single-field inflation model, in particular for the DBI inflation [5, 6]. On the other hand, in order to achieve a bispectrum which is large not only for the non-local shape but also for the local shape, we suggest a mixed scenario where the entropy fluctuations are assumed to be converted into a part of total curvature perturbation and produce a local form bispectrum at/after the end of inflation [9–16]. We find that the consistency relations derived in the single-field model are also valid in this mixed scenario. Comparing to the WMAP 7yr data, the DBI inflation has been disfavored at more than  $1\sigma$  level.

The CMB temperature anisotropy in the Sachs-Wolfe limit is given by  $\Delta T/T = -\frac{1}{3}\Phi$ , where  $\Phi$  is the Bardeen's curvature perturbation. The power spectrum of the curvature perturbation is defined by

$$\langle \Phi_{\mathbf{k}_1} \Phi_{\mathbf{k}_2} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \frac{\Delta_{\Phi}}{k_1^s}, \quad (1)$$

where

$$s = 4 - n_s. \quad (2)$$

A general description of non-Gaussianity at the leading

order is the bispectrum of curvature perturbation

$$\langle \Phi_{\mathbf{k}_1} \Phi_{\mathbf{k}_2} \Phi_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta^{(3)} \left( \sum_{i=1}^3 \mathbf{k}_i \right) F(k_1, k_2, k_3), \quad (3)$$

where  $\Phi_{\mathbf{k}}$  is the Fourier mode of curvature perturbation in momentum space and  $F(k_1, k_2, k_3)$  depends on the configuration in momentum space. For the local, equilateral, and orthogonal form bispectrum,  $F(k_1, k_2, k_3)$  are respectively given by

$$F^{loc.}(k_1, k_2, k_3) = \Delta_{\Phi}^2 f_{NL}^{loc.} \tilde{F}^{loc.}(k_1, k_2, k_3), \quad (4)$$

$$F^{equil.}(k_1, k_2, k_3) = \Delta_{\Phi}^2 f_{NL}^{equil.} \tilde{F}^{loc.}(k_1, k_2, k_3), \quad (5)$$

$$F^{orth.}(k_1, k_2, k_3) = \Delta_{\Phi}^2 f_{NL}^{orth.} \tilde{F}^{loc.}(k_1, k_2, k_3), \quad (6)$$

where

$$\tilde{F}^{loc.}(k_1, k_2, k_3) = 2 \left[ \frac{1}{k_1^s k_2^s} + (2 \text{ perm.}) \right], \quad (7)$$

$$\tilde{F}^{equil.}(k_1, k_2, k_3) = 6 \left[ -\frac{1}{k_1^s k_2^s} - \frac{1}{k_2^s k_3^s} - \frac{1}{k_3^s k_1^s} - \frac{2}{(k_1 k_2 k_3)^{2s/3}} + \left( \frac{1}{k_1^{s/3} k_2^{2s/3} k_3^s} + (5 \text{ perm.}) \right) \right], \quad (8)$$

$$\tilde{F}^{orth.}(k_1, k_2, k_3) = 6 \left[ -\frac{3}{k_1^s k_2^s} - \frac{3}{k_2^s k_3^s} - \frac{3}{k_3^s k_1^s} - \frac{8}{(k_1 k_2 k_3)^{2s/3}} + \left( \frac{3}{k_1^{s/3} k_2^{2s/3} k_3^s} + (5 \text{ perm.}) \right) \right]. \quad (9)$$

These three forms are nearly orthogonal to one another. They probe different aspects of the physics of inflation.

