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FOURTH-ORDER GRAVITY AS THE INFLATIONARY MODEL REVISITED [1]

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Abstract

We revisit the simplest (fourth-order or quadratically generated) modified gravity model in four space-time dimensions. It is equivalent to the certain quintessence model via a Legendre-Weyl transform. By using the quintessence scalar potential, we compute the (CMB) observables of inflation associated with curvature perturbations (namely, the scalar and tensor spectral indices, and the tensor-to-scalar ratio) by using the most recent WMAP5 experimental bound. Our results include the next-to-leading terms with respect to the inverse number of e-foldings.

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1 Introduction

Inflation is a proposal (cosmological paradigm) about the existence of a short but fast (exponential, or de-Sitter-type) accelerated grow of the FLRW scale factor $a(t)$ in the early Universe, after the Big-Bang but before the radiation-dominated epoch [1]. It implies

$$\ddot{a}(t) > 0 \tag{1.1}$$

Though the whole idea of inflation remains to be a speculation, there is the significant (indirect) evidence for it. In the first place, it is the correct prediction of CMB fluctuations and large scale structure in remarkable agreement with the WMAP observations of CMB — see eg., ref. [2]. Inflation can generate irregularities in the Universe that may lead to the formation of structure. The main discriminators among various inflationary models are the *spectral indices* associated with the primordial power spectrum of curvature perturbations [3]. For instance, the on-going PLANCK satellite mission is going to provide tight constraints on the observable spectral indices with the accuracy of under 0.5 percent [4]. It is therefore of importance to reconsider primary candidates among the inflationary models, as to whether they can survive those precision tests in a near future.

One of the well known inflationary models is given by the fourth-order gravity [5, 6]. It is the simplest version of modified $f(R)$ gravity, whose extra terms beyond the standard Einstein-Hilbert term are merely *quadratic* in the scalar curvature, so that the equations of motion are of the fourth-order in the derivatives of a metric. We briefly review that model in Sec. 2. An $f(R)$ gravity model is known to be equivalent to the certain quintessence model via a Legendre-Weyl transform. We review that procedure in Sec. 3. Also in Sec. 3 we derive the quintessence scalar potential in the case of our specific modified gravity model. Both Secs. 2 and 3 serve as the technical introduction and setup. The main part is given by Sec. 4 where we compute the spectral indices of the slow-roll inflation in the model under consideration, including the next-to-leading-order terms with respect to the inverse number of e-foldings, and solve the equations of motion in the slow-roll approximation, in the dual (scalar-tensor or quintessence) picture.

2 Definition of the model

There is *a priori* no reason of restricting the gravitational Lagrangian to the standard Einstein-Hilbert term that is linear in the scalar curvature, as long as it does not contradict an experiment. The first attempt of this kind was made by Weyl as early as 1921. Nowadays, there is no doubt that the extra terms of the higher-order in the curvature should appear in the gravitational effective action of *any* Quantum Theory of Gravity. For instance, they do appear in String Theory — see eg., ref. [7] for a review. Since the scale of inflation is just a few orders less than the Planck scale [3], it is conceivable that the higher-order gravitational terms may be instrumental for inflation. It is already the case in the simplest modified gravity model having only the terms quadratic in the curvature [6].

As is well known, there exist only three independent *quadratic* curvature invariants, $R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}$, $R^{\mu\nu}R_{\mu\nu}$ and R^2. In addition, in four space-time dimensions,

$$\int d^4x\,\sqrt{-g}\left(R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}-4R^{\mu\nu}R_{\mu\nu}+R^2\right) \tag{2.2}$$

is topological for any metric, whereas

$$\int d^4x\,\sqrt{-g}\left(3R^{\mu\nu}R_{\mu\nu}-R^2\right) \tag{2.3}$$

is topological for any FLRW metric. Those combinations do not contribute to the (Friedmann) equation of motion for the scale factor, indicating that the scalar curvature models play the most important role in cosmological dynamics. Hence, the most general gravitational action of the highest order 2 in the curvature, which may be relevant for inflation, is given by

$$S=\frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\left(2\Lambda-R+\alpha R^2\right) \tag{2.4}$$

where we have introduced the cosmological constant Λ and the dimensional parameter $\alpha\equiv M^{-2}$ of mass dimension (-2). We use the spacetime signature $(+,-,-,-)$ and the units $\hbar=c=1$. The Einstein-Hilbert term in eq. (2.4) has the standard normalization with $\kappa=M_{\rm Pl}^{-1}$ in terms of the reduced Planck mass $M_{\rm Pl}^{-2}=8\pi G_N$. The rest of our notation for space-time (Riemann) geometry is the same as in ref. [8].

