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Boundary stress tensors for spherically symmetric conformal Rindler observers

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Abstract

The boundary energy - momentum tensors for a static observer in the conformally flat Rindler geometry are considered. We found the surface energy is positive far form the Planck world but the transversal pressures are negative.

The kinematical parameters associated to a nongeodesic congruence of static observers are computed. The entropy S corresponding to the degrees of freedom on the two surface of constant ρ and t equals the horizon entropy of a black hole with a time dependent mass and the Padmanabhan expression $E = 2ST$ is obeyed.

The two surface shear tensor is vanishing but the coefficient of the bulk viscosity ζ is $1/16\pi$ and therefore the negative pressure due to it acts as a surface tension.

Keywords : surface energy, conformal Rindler geometry, horizon entropy, equipartition law.

1 Introduction

There has been renewed interest in recent years in boundary matter. General Relativity, like Newtonian gravitation, indicate that matter is the source for gravity. Khoury and Parikh [1] asked the question whether the acceleration could be attributed to matter. But the matter distribution is encoded in the stress tensor. Therefore, where the energy and stresses are localized as we could accelerated even in Minkowski space (inertial frames exist even in the total absence of matter) ? Usually the boundary conditions for the metric are in the form of an induced metric and extrinsic curvature for some hypersurface.

The boundary matter concept appears, for example, in the so-called "membrane paradigm" for the black hole [2] [3] [4] stretched horizon, viewed as an inner boundary (it is worth to note that the Gibbons - Hawking term in the

Einstein - Hilbert action is a surface integral over the outer boundary of spacetime).

As Khoury and Parikh have conjectured, "matter refers to both bulk and boundary stress tensors". They uniquely specify the geometry of spacetime. This may be recognized from the fact that the bulk stress tensor does not fully determine the Riemann tensor which, besides the Ricci tensor, contains the Weyl tensor (its free of trace part). In addition, the Weyl tensor is independent of matter. Therefore, the authors of [1] emphasized that , to determine the solutions of Einstein's equations, one needs to add boundary conditions.

Other contexts where boundary matter arises are the brane-world scenarios , AdS - CFT correspondence [5], and the Brown - York construction of a gravitational boundary stress tensor [6].

The purpose of this paper is to compute the boundary stress tensors for a uniformly accelerating distribution of conformal Rindler observers [7] [8]. Our "boundary sources" are located on any $\rho = const.$ hypersurface , where ρ is the radial coordinate. As Padmanabhan [9] has observed, one can associate to the area A of a spherical surface around a massive body the degrees of freedom corresponding to the entropy $S = A/4$ because observers at rest on the surface will have an acceleration produced by the body. We have in our situation local Rindler observers who will perceive any timelike surface as a stretched horizon. In addition, for accelerated observers one can in principle locate holographic screens (equipotential surfaces) anywhere in space [10].

We further analyze the spacelike two-section of the $\rho = const.$ hypersurface (stretched horizon). The corresponding surface stress tensor is proportional to the induced metric tensor. The shear tensor is zero but the bulk viscosity coefficient $\zeta = 1/16\pi$.

Throughout the paper we use geometrical units : $G = c = \hbar = 1$.

2 The conformally flat Rindler metric

Let us consider the 4 - dimensional subspace of the Witten bubble spacetime [12]

$$ds^2 = -g^2r^2dt^2 + (1 - \frac{4b^2}{r^2})^{-1}dr^2 + r^2cosh^2gt\ d\Omega^2 \tag{2.1}$$

with b and g - constants and $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$ is the metric on the unit two sphere .

