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TYURINA COMPONENTS AND RATIONAL CYCLES FOR RATIONAL SINGULARITIES

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Abstract

In this paper, we give a geometric proof of Pinkham's theorem on the positive cycles supported on the exceptional divisor of a rational singularity. In order to do this, we give several properties of the Tyurina components of the exceptional divisor and of the points of blowing-up surface of a rational singularity.

Introduction

An isolated singularity of a complex surface S is rational if the stalk at the singularity of the coherent sheave $R^1\pi_*\mathcal{O}_X$ is equal to zero where $\pi : X \to S$ is a resolution of S at the singularity. The numerical characterization of a rational singularity, given by M. Artin in [1] (see theorem 2.4 below), permits us to study these singularities by the exceptional divisor of a resolution of the singularity (see [15] or [17]). Here we are interested in the positive cycles supported on the exceptional divisor of a resolution divisor of a rational singularity. In [1] and [11], it has been shown that these cycles correspond to some special functions on S. In section 4, we use this correspondance to prove Pinkham's theorem given in [12] (see theorem 4.4 below).

We start our paper by introducing some notations. In section 3, following the general case of a theorem of M. Artin [1] (see theorem 2.4 below), we give a proof on the nature of the exceptional divisor of a resoution of a rational singularity (see corollary 3.2 below). After giving some properties on the blowing up surface of a rational singularity, we finish the section by giving a bound on the non-Tyurina components of the exceptional divisor of a resolution of a rational singularity.

Recall

Let (S, ξ) be a germ of a normal analytic surface embedded in C^N. Denote by S a sufficiently small representative of the germ (S, ξ). A *resolution* of S is a complex analytic surface X and a proper holomorphic map $\pi : X \to S$ such that its restriction to $X - \pi^{-1}(\xi)$ is a biholomorphic map and $X - \pi^{-1}(\xi)$ is dense in X. By the Main Theorem of Zariski, the exceptional divisor $E := \pi^{-1}(\xi)$ is connected and of dimension 1. Let $E_1, \ldots, E_n$ denote its irreducible components.

We call *positive cycle* a formal sum of the irreducible components E_i of E with non-negative integral coefficients and with at least one non-zero coefficient. We denote by $\mathcal{E}^+$ the set of the positive cycles Y such that $(Y.E_i) \leq 0$ for all i (see [11], ∮18). The existence of such cycles is due to O. Zariski (see [19]). We define a partial ordering on $\mathcal{E}^+$ as following : For $Y, Y' \in \mathcal{E}^+$ with $Y = \sum_{i=1}^n m_i' E_i$, we have $Y \geq Y'$ if $m_i \geq m_i'$ for all $i, (i = 1, \ldots, n)$. Since E is connected, we have:

Remark 2.1. *For a positive cycle $Y = \sum_{i=1}^n m_i E_i$, if $(Y.E_i) \leq 0$ for all i, then we have $m_i \geq 1$ for all $i, (i = 1, \ldots, n)$.*

Definition 2.2 *Let A be a set of positive cycles $Y = \sum_{i=1}^n m_i E_i$. We define $lnfA$ as $Z_0 = \sum_{i=1}^n a_i E_i$ with*

$$a_i = inf_{Y \in A}\{m_i \mid m_i = mult_Y E_i\}$$

where $mult_Y E_i$ is the coefficient of E_i in Y. The cycle Z_0 is a positive cycle since $m_i \in \mathbf{N}^*$ for all i.

Theorem 2.3 *For all subset A of $\mathcal{E}^+$, we have $lnfA \in \mathcal{E}^+$.*

Proof. Let $Z_0 = \sum_{i=1}^n a_i E_i = lnfA$. We will show that $(Z_0.E_i) \leq 0$ for all i:

$$(Z_0.E_j) = a_j(E_j.E_j) + \sum_{i \neq j} a_j(E_i.E_j)$$

Let $Y^0 = \sum_{i=1}^n m_i^0 E_i$ be a positive cycle in A such that $m_j^0 = a_j$. We have then

$$(Z_0.E_j) = m_j^0(E_j.E_j) + \sum_{i\neq j} a_i(E_i.E_j)$$

$$(Z_0.E_j) \leq m_j^0(E_j.E_j) + \sum_{i\neq j} m_i^0(E_i.E_j) = (Y^0.E_j) \leq 0.$$

Hence we have $Z_0 \in \mathcal{E}^+$. □

Theorem 2.4 *For a resolution of S, the arithmetic genus of $lnf\mathcal{E}^+$ is greater than or equal to zero. In particular, the singularity E of S is rational if and only if the arithmetic genus of $lnf\mathcal{E}^+$ is zero.*

By [1] and [19], the cycle $Z = lnf\mathcal{E}^+ = \sum_{i=1}^n a_iE_i$ is called the *fundamental cycle* of the resolution π and it can be computed by Laufer algorithm (see [8], proposition 4.1).