The simple model (2.4) is the particular case of the Starobinsky models [5]. As was shown in refs. [5, 6], the equations of motion for the action (2.4) have an inflationary solution with $\alpha\neq 0$ (even when $\Lambda=0$), and it is stable provided that $\alpha>0$. The stability is confirmed by our method in Sec. 3.

3 f(R) gravity and quintessence

The model (2.4) is the simplest particular case of the modified $f(R)$ gravity models characterized by an action

$$S_f=-\frac{1}{2\kappa^2}\int d^4x\,f(R) \tag{3.5}$$

with some function $f(R)$ of the scalar curvature. Those models are quite popular in the current literature — see eg., the recent reviews [9] and the references therein — mainly due to their theoretical applications to inflation and dark energy.

The gravitational equations of motion derived from the action (3.5) read

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + g_{\mu\nu}\Box f'(R) - \nabla_\mu\nabla_\nu f'(R) = 0 \tag{3.6}$$

where the primes denote differentiation. Those equations of motion are the 4th-order differential equations with respect to the metric $g_{\mu\nu}$ (ie. with the higher derivatives). Taking the trace of eq. (3.6) yields

$$\Box f'(R) + \tfrac{1}{3}f'(R)R - \tfrac{2}{3}f(R) = 0 \tag{3.7}$$

Hence, in contrast to General Relativity having $f'(R) = const.$, in f(R) gravity the field $A = f'(R)$ is *dynamical*, ie. it represents the independent propagating (scalar) degree of freedom. In terms of the fields $(g_{\mu\nu}, A)$ the equations of motion are of the 2nd order in the derivatives of the fields.

In fact, any $f(R)$ gravity is classically (mathematically) equivalent to a scalar-tensor gravity [6]. The equivalence is established by applying a Legendre-Weyl transform. The action (3.5) is equivalent to

$$S_A = \frac{-1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\{AR - Z(A)\} \tag{3.8}$$

where the real scalar $A(x)$ is related to the scalar curvature R by the Legendre transformation

$$R = Z'(A) \qquad \text{and} \qquad f(R) = RA(R) - Z(A(R)) \tag{3.9}$$

A Weyl transformation of the metric

$$g_{\mu\nu}(x) \to \exp\left[\frac{2\kappa\phi(x)}{\sqrt{6}}\right] g_{\mu\nu}(x) \tag{3.10}$$

with the arbitrary field parameter $\phi(x)$ yields

$$\sqrt{-g}\,R \to \sqrt{-g}\,\exp\left[\frac{2\kappa\phi(x)}{\sqrt{6}}\right]\left\{R - \sqrt{\frac{6}{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)\kappa - \kappa^2 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\} \tag{3.11}$$

Hence, when choosing

$$A(\kappa\phi) = \exp\left[\frac{-2\kappa\phi(x)}{\sqrt{6}}\right] \tag{3.12}$$

and ignoring the total derivative, we can rewrite the action (3.8) to the form

$$S_\phi = \int d^4x\,\sqrt{-g}\left\{\frac{-R}{2\kappa^2} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2\kappa^2}\exp\left[\frac{4\kappa\phi(x)}{\sqrt{6}}\right] Z(A(\kappa\phi))\right\} \tag{3.13}$$

in terms of the physical (and canonically normalized) scalar field $\phi(x)$.

Equation (3.13) is the standard action of the real dynamical scalar field $\phi(x)$ minimally coupled to Einstein gravity and having the scalar potential

$$V(\phi) = -\frac{M_{\rm Pl}^2}{2} \exp\left\{ \frac{4\phi}{M_{\rm Pl}\sqrt{6}} \right\} Z\left(\exp\left[\frac{-2\phi}{M_{\rm Pl}\sqrt{6}} \right] \right) \tag{3.14}$$