First of all, let's focus on the general single-field inflation. In [7] the bispectrum was calculated in the single-field inflation model with action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_P^2}{2} R + P(X, \phi) \right], \quad (10)$$

where  $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ . This action is the most general Lorentz invariant action for inflaton  $\phi$  minimally coupled to Einstein gravity. For the case with large bispectrum ( $c_s \ll 1$  and/or  $\lambda/\Sigma \gg 1$ ),

$$F(k_1, k_2, k_3) = \Delta_{\Phi}^2 \mathcal{B}(k_1, k_2, k_3), \quad (11)$$

where

$$\begin{aligned} \mathcal{B}(k_1, k_2, k_3) &\simeq 10 \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right) \frac{1}{k_1 k_2 k_3 K^3} \\ &+ \frac{20}{3} \left( \frac{1}{c_s^2} - 1 \right) \frac{1}{k_1^3 k_2^3 k_3^3} \left( -\frac{1}{K} \sum_{i>j} k_i^2 k_j^2 \right. \\ &\left. + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_i k_i^3 \right), \quad (12) \end{aligned}$$

and

$$K = k_1 + k_2 + k_3, \quad (13)$$

$$c_s^2 = \frac{P_{,X}}{P_{,X} + 2X P_{,XX}}, \quad (14)$$

$$\lambda = X^2 P_{,XX} + \frac{2}{3} X^3 P_{,XXX}, \quad (15)$$

$$\Sigma = X P_{,X} + 2X^2 P_{,XX}. \quad (16)$$

Projecting the bispectrum in general Lorentz invariant single-field inflation to the local, equilateral, and orthogonal templates, we find  $f_{NL}^{loc.} \simeq 0$  and

$$f_{NL}^{equil.} \simeq -0.2728 \left( \frac{1}{c_s^2} - 1 \right) - 0.1494 \frac{\lambda}{\Sigma}, \quad (17)$$

$$f_{NL}^{orth.} \simeq -0.02831 \left( \frac{1}{c_s^2} - 1 \right) + 0.008114 \frac{\lambda}{\Sigma}. \quad (18)$$

The parameters  $c_s^2$  and  $\lambda/\Sigma$  can be fixed once  $f_{NL}^{equil.}$  and  $f_{NL}^{orth.}$  are detected,

$$\frac{1}{c_s^2} - 1 = -1.260 f_{NL}^{equil.} - 23.19 f_{NL}^{orth.}, \quad (19)$$

$$\frac{\lambda}{\Sigma} = -4.394 f_{NL}^{equil.} + 42.34 f_{NL}^{orth.}. \quad (20)$$

Stability of the field theory implies  $c_s^2 \geq 0$  which leads to

$$f_{NL}^{orth.} \lesssim -0.054 f_{NL}^{equil.}. \quad (21)$$

It is the consistency condition for the general single-field inflation without breaking Lorentz symmetry.

Nowadays string theory is supposed to be the only one self-consistent theory of quantum gravity. A very popular inflation model in string theory is brane inflation [18]. In KKLT [19] set up, brane can move very fast (close to the speed of light) in the Klebanov-Strassler (KS) throat, and the full DBI action for the inflaton field should be taken into account. For DBI inflation [5, 6], the action of  $\phi$  takes the form

$$P = -f^{-1}(\phi) \sqrt{1 - 2X f(\phi)} + f^{-1}(\phi) - V(\phi). \quad (22)$$

Therefore

$$c_s = \sqrt{1 - 2X f(\phi)} \ll 1, \quad (23)$$

$$\frac{\lambda}{\Sigma} = \frac{1}{2} \left( \frac{1}{c_s^2} - 1 \right). \quad (24)$$

The sizes of equilateral and orthogonal form bispectrum are given by

$$f_{NL}^{equil.} \simeq -0.3475 \left( \frac{1}{c_s^2} - 1 \right), \quad (25)$$

$$f_{NL}^{orth.} \simeq -0.02425 \left( \frac{1}{c_s^2} - 1 \right). \quad (26)$$

Because the radial velocity of brane in the KS throat is limited by the speed of light,  $c_s^2 \geq 0$  which is nothing but the condition for stability of the field theory. Therefore we obtain

$$f_{NL}^{orth.} \simeq 0.070 f_{NL}^{equil.} \lesssim 0. \quad (27)$$

As we know, the single-field inflation can only generate non-local form non-Gaussianity. Comparing to the WMAP 7yr data (constraints on  $f_{NL}^{equil.}$  and  $f_{NL}^{orth.}$ ), the allowed region for the general Lorentz invariant single-field inflation shows up in Fig. 1.