Tyurina Components

The purpose of this section is to understand the nature of the exceptional divisor of a resolution of a rational singularity and the points of the blowing-up surface of a rational singularity. We will finish this section by giving a bound on the non-Tyurina components of the exceptional divisor of a resolution of a rational singularity.

From the proof of proposition 1 in [1], we deduce following theorem:

Theorem 3.1 *Let S be a sufficiently small representation of a germ (S, ξ) of a complex analytic normal surface having rational singularity at ξ. Let S' be a normal surface and let $\rho : S' \to S$ be a bimeromorphic proper map which is not the identity map. Let Y be a positive cycle supported on the divisor $\rho^{-1}(\xi)$. Then we have $H^1(\mid Y \mid, \mathcal{O}_Y) = 0$.*

Proof. The Main Theorem of Zariski says that the divisor $\rho^{-1}(\xi)$ is connected and of dimension 1. Let $E_1, \ldots, E_n$ be the irreducible components of $\rho^{-1}(\xi)$ and let $Y_{(r)} = \sum r_jE_j$ with $(r) = (r_1, \ldots, r_k)$ be a positive cycle supported on $\rho^{-1}(\xi)$. By the analytic comparison theorem of H. Grauert ([2], p.15-02), we have:

$$(R^1\rho_*\mathcal{O}_{S'})_\xi{}^\wedge = \lim_{(r) \leftarrow (\infty)} H^1(\mid Y_{(r)} \mid, \mathcal{O}_{Y_{(r)}})$$

where $(R^1\rho_*\mathcal{O}_{S'})_\xi{}^\wedge$ is the completion of the module of finite type $(R^1\rho_*\mathcal{O}_{S'})$ on $\mathcal{O}_{S,\xi}$ for the $\mathcal{M}$-adic topology where $\mathcal{M}$ is the maximal ideal of the local ring $\mathcal{O}_{S,\xi}$. Since the spaces $\mid Y_{(r)} \mid$ have dimension 1, the map

$$H^1(\mid Y_{(r)} \mid, \mathcal{O}_{Y_{(r)}}) \to H^1(\mid Y_{(r')} \mid, \mathcal{O}_{Y_{(r')}})$$

is surjective when $r_j \geq r'_j$ for all j. This gives $H^1(\mid Y_{(r)} \mid, \mathcal{O}_{Y_{(r)}}) = 0$ for all (r).

Since for all positive cycles Y supported on the divisor $\rho^{-1}(\xi)$, there exists (r) such that $Y \subset Y_{(r)}$, we have $H^1(Y, \mathcal{O}_Y) = 0$. □

Corollary 3.2 *With the same hypothesis as in the theorem 3.1, all irreducible components of the divisor $\rho^{-1}(\xi)$ are non-singular rational curves.*

Proof. We have to prove that E_i is non-singular and $p(E_i) = 0$ where $p(E_i)$ is the arithmetic genus of E_i. By the theorem 3.1, we have $H^1(E_i, \mathcal{O}_{E_i}) = 0$, and since E_i is a reduced irreducible curve, $H^0(E_i, \mathcal{O}_{E_i}) = 1$ (see [6], theorem I.3.4). Since 1-$p(E_i) = \chi(\mathcal{O}_{E_i}) = dimH^0(E_i, \mathcal{O}_{E_i}) - dimH^1(E_i, \mathcal{O}_{E_i}) = 1$ where $X(\mathcal{O}_{E_i})$ is the Euler characteristic of $\mathcal{O}_{E_i}$, we obtain $p(E_i) = 0$.

Let $n : \bar{E}_i \to E_i$ be the normalization of E_i. Since $\bar{E}_i$ is non-singular, we have the following exact sequence of coherent sheaves on E_i (see [6], exercise IV.1.8):

$$0 \to \mathcal{O}_{E_i} \to n_*\mathcal{O}_{Ei} \to \sum_{p \in E_i} \bar{\mathcal{O}}_p/\mathcal{O}_p \to 0$$

where $\bar{\mathcal{O}}_p$ is the integral closure sheaf of $\mathcal{O}_p$ is a coherent sheave concentrated on the singular points of E_i. Moreover we have $X(n_*\mathcal{O}_{E_i}) = X(\mathcal{O}_{E_i})$ (see [6], exercise III.4.1). Then

$$X(\mathcal{O}_{E_i}) = -dimH^1(\bar{E}_i, \mathcal{O}_{\bar{E}_i})$$

Let δ_p=length $(\bar{\mathcal{O}}_p/\mathcal{O}_p)$. We have :

$$X(\sum_{p \in E_1} \bar{\mathcal{O}}_p/\mathcal{O}_p) = dim_C H^0(E_i, \sum_{p \in E_i} \bar{\mathcal{O}}_p/\mathcal{O}_p) = \sum_{p \in E_i} \delta_p$$