The metric (2.1) is the ordinary Minkowski space provided $r >> 2b$, but written in spherical Rindler coordinates [13] (a spherical distribution of uniformly accelerated observers , with the rest - system acceleration g, uses this type of hyperbolically expanding coordinates). The singularity at $r = 2b$ is only a coordinate singularity, as can be seen from the isotropical form of (2.1), with the help of a new radial coordinate ρ

$$r = \rho + \frac{b^2}{\rho}. \tag{2.2}$$

Therefore, the spacetime (2.1) appears now as

$$ds^2 = \left(1 + \frac{b^2}{\rho^2}\right)^2 (-g^2\rho^2 dt^2 + d\rho^2 + \rho^2 cosh^2 gt\ d\Omega^2). \qquad (2.3)$$

As Cho and Pak [14] have observed, the singularity at $\rho = 0$ is a coordinate singularity since the curvature invariant $R_{\alpha\beta}R^{\alpha\beta}$ and the Kretschmann scalar $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ are finite there. Therefore, the conformally flat metric (2.3) can be interpreted as an instanton - antiinstanton pair in conformal gravity [14].
We note that the coordinate transformation

$$\bar{x} = \rho cosh gt sin\theta cos\phi, \quad \bar{y} = \rho cosh gt sin\theta sin\phi, \quad \bar{z} = \rho cosh gt cos\theta, \quad \bar{t} = \rho sinh gt \qquad (2.4)$$

changes the previous metric into

$$ds^2 = (1 + \frac{b^2}{\bar{x}_\alpha \bar{x}^\alpha})^2\ \eta_{\mu\nu}\ d\bar{x}^\mu\ d\bar{x}^\nu, \qquad (2.5)$$

which is conformally flat ($\eta_{\mu\nu} = diag(-1, +1, +1, +1)$, the Greek indices run from 0 to 3) and $\rho^2 = \bar{x}^i \bar{x}_i - \bar{t}^2, i = 1, 2, 3$. From now on we take the constant b to be of the order of the Planck length.

Let us observe that the geometry (2.3) becomes flat provided $\rho >> b$ (the conformal factor tends to unity) or $\rho << b$, when the first term in the conformal factor may be neglected. Therefore, (2.3) represents the Lorentzian version of the euclidean Hawking wormhole [15] [16].

It is interesting to treat the region $\rho < b$ by means of the coordinate transformation

$$\bar{\rho} = \frac{b^2}{\rho}. \qquad (2.6)$$

Inserting (2.6) in (2.3), we conclude that the metric (2.3) is invariant under the inversion (2.6). Therefore, we may say that the $\rho\ <\ b$ region is the image of $\rho\ >\ b$ region obtained by inversion.

We know that the spacetime (2.3) is not a solution of the vacuum Einstein's equations. Therefore, a source was introduced on the r.h.s. of them [8] in order that (2.3) to become an exact solution.

3 The three - surface stress tensor

From the surface source term in the gravitational action we know that the following energy - momentum tensor emerges [3] [17]

$$T^{(s)}_{\alpha\beta} = \frac{1}{8\pi}(K_{\alpha\beta} - \gamma_{\alpha\beta}K), \qquad (3.1)$$

where (s) refers to "surface", $K_{\alpha\beta}$ is the extrinsic curvature tensor corresponding to $\rho = \rho_0 = const.$ timelike hypersurface, K is its trace, $\gamma_{\alpha\beta}$ is the induced metric given by

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta. \qquad (3.2)$$

The normal 4 - vector to the surface has the components $n^\alpha = (0, 1/\omega, 0, 0)$, where $\omega = 1 + (b^2/\rho^2)$. With the help of the well-known expression

$$K_{\alpha\beta} = \gamma^\nu_\beta \nabla_\nu n_\alpha, \tag{3.3}$$

one obtains

$$K^t_t = K^\theta_\theta = K^\phi_\phi = \frac{\rho^2 - b^2}{\omega^2 \rho^3} \tag{3.4}$$

and the trace gives us

$$K \equiv \gamma^{\alpha\beta} K_{\alpha\beta} = \frac{3(\rho^2 - b^2)}{\omega^2 \rho^3}. \tag{3.5}$$

The Eq. (3.1) yields now

$$T^{(s)t}_t = T^{(s)\theta}_\theta = T^{(s)\phi}_\phi = -\frac{(\rho^2 - b^2)}{4\pi\omega^2\rho^3}. \tag{3.6}$$