We are now going to employ it as the quintessence model of inflation. In order to explicitly derive the quintessence scalar potential (3.14), one has to solve for R in terms of ϕ by inverting the relation

$$f'(R) = A(\phi) \tag{3.15}$$

that follows from eq. (3.9) by differentiation. In the special case of

$$f(R) = R - 2\Lambda - \frac{1}{M^2} R^2 \tag{3.16}$$

we find

$$V(\phi) = \left(\frac{M_{\rm Pl}^2 M^2}{8} + \tilde{\Lambda} \right) \exp\left\{ \frac{2\sqrt{2}\phi}{M_{\rm Pl}\sqrt{3}} \right\} - \frac{M_{\rm Pl}^2 M^2}{4} \exp\left\{ \frac{\sqrt{2}\phi}{M_{\rm Pl}\sqrt{3}} \right\} + \frac{M_{\rm Pl}^2 M^2}{8} \tag{3.17}$$

where the notation $\tilde{\Lambda} = M_{\rm Pl}^2 \Lambda$ has been introduced. In terms of the new variable and the parameter

$$y = \sqrt{\frac{2}{3}} \frac{\phi}{M_{\rm Pl}} \qquad \text{and} \qquad V_0 = \frac{1}{8} M_{\rm Pl}^2 M^2 \tag{3.18}$$

respectively, the potential (3.17) reads

$$v(y) = \frac{V(y)}{V_0} = \left(1 + \frac{\tilde{\Lambda}}{V_0} \right) e^{2y} - 2e^y + 1 \tag{3.19}$$

The scalar potential appears to be bounded from below with the only minimum at $y = 0$ (stability!). It is also sufficiently steep for a slow-roll inflation. It is the last (third) cosmological term on the right-hand-side of eq. (3.17) that dominates in the potential during the slow-roll inflation (when taken alone, it gives rise to a de-Sitter inflationary solution), the second term represents the 1st-order (leading) correction, and the first term is the 2nd-order (subleading) correction. [2] In what follows we ignore $\tilde{\Lambda}$. Then the scalar potential for the slow-roll inflation gets simplified to

$$V(y) = V_0 \left(e^y - 1 \right)^2 \tag{3.20}$$

[2] By the same token we find that the cosmological term is unimportant during the slow-roll inflation. The ratio $\tilde{\Lambda}/V_0$ is also negligible from physical (scale) arguments.

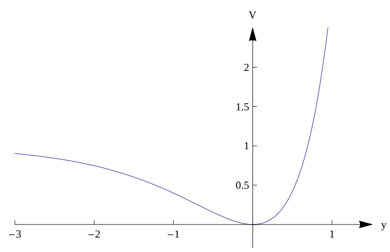

Figure 1: Graph of the function $v(y) = e^{2y} - 2e^{y} + 1$

A graph of the function $v(y) = e^{2y} - 2e^{y} + 1$ near its minimum $y = 0$ is given in Fig. 1. After a shift $\phi \to \phi + \phi_0$ with $2\exp\left[\sqrt{\frac{2}{3}}\frac{\phi_0}{M_{\rm Pl}}\right] = 1$, the potential (3.20) for the sufficiently negative values of y can be approximated as

$$V_{\rm eff}(\phi) \approx V_0 \left[1 - \exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_{\rm Pl}}\right)\right] \tag{3.21}$$

where we have ignored the subleading contribution. This scalar potential is known in the quintessence (inflationary) model building [3]. In our treatment of Sec. 4 we use the potential (3.20).

The $R + R^2$ gravity model is known as the excellent model of chaotic inflation in early Universe, and its spectral indices in the leading approximation are also known [10]. In the next Sec. 4 we re-derive those indices in the dual (quintessence) picture, and calculate the sub-leading terms.

4 Spectral indices

The slow-roll inflation parameters are defined by [3]

$$\varepsilon(\phi) = \frac{1}{2}M_{\rm Pl}^2\left(\frac{V'}{V}\right)^2 \tag{4.22}$$

and

$$\eta(\phi) = M_{\rm Pl}^2\frac{V''}{V} \tag{4.23}$$

where the primes denote derivatives with respect to the quintessence (inflaton) field ϕ. A necessary condition for the slow-roll approximation is the smallness of the inflation parameters [3],

$$\varepsilon(\phi) \ll 1 \qquad \text{and} \qquad |\eta(\phi)| \ll 1 \tag{4.24}$$