Hence $p(E_i) = p(\bar{E}_i) + \sum_{p \in E_i} \delta_p$. Here $p(E_i) \geq 0$ (see [6], p.181) and $\delta_p \geq 0$ for all $p \in E_i$. Since $p(E_i) = 0$ we have $p(\bar{E}_i) = 0$ and $\sum_{p \in E_i} \delta_p = 0$. Then E_i is a non-singular curve.

By the theorem 3.1 and corallary 3.2, we obtain: □

Corollary 3.3 *With the same notations above, we have:*
(i) $(E_i.E_j) = 0$ *or 1 if* $i \neq j$,

(ii) $E_i \cap E_j \cap E_k = \theta$ *if, i, j and k are three integres pairwise distincts,*

(iii) $E = \cup E_i, (1 \leq i \leq n)$ *doesn't contain any cycle.*

Proof. (i) If $(E_i \cap E_j) = \theta$ we have $(E_i.E_j) = 0$. If $(E_i \cap E_j) \neq \theta$ we have:

$$H^0(\mid E_i + E_j \mid, \mathcal{O}_{E_i+E_j}) \cong \mathbf{C}$$

and, by theorem 3.1, $H^1(E_i + E_j, \mathcal{O}_{E_i+E_j}) = 0$. So $p(E_i + E_j) = 0$. Moreover, by Riemann Roch theorem, we have:

$$p(E_i + E_j) = p(E_i) + p(E_j) + (E_i.E_j) - 1$$

Then we obtain $(E_i.E_j) = 1$.

(ii) Let E_i, E_j and E_k be three components of $\rho^{-1}(\xi)$ which are pairwise distincts. We have :

$$p(E_i + E_j + E_k) = p(E_i) + p(E_j + E_k) + (E_i.(E_j + E_k)) - 1$$

Assume $E_i \cap E_j \cap E_k \neq \theta$. This implies $(E_i.E_j) \neq 0$ and $(E_i, E_k) \neq 0$, which means, by (i) above, $(E_i.E_j) = (E_i.E_k) = 1$. Since $p(E_i) = 0$, and $E_i + E_j + E_k$ and $E_j + E_k$ are connected, by the proof of the corollary 3.2, we obtain:

$$p(E_i + E_j + E_k) = p(E_i) = p(E_j + E_k) = 0$$

Hence $(E_i.E_j) + (E_i.E_k) - 1 = 0$. This contradicts $(E_i.E_j) = 1$ and $(E_i.E_k) = 1$. So we have $E_i \cap E_j \cap E_k = \theta$.

(iii) Let $E_{i_1}, \ldots, E_{i_p}$ the irreducible components of $\rho^{-1}(\xi)$ pairwise distincts. Assume $(E_{i_j}.E_{i_{j+1}}) \neq 0$ and $(E_{i_1}.E_{i_p}) \neq 0$. By (i) above, $(E_{i_j}.E_{i_{j+1}}) = (E_{i1}.E_{i_p}) = 1$. Now consider

$$p(E_{i1} + \ldots + E_{i_p}) = p(E_{i_1}) + p(E_{i_2} + \ldots + E_{i_p}) + (E_{i_1}.(E_{i_2} + \ldots + E_{i_p})) - 1$$

Since $E_{i_1} + \ldots + E_{i_p}$ and $E_{i_2} + \ldots + E_{i_p}$ are connected, we obtain $p(E_{i1} + \ldots + E_{i_p}) = p(E_{i_1} = p(E_{i_2} + \ldots + E_{i_p}) = 0$. Hence $(E_{i_1}.E_{i_2} + \ldots + E_{i_p}) - 1 = 0$

Since $(E_{i_1}.E_{i_2}) = (E_{i_1}.E_{ip}) = 1$ and $(E_{i_1}.E_{ij}) \geq 0$ for $2 \leq j \leq p$, this is a contradiction. □