We see that the energy - momentum tensor is proportional to the metric tensor

$$T^{(s)\alpha}_\beta = -\frac{(\rho^2 - b^2)}{4\pi\omega^2\rho^3} \gamma^\alpha_\beta. \tag{3.7}$$

We have therefore

$$\Sigma \equiv -T^{(s)t}_t = -p_\theta = -p_\phi = \frac{\rho^2 - b^2}{4\pi\omega^2\rho^3} \tag{3.8}$$

where $\Sigma > 0$ is the surface energy density and $p_\theta < 0,\ p_\phi < 0$ are the pressures (in fact tensions) on the angular directions. Note that, far from the Planck world ($\rho >> b$), $\Sigma = 1/4\pi\rho$. As a function of ρ, Σ vanishes at $\rho = 0$ *and* $\rho = b$. In Cartesian coordinates, these correspond to the light cones $\bar{x} = \pm\bar{t}$ or the hyperbolic trajectories $\bar{x} = \pm\sqrt{\bar{t}^2 + b^2}$.

At $\rho_{01} = b(\sqrt{2} - 1)$ we have

$$\Sigma_{min} = -\frac{2 - \sqrt{2}}{8\pi b} < 0 \tag{3.9}$$

and at $\rho_{02} = b(\sqrt{2} + 1)$ one obtains

$$\Sigma_{max} = \frac{2 + \sqrt{2}}{8\pi b} > 0. \tag{3.10}$$

Σ is positive for $\rho > b$ and negative for $0 < \rho < b$. The region around $\rho = b$ acts as a domain wall of thickness $\rho_{02} - \rho_{01} = 2b$. The fluctuations of the surface energy are completely negligible far from Planck's world.

4 The nongeodesic congruence of the static observer

Let us consider now "static" observers in the spacetime (2.3). Their 4 - velocity field is given by

$$u_\alpha = (-g\rho\omega, 0, 0, 0) \tag{4.1}$$

The scalar expansion Θ associated to u_α appears as

$$\Theta \equiv \nabla_\alpha u^\alpha = \frac{2\ tanh\ gt}{\omega\rho} \tag{4.2}$$

It is always finite and varies between $-2/\omega\rho$ and $2/\omega\rho$, when $t \to \pm\infty$.

The acceleration vector of the fluid world lines is given by

$$a^\alpha = (0, \frac{\rho^2 - b^2}{\omega^3\rho^3}, 0, 0), \tag{4.3}$$

with

$$\sqrt{a_\alpha a^\alpha} \equiv a(\rho) = \frac{|\rho^2 - b^2|}{\omega^2\rho^3}. \tag{4.4}$$

It is worth to note that the relation

$$a(\rho) = 4\pi|\Sigma(\rho)| \tag{4.5}$$

is obeyed, in accordance with the Gauss law. A similar expression has been obtained in [18], for the Rindler horizon, in Cartesian coordinates. For example, with $\rho_0 = 1cm$ ($\rho_0 >> b$), we obtain $\Sigma \approx 1/4\pi\rho = 10^{48} ergs/cm^2$. One means our hyperbolic observer , located at $\rho_0 = 1cm$ from the origin , perceives an enormous surface energy density. However, we must remind that the acceleration is also very large. On the contrary, we have $\Sigma = 10^{28}\ ergs/cm^2$ for $a = 10\ cm/s^2$ $(with\ \rho_0 \approx 10^{20}cm)$.