The first condition implies eq. (1.1), whereas the second condition guarantees that inflation lasts long enough, via domination of the friction term in the inflaton equation of motion (in the slow-roll case):

$$3H \dot{\phi} = -V' \tag{4.25}$$

Here H stands for the Hubble 'constant' $H(t) = \dot{a}/a$. Equation (4.25) is to be supplemeted by the Friedmann equation

$$H^2 = \frac{V}{3M_{\rm Pl}^2} \tag{4.26}$$

It follows from eqs. (4.25) and (4.26) that

$$\dot{\phi} = -M_{\rm Pl}\frac{V'}{\sqrt{3V}} < 0 \tag{4.27}$$

whose solution during the slow-roll inflation ($t_0 < t_{\rm start} \leq t \leq t_{\rm end}$) is

$$\phi(t) = -\sqrt{\frac{3}{2}}M_{\rm Pl}\ln\left[\frac{4\sqrt{V_0}}{3\sqrt{3}M_{\rm Pl}}(t-t_0)\right] \tag{4.28}$$

Substituting it into eq. (4.26) and using the definition $H = \dot{a}/a$ gives rise to a differential equation on the scale factor $a(t)$. Its solution is

$$a(t) = e^{H_0 t}\left[\frac{t-t_0}{\rm const.}\right]^{-3/4} \tag{4.29}$$

where we have introduced the notation $H_0 = M/\sqrt{24}$. The presence of a singularity at $t = t_0$ in eq. (4.29) is harmless because our inflationary solution is only valid during the slow-roll inflation when $t \geq t_{\rm start} > t_0$, so that it does not apply to the Big Bang. A resolution of the Big Bang singularity is supposed to require the higher-order curvature terms in the gravitational effective action (2.4).

The amount of inflation is measured by the e-foldings number

$$N_e = \int_t^{t_{\rm end}} H dt \approx \frac{1}{M_{\rm Pl}^2}\int_{\phi_{\rm end}}^{\phi}\frac{V}{V'}d\phi \tag{4.30}$$

where the $t_{\rm end}$ stands for the (time) end of inflation when one of the slow-roll parameters becomes equal to 1. The number of e-foldings between 50 and 100 is usually considered to be acceptable.

In the case of the slow-roll inflation with the scalar potential (3.20), we find that $\varepsilon(\phi)$ *first* approaches 1 at $\phi_{\rm end} = \sqrt{\frac{3}{2}}M_{\rm Pl}\ln\left(2\sqrt{3}-3\right) \approx -0.94\ M_{\rm Pl}$, since $|\eta(\phi)|$ approaches 1 later, at $\phi_{\rm end} = -\sqrt{\frac{3}{2}}M_{\rm Pl}\ln\frac{5}{3} \approx -0.62\ M_{\rm Pl}$. Then eq. (4.30) yields

$$N_e = \frac{3}{4}\left(e^{-y}+y\right) - \frac{3}{4}\left(\exp\left[\sqrt{\frac{2}{3}}\cdot 0.94\right] - \sqrt{\frac{2}{3}}\cdot 0.94\right) \approx \frac{3}{4}\left(e^{-y}+y\right) - 1.04 \tag{4.31}$$

where we have used the notation (3.18). Similarly, we find

$$\varepsilon = \frac{4e^{2y}}{3\left(1-e^{y}\right)^{2}} \qquad \text{and} \qquad \eta = \frac{-4e^{y}(1-2e^{y})}{3\left(1-e^{y}\right)^{2}} \tag{4.32}$$

Equation (4.31) can now be used to get y in terms of N_e, while a substitution of $y(N_e)$ into eq. (4.32) yields both $\varepsilon(N_e)$ and $\eta(N_e)$. The results of our numerical calculations (by using MATHEMATICA) are summarized in Table 1.