3.4 Let S be a normal surface having a singularity at ξ (not necessarily rational). Let $\mathcal{J}$ be the ideal of $\mathbf{C}\{x_1, \ldots, x_N\}$ which defines the surface S in a neighbourhood U in $\mathbf{C}^N$. Let $f \in \mathcal{J}$. We call *initial* form of f at ξ, noted by $ln_\xi f$, the homogeneous polynomial of lowest degree in the Taylor expansion of f at ξ. Let $\mathcal{J}$ be the ideal of $\mathbf{C}\{x_1, \ldots, x_N\}$ generated by the set $\{ln_\xi f \mid f \in \mathcal{J}\}$. *The tangent cone* $C_{S,\xi}$ of S at ξ is the 2-dimensional algebraic subvariety of $\mathbf{C}^N$ defined by the homogeneous ideal $\mathcal{L}$. We denote by $ProjC_{S,\xi}$ the projectivized curve in $\mathbf{P}^{N-1}$ associated with $C_{S,\xi}$. The set $Proj \mid C_{S,\xi} \mid = \mid ProjC_{S,\xi}$ is the set of lines of the tangent cone $C_{S,\xi}$. If $\sigma : \bar{S} \to S$ denotes the blowing up of S at ξ then we have $\sigma^{-1}(\xi) \cong ProjC_{S,\xi} \subset \mathbf{P}^{N-1}$ (see [18], theorem 5.8).

In what follows, we assume that S has a rational singularity at ξ. Then $\mathcal{M}\mathcal{O}_X$ is locally principal (see [1], theorem 4). Let $\sigma : \bar{S} \to S$ be the blowing up of S at ξ. The surface $\bar{S}$ is normal (see [16], theorem 1). If $\pi : X \to S$ denotes a resolution of S, by the universal property of blowing up, there exists a map $\bar{\pi} : X \to \bar{S}$ such that $\sigma \circ \bar{\pi} = \pi$. Let us denote by $E = \cup E_i$ and by $Z = \sum_{i=1}^{n} a_i E_i$ the fundamental cycle of π.

Definition 3.5 *A Tyurina component of E is a maximal connected set B of irreducible components of E such that $(Z.E_i) = 0$ for all irreducible components E_i in B.*

By [15] and [16], the Tyurina components have the following geometric interpretation: Consider the non-Tyurina components of E, *i.e.* the irreducible components E_i of E such that $(Z.E_i) < 0$. Suppose that the number of these components is equal to s. We have $s \leq n$. We may assume that $(Z.E_i) < 0$ for all $1 \leq i \leq s$. Now we consider the closure of the curve $E \setminus (E_i \cup \ldots \cup E_s)$; it is not necessarily connected. Denote by $B_1, \ldots, B_k$ its connected components. So each $B_j, (1 \leq j \leq k)$, is a Tyurina component of E. As in [15] (remark 3.2), we have:

Proposition 3.6 *An irreducible component E_i of E is contained in a Tyurina component of E if and only if $\bar{\pi}(E_i)$ is a point of $\bar{S}$.*

Proof. Let Z_1 be the positive cycle defined by $\mathcal{M}\mathcal{O}_{\bar{S}}$ where $\mathcal{M}$ is the maximal ideal of the local ring $\mathcal{O}_{S,\xi}$. We have $\mathcal{M}\mathcal{O}_X = \bar{\pi}^*(\mathcal{M}\mathcal{O}_{\bar{S}})$. Since $\bar{\pi}$ is a proper map, the projection formula (see [3], paragraph 2.6) says that:

$$(Z.E_i) = (Z_1.\bar{\pi}(E_i)).$$

It is clear that if $\bar{\pi}(E_i)$ is a point in $\bar{S}$ then we have $(Z.E_i) = 0$.

If $\bar{\pi}(E_i)$ is not a point, then it is an irreducible component of $\mid Z_1 \mid$ where $\mid Z_1 \mid$ is the reduced curve associated with Z_1. Let C_h be a generic hyperplane section of S at ξ defined by the equation $(h = 0)$ with $h \in \mathcal{M}/\mathcal{M}^2$ such that its strict transform h' by σ intersects $\mid Z_1 \mid$ transversely. Let h'' be the strict transform by π of C_h. The divisor of h in X can be written as $(\pi^* h) = Z + h''$. Since $((\pi^* h).E_i) = 0$ for all i, the projection formula gives:

$$(h''.E_i) = (h'.\bar{\pi}(E_i)).$$

Since $(h''.\bar{\pi}(E_i)) > 0$, we deduce $(Z.E_i) < 0$. This implies that E_i is a non-Tyurina component of E.