One can associate to the $\rho = \rho_0$ observer in the accelerated system a temperature $T(\rho_0)$ which is given by $T(\rho_0) = a(\rho_0)/2\pi$, according to Unruh's formula. Even though the $\rho = 0$ hypersurface is no longer a horizon - with respect to the $b = 0$ case - (the geometry (2.3) is time dependent and there is no a timelike Killing vector) , we might formally construct a "surface gravity"

$$\sqrt{a_\alpha a^\alpha}\ \sqrt{-g_{tt}}|_{\rho=0} = g. \tag{4.6}$$

Our uniformly accelerating system is not static. However, for $\rho >> b\ or\ \rho << \ b$, the "proper" temperature $T(\rho)$ observes the Tolman relation [19]

$$T(\rho)\sqrt{-g_{tt}} = \frac{g}{2\pi}\frac{|\rho^2 - b^2|}{\rho^2 + b^2} \approx \frac{g}{2\pi} = const., \tag{4.7}$$

where $T(\rho)$ is measured by a local observer. The temperature gradient is necessary to prevent the heat flow from regions with different gravitational potential. We have, indeed [20]

$$Q_\alpha = -\kappa h^\nu_\alpha (T_{,\nu} + Ta_\nu) = 0, \tag{4.8}$$

where a_ν is given by (4.3) , κ is the coefficient of thermal conductivity and Q_α is the spacelike heat flux.

The shear tensor of the flow

$$\sigma_{\mu\nu} = \frac{1}{2}(h^\alpha_\nu \nabla_\alpha u_\mu + h^\alpha_\mu \nabla_\alpha u_\nu) - \frac{1}{3}\Theta h_{\mu\nu} + \frac{1}{2}(a_\mu u_\nu + a_\nu u_\mu) \tag{4.9}$$

has the nonzero components

$$\sigma^\rho_\rho = -\frac{2\ tanh\ gt}{3\rho\omega}, \quad \sigma^\theta_\theta = \sigma^\phi_\phi = \frac{tanh\ gt}{3\rho\omega}, \tag{4.10}$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projection tensor in the direction perpendicular to u_α. However, the rotation tensor $\omega_{\mu\nu}$ is vanishing.

The time - evolution of the scalar expansion is given by the Raychaudhuri equation

$$\frac{d\Theta}{d\lambda} - \nabla_\alpha a^\alpha + \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \omega^{\alpha\beta}\omega_{\alpha\beta} + \frac{1}{3}\Theta^2 = -R_{\alpha\beta}u^\alpha u^\beta \tag{4.11}$$

where λ is the parameter along the worldlines, the Ricci tensor is defined by $R_{\alpha\beta} = g^{\mu\nu}R_{\alpha\mu\beta\nu}$ and

$$\frac{d\Theta}{d\lambda} \equiv \dot{\Theta} = u^\alpha \nabla_\alpha \Theta = \frac{2}{\omega^2\rho^2 cosh^2 gt} \tag{4.12}$$

Keeping in mind that

$$\sigma^{\alpha\beta}\sigma_{\alpha\beta} = \frac{2\ tanh\ gt}{3\omega\rho} \tag{4.13}$$

and

$$\nabla_\alpha a^\alpha = \frac{2}{\omega^4\rho^2}(1 + \frac{b^4}{\rho^4}) \tag{4.14}$$

one obtains for the l. h. s. of (4.11) the expression $-\ 4b^2/(\omega^4\rho^4)$. But R_{tt} for the metric (2.3) is given by $-4b^2/(\omega^2\rho^2)$ [8]. Using (4.1) we find that the Raychaudhuri equation is satisfied.

Let us find now the energy E enclosed by the 2 - surface $H : \rho = \rho_0, t = t_0$. The metric tensor on H is

$$s_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta + u_\alpha u_\beta \tag{4.15}$$

The energy is obtained from

$$E = \int_H \Sigma(\rho)\sqrt{s}\ d\theta\ d\phi \tag{4.16}$$

With $\sqrt{s} = \omega^2\rho^2 cosh^2 gt$ and $\Sigma(\rho)$ from (3.8), Eq. (4.16) yields

$$E = \rho(1 - \frac{b^2}{\rho^2})cosh^2\ gt, \tag{4.17}$$

taken at $\rho = \rho_0, \ t = t_0$.