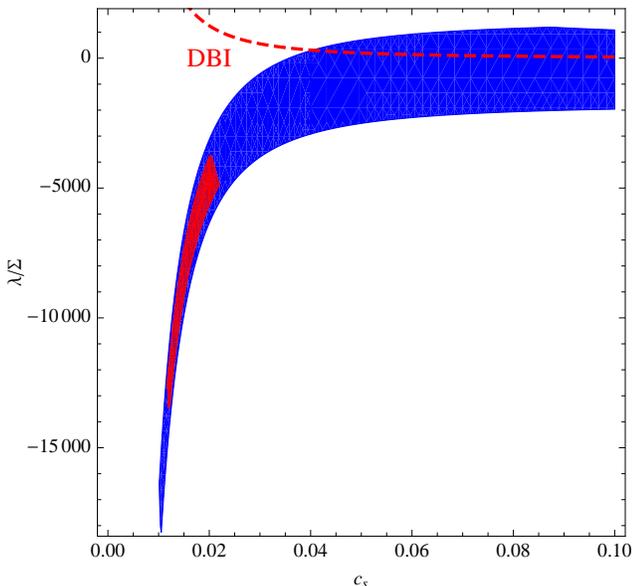


FIG. 1: (Color on line). The allowed parameter space for the general Lorentz invariant single-field inflation. The red and blue patches are the allowed regions at 68% and 95% CL respectively. The red dashed line corresponds to the prediction of DBI inflation.

However, a convincing detection of a large local form bispectrum will rule out all single-field inflation models (not only the slow-roll model). If one wants to obtain a bispectrum which is large not only for the non-local shape but also for the local shape, one needs a mixed scenario like that for the curvaton model suggested in [16]. Here we give a more general discussion about the bispectrum in the multi-field inflation where the trajectory of inflaton fields is a straight line in the field space during inflation. Denote the scalar field  $\phi_s$  to be transverse to the inflaton field  $\phi$  along the adiabatic direction. In the mixed scenario, both quantum fluctuations of  $\phi$  and  $\phi_s$  are assumed to contribute to the total curvature perturbation which seeds the temperature anisotropies in the CMBR. For simplicity, we assume that  $\phi_s$  is decoupled to  $\phi$ , namely  $\langle \delta\phi_{\mathbf{k}_1}, \delta\phi_{\mathbf{s}, \mathbf{k}_2} \rangle = 0$ . Considering

$$\Phi = \Phi^\phi + \Phi^{\phi_s}, \quad (28)$$

and  $\langle \Phi_{\mathbf{k}_1}^\phi, \Phi_{\mathbf{k}_2}^{\phi_s} \rangle = 0$ , we obtain

$$\Delta_\Phi^{tot.} = \Delta_\Phi^\phi + \Delta_\Phi^{\phi_s}. \quad (29)$$

For convenience, we introduce a new parameter  $\beta$  which is defined by

$$\beta = \Delta_\Phi^{\phi_s} / \Delta_\Phi^{tot.}. \quad (30)$$

Here  $\beta \in [0, 1]$ , and  $\Delta_\Phi^\phi = (1 - \beta)\Delta_\Phi^{tot.}$ . The index of power spectrum becomes

$$n_s^{tot.} = (1 - \beta)n_s^\phi + \beta n_s^{\phi_s}. \quad (31)$$

Since  $\phi_s$  is orthogonal to the adiabatic direction and hence the fluctuation of  $\phi_s$  only perturbs the value of  $\phi_s$ , not the energy density, it does not generate curvature perturbation during inflation, but its fluctuation can be converted into the adiabatic one at/after the end of inflation on the super-horizon scale [9–16]. So the non-Gaussianity caused by the fluctuation of  $\phi_s$  must have a local form. Denoting that the size of local form bispectrum generated by the quantum fluctuation of  $\phi_s$  as  $f_{NL}^{\phi_s}$ , we obtain

$$F(k_1, k_2, k_3) \simeq (\Delta_\Phi^{\phi_s})^2 f_{NL}^{\phi_s} \tilde{F}^{loc.}(k_1, k_2, k_3) + (\Delta_\Phi^\phi)^2 \mathcal{B}(k_1, k_2, k_3). \quad (32)$$