An analytic approximation can be obtained by using the expansion with respect to the *inverse* number of e-foldings. For instance, eq. (4.31) yields

$$e^{y} = \frac{3}{4N_e} - \frac{9\ln N_e}{16N_e^2} - \frac{0.94}{N_e^2} + \mathcal{O}\left(\frac{\ln^2 N_e}{N_e^3}\right) \tag{4.33}$$

Equation (4.32) now implies

$$\varepsilon = \frac{3}{4N_e^2} + \mathcal{O}\left(\frac{\ln^2 N_e}{N_e^3}\right) \tag{4.34}$$

and

$$\eta = -\frac{1}{N_e} + \frac{3\ln N_e}{4N_e^2} + \frac{5}{4N_e^2} + \mathcal{O}\left(\frac{\ln^2 N_e}{N_e^3}\right) \tag{4.35}$$

We are now ready for a calculation of the CMB *observable* quantitites in our inflationary model, ie. for its specific physical predictions. The primordial spectrum in a power-law approximation takes the form of k^{n-1} in terms of the comoving wave number k and the spectral index n. In particular, the slope n_s of the *scalar* power spectrum, associated with the density perturbations, is given by [3]

$$n_s = 1 + 2\eta - 6\varepsilon \quad , \tag{4.36}$$

the slope of the *tensor* primordial spectrum, associated with the gravitational waves, is given by [3]

$$n_t = -2\varepsilon \quad , \tag{4.37}$$

whereas the scalar-to-tensor ratio is given by [3]

$$r = 16\varepsilon \quad . \tag{4.38}$$

For instance, eqs. (4.34), (4.35) and (4.36) in our model imply

$$n_s = 1 - \frac{2}{N_e} + \frac{3\ln N_e}{2N_e^2} - \frac{2}{N_e^2} + \mathcal{O}\left(\frac{\ln^2 N_e}{N_e^3}\right) \tag{4.39}$$

The spectral indices are constrained by cosmological observations — see eg., the recent WMAP5 data [11] that implies

$$n_s = 0.960 \pm 0.013 \qquad \text{and} \qquad r < 0.22 \tag{4.40}$$

Table 1: The slow-roll parameters and spectral indices for some values of N_e

N_e	$\varepsilon(\times 10^{-4})$	$\eta(\times 10^{-2})$	$r(\times 10^{-3})$	$n_t(\times 10^{-4})$	n_s
35	5.13	- 2.56	8.20	- 10.3	0.946
40	3.99	- 2.27	6.39	- 7.98	0.952
45	3.20	- 2.03	5.12	- 6.40	0.957
50	2.62	- 1.84	4.19	- 5.24	0.962
55	2.19	- 1.69	3.50	- 4.37	0.965
60	1.85	- 1.55	2.96	- 3.71	0.968
65	1.59	- 1.44	2.54	- 3.18	0.970
70	1.38	- 1.34	2.21	- 2.76	0.972
75	1.21	- 1.26	1.93	- 2.42	0.974

In addition, the amplitude of the initial perturbations, $\Delta_R^2 = M_{\rm Pl}^4 V/(24\pi^2\varepsilon)$, is yet another physical observable, whose experimental value is [3]

$$\left(\frac{V}{\varepsilon}\right)^{1/4} = 0.027\, M_{\rm Pl} = 6.6 \times 10^{16} \text{ GeV} \tag{4.41}$$

Equation (4.41) determines the normalization of the R^2-term in eq. (2.4) as

$$\frac{M}{M_{\rm Pl}} = 4 \cdot \sqrt{\frac{2}{3}} \cdot (2.7)^2 \cdot \frac{e^y}{(1-e^y)^2} \cdot 10^{-4} = (3.5 \pm 1.2) \cdot 10^{-5} \tag{4.42}$$

where, in the last step, we have used the value of $N_e = 53.8 \pm 18$, as it follows from eqs. (4.39) and (4.40). The results of our numerical calculations of the spectral indices are collected in Table 1. In particular, we find that the WMAP5 experimental bounds on the scalar spectral index in eq. (4.40) are satisfied in the cosmological model (2.4) provided that the e-foldings number N_e lies between 35.9 and 71.8, with the middle value of $\bar{N}_e = 53.8$. We also find the noticable suppression of tensor fluctuations as $|r| < 8.2 \cdot 10^{-3}$ and $|n_t| < 10^{-3}$. There is a possibility of further theoretical modification, which would imply more tuning of the spectral indices, when more terms of the higher-order in the curvature are added into the action (2.4).

5 Conclusion

Our main results are given by eqs. (4.29), (4.31), (4.32), (4.39), (4.42) and Table 1. The leading terms agree with the known results [10, 12]. We confirm that the simplest model of modified gravity with the single new parameter M may describe inflation and agree with experimental (CMB) observations. As regards possible generalizations to the quartic curvature terms, see eg., ref. [13]. Modified gravity is extendable to modified supergravity [14].

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