By means of the Padmanabhan formula $E = 2ST$ [9] one obtains for the entropy S corresponding to the degrees of freedom on H (or, adopting the Holographic Principle, in the space enclosed by H)

$$S = \frac{E}{2T} = \pi \omega^2 \rho^2 cosh^2 \ gt. \tag{4.18}$$

Hence, $S = A/4$, where A is the area of H. This expression corresponds exactly to the horizon entropy of a black hole. That is not surprising since the $\rho = const.$ observer is at rest in the accelerated system, which is equivalent to a gravitational field. As Padmanabhan has noticed, one can construct a local Rindler observer who will perceive $\rho = \rho_0$ surface as a stretched horizon. In addition, we can check that the expression (4.17) is in accordance with the equipartition law [9]

$$E = \frac{1}{2} nT, \tag{4.19}$$

where n is the number of elementary cells of area l_P^2 (l_P - the Planck length). (4.17) and (4.18) are, of course, valid for $\rho > b$, otherwise E becomes negative.

The metric tensor $s_{\alpha\beta}$ of the spacelike 2 - section H of the constant ρ hypersurfaces, to which the velocity vector u^α is normal, is given by

$$s_{\alpha\beta} = (0, 0, \omega^2 \rho^2 cosh^2 gt, \omega^2 \rho^2 cosh^2 gt \ sin^2 \theta). \tag{4.20}$$

We may decompose the extrinsic curvature tensor $k^\alpha_\beta = s^\nu_\beta \nabla_\nu u^\alpha$ of H into a traceless part and a trace [3]

$$k_{\alpha\beta} = \sigma^{(2)}_{\alpha\beta} + \frac{1}{2} s_{\alpha\beta} \Theta^{(2)} \tag{4.21}$$

where $\sigma^{(2)}_{\alpha\beta}$ and $\Theta^{(2)}$ are, respectively, the shear tensor and the expansion of the surface elements. Using the space (4.20) we immediately find that $\sigma^{(2)}_{\alpha\beta}$ is vanishing and $\Theta^{(2)}$ appears as

$$\Theta^{(2)} = s^\nu_\alpha \nabla_\nu u^\alpha = \frac{2 \ tanh \ gt}{\omega \rho} \tag{4.22}$$

Therefore, the surface stress tensor

$$t^H_{\alpha\beta} = \frac{1}{8\pi} (k_{\alpha\beta} - k s_{\alpha\beta}) = -\frac{1}{16\pi} \Theta s_{\alpha\beta} \tag{4.23}$$

has two nonzero components

$$t^{(H)\theta}_\theta = t^{(H)\phi}_\phi = -\frac{tanh \ gt}{16\pi \omega \rho} \tag{4.24}$$

A comparison with a viscous Newtonian fluid leads to $\zeta = 1/16\pi$, where ζ stands for the bulk viscosity coefficient. In other words, on the 2 - surface H the pressure $(-\zeta \Theta^{(2)}) < 0$ and , therefore, it acts as a surface tension. In spite of the fact that the authors of [1] [3] [11] reached a negative value for ζ, our result is different because the 2 - surface H is not lightlike.

5 Conclusions

We consider in this paper a uniformly accelerated distribution of conformal Rindler observers and compute the stress tensors on 3 - and 2 - surfaces viewed as boundary of the spacetime. To any $\rho = \rho_0$ static observer in the accelerated system we may associate a temperature proportional to its acceleration since the local Rindler observers will perceive the timelike surface as a stretched horizon.

We have calculated the energy enclosed by a 2 - surface H of constant ρ and t and found that it obeys the equipartition law $E = (1/2)nT$. In addition, we established that entropy obtained from Padmanabhan's formula $S = E/2T$ corresponds to the horizon of a black hole. That supports our view that $\rho = \rho_0$ observers are equivalent to those located near the black hole horizon.

The stress tensor on H corresponds to a Newtonian viscous fluid with a bulk viscosity $\zeta = 1/16\pi$ leading to a negative pressure which acts as a surface tension.

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