Considering  $\Delta_\Phi^{\phi_s} = \beta \Delta_\Phi^{tot.}$  and  $\Delta_\Phi^\phi = (1 - \beta)\Delta_\Phi^{tot.}$ , the effective sizes of local, equilateral and orthogonal form bispectrum are respectively given by

$$f_{NL}^{loc.} \simeq \beta^2 f_{NL}^{\phi_s}, \quad (33)$$

$$f_{NL}^{equil.} \simeq (1 - \beta)^2 f_{NL}^{\phi_s, equil.}, \quad (34)$$

$$f_{NL}^{orth.} \simeq (1 - \beta)^2 f_{NL}^{\phi_s, orth.}. \quad (35)$$

Since  $f_{NL}^{equil.}$  and  $f_{NL}^{orth.}$  are rescaled by a same factor, the consistency relations (21) and (27) are still valid in the mixed scenario.

Because  $\beta$  is a free parameter, it is better to use the consistency relations to constrain the inflation model. Comparing to the bounds on  $f_{NL}^{equil.}$  and  $f_{NL}^{orth.}$  from WMAP 7yr data, we find that DBI inflation is disfavored at more than 68% CL. See Fig. 2.

In [16] we first pointed out the enhancement of  $\tau_{NL}^{loc.}$  comparing to  $(\frac{6}{5} f_{NL}^{loc.})^2$  in the mixed curvaton model. Actually it is a generic result. The curvature perturbation generated by  $\phi_s$  on the super-horizon scale can be expanded as follows

$$\Phi^{\phi_s} = \Phi_L^{\phi_s} + f_{NL}^{\phi_s} \left( \Phi_L^{\phi_s} \right)^2 + g_{NL}^{\phi_s} \left( \Phi_L^{\phi_s} \right)^3 + \dots, \quad (36)$$

where  $\Phi_L^{\phi_s}$  is the linear part of the curvature perturbation. Comparing to the definition of  $\tau_{NL}^{loc.}$

$$\begin{aligned} & \langle \Phi_{\mathbf{k}_1} \Phi_{\mathbf{k}_2} \Phi_{\mathbf{k}_3} \Phi_{\mathbf{k}_4} \rangle \\ &= (2\pi)^3 \delta^{(3)} \left( \sum_{i=1}^4 \mathbf{k}_i \right) \cdot \left[ 6g_{NL}^{loc.} \Delta_\Phi^3 \cdot \frac{\sum_{i=1}^4 k_i^s}{\prod_{i=1}^4 k_i^s} \right. \\ & \left. + \frac{25}{18} \tau_{NL}^{loc.} \Delta_\Phi^3 \cdot \left( \frac{1}{k_{12}^s k_2^s k_3^s} + 23 \text{ perms.} \right) \right], \end{aligned} \quad (37)$$

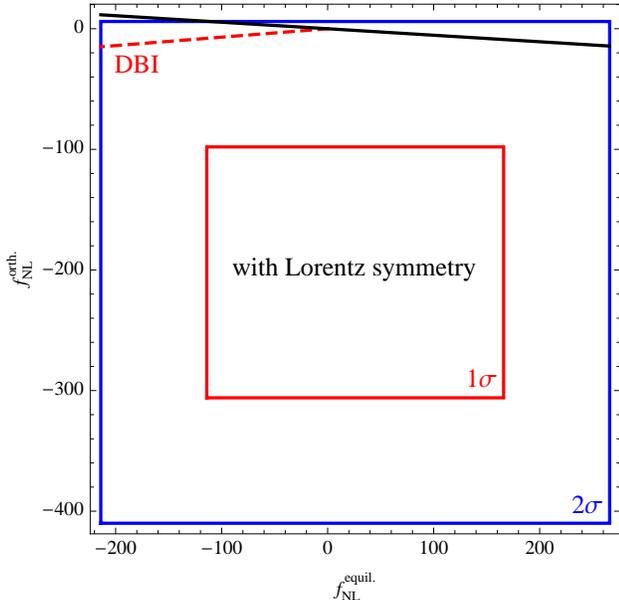


FIG. 2: (Color on line). Constraint on the prediction of the general single-field inflation without breaking Lorentz symmetry comparing to the WMAP 7yr data. The patch below the black solid line corresponds to the allowed region for the general single-field inflation with Lorentz symmetry, and the red dashed line corresponds to prediction of DBI inflation.

we find

$$\tau_{NL}^{loc.} \simeq \beta^3 \left( \frac{6}{5} f_{NL}^{\phi_s} \right)^2. \quad (38)$$

Here we consider that the quantum fluctuation of  $\phi$  does not generated local shape bispectrum. Since  $f_{NL}^{loc.} \simeq \beta^2 f_{NL}^{\phi_s}$ ,

$$\tau_{NL}^{loc.} \simeq \frac{1}{\beta} \left( \frac{6}{5} f_{NL}^{loc.} \right)^2 \quad (39)$$

which is enhanced by a factor  $1/\beta$ . If both local and non-local shape bispectrum are detected by the upcoming cosmological observations, such as Planck satellite,  $\beta$  should be smaller than one and hence  $\tau_{NL}^{loc.} > (\frac{6}{5} f_{NL}^{loc.})$ . This is a distinguishing prediction of the mixed scenario. As we know, a local form trispectrum with  $\tau_{NL}^{loc.} > 560$  can be detected by Planck at  $2\sigma$  level [20]. It is very exciting to check the consistency relation between  $\tau_{NL}^{loc.}$  and  $f_{NL}^{loc.}$  in the near future.

In addition, one may worry that the isocurvature perturbations in the mixed scenario might be too big to fit the WMAP data [17]. However, whether the isocurvature perturbations are generated in the multi-field inflation depends on the detail of reheating. For example, if the cold dark matter is not the direct decay product of the adiabatic and entropic fields and the cold dark matter is generated after all of these fields decay completely, the perturbations from multi fields do not gen-

erate the detectable isocurvature perturbations, and the mixed scenario is free from the constraint on the isocurvature perturbations.

To summarize, the stability of the field theory leads to a consistency relation (21) between the equilateral and orthogonal shape bispectrum in the general Lorentz invariant single-field inflation. Even though there is a big room for the general Lorentz invariant single-field inflation, some Lorentz invariant single-field inflation models, such as DBI inflation, have been tightly constrained by the WMAP 7yr data. The consistency relations (21) and (27) are also valid even in the mixed scenario where the fluctuations along the entropy directions also make contributions to the total curvature perturbation on the super-horizon scale and generate a local form bispectrum. We conclude that the DBI inflation is disfavored by WMAP 7yr data at more than 68% CL. So the inflation driven by a D-brane in string theory seems unlikely. Similarly, the consistency relation for the power-law K-inflation [21] is  $f_{NL}^{orth.} \simeq 0.1 f_{NL}^{equil.} < 0$  which is also disfavored.

As we know, the mixed scenario provides the only mechanism to achieve an inflation model with not only a large local form but also large non-local shape bispectrum. A distinguishing prediction of this scenario is  $\tau_{NL}^{loc.} > (\frac{6}{5} f_{NL}^{loc.})^2$ .

**Acknowledgments.** We would like to thank P. Chingangbam and X. Gao for useful discussions. This work is supported by the project of Knowledge Innovation Program of Chinese Academy of Science.

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