Then the normal surface $\bar{S}$ has k singularities each of which is obtained by the contraction of a Tyurina component $B_j, (1 \leq j \leq k)$, of E by $\bar{\pi}$ to a point of $\bar{S}$. Let us denote by $\xi_1, \ldots, \xi_k$ these singularities. Let V_j be a small neighbourhood of ξ_j in $\bar{S}$. We have: □

Corollary 3.7 *With the preceding notations, we have:*

(1) The restriction map $\bar{\pi} \mid_{\bar{\pi}-1(V_j)}$ is a resolution of the germ $(\bar{S}, \xi_j)$.

(2) If X is the minimal resolution of S then X is the minimal resolution of $\bar{S}$, and a Tyurina component B_j of E is the exceptional divisor $\bar{\pi}^{-1}(\xi_j)$ of the minimal resolution of $\bar{S}$ at ξ_j.

(3) A point of $\bar{S}$ which is not the contraction of a Tyurian component is a non-singular point of $\bar{S}$.

(4) The singular points of $\mid \sigma^{-1}(\xi) \mid$ are necessarily the intersection points of irreducible components of $\mid \sigma^{-1}(\xi) \mid$ (see corollary 3.2).

Moreover we have:

Proposition 3.8 *The singularities $\xi_1, \ldots, \xi_k$ of $\bar{S}$ are all rational.*

Proof. The contraction of each B_j gives a normal surface singularity (see [7], lemma 5.11 and [5], theorem 1). By the theorem 2.4, we have $p(Z_{B_j}) \geq 0$ where Z_{B_j} is the fundamental cycle associated with B_j. Moreover, theorem 3.1 implies $p(Z_{B_j}) \leq 0$. This gives the proposition. □

Proposition 3.9 *Suppose that ξ is a rational singularity and the multiplicity of S at ξ is m. Let E be the exceptional divisor of a resolution $\pi : X \to S$ of S. Then the number of the non-Tyurina components of E is less than or equal to m.*

Proof. Let $E_1, \ldots, E_n$ be the irreducible components of E. Assume that $(Z.E_i) < 0$ if and only if $i \in \{1, \ldots, s\}$. It is well known that $(Z.Z) = -m$ (see [1], corollary 6). If we denote $(Z.E_i) = -d_i$ where d_i is a positive integer for all $i, (1 \leq i \leq s)$, we obtain :

$$(Z.Z) = \sum_{i=1}^{n} a_i (Z.E_i) = -\sum_{i=1}^{s} d_i a_i = -m$$

By definition, we have $a_i \geq 1$ and $d_i \geq 1$; so $a_i d_i \geq 1$. Then $\sum_{i=1}^{s} a_i d_i \geq s$.

The non-Tyurina components of the exceptional divisor of a resolution of S correspond exactly to the strict transform by $\bar{\pi}$ of the components of $Proj \mid C_{S,\xi} \mid$. We will use this fact in the next section to associate the Tyurina components with the functions on S. □

Rational cycles

We will call *rational cycle* an element of $\mathcal{E}^+$. In this section, we will construct the elements of $\mathcal{E}^+$ by using the fundamental cycle Z (see [16] or [12]), and some special functions on S which correspond to these rational cycles. In order to do this, we will prove theorem 4.2 that we call Pinkham's theorem.

Let $\pi : X \to S$ be a resolution of S having a rational singularity at ξ. By [11], there is a one-to-one correspondence between the $\mathcal{M}$-primary complete ideals I in the local ring $\mathcal{O}_{S,\xi}$ such that $I\mathcal{O}_X$ is invertible and the rational cycles. In other words, there exists a rational cycle D on X and a $\mathcal{M}$-primary ideal I in $\mathcal{O}_{S,\xi}$ such that $I\mathcal{O}_X = \mathcal{O}(-D)$. In particular, we have $\mathcal{M}\mathcal{O}_X = \mathcal{O}(-Z)$ where Z is the smallest rational cycle of π.

Here we will speak on the functions on S rather than on the $\mathcal{M}$-primary complete ideals. In fact, the rational cycles correspond to the elements of these ideals in $\mathcal{O}_{S,\xi}$. By [1], a positive cycle D supported on E is an element of $\mathcal{E}^+$ if and onl if there exists a function f in $\mathcal{M}$ on S such that the compact part of the divisor in X corresponding to f is D *i.e.*$(\pi^* f) = D + f''$, where f'' is the strict transform by π of f. Since $(\pi^* f).E_i = 0$ for all irreducible components E_i of the exceptional divisor E of π, we obtain $(f''.E_i) = -(D.E_i)$. This means that D is a rational cycle since $(f''.E_i) \geq 0$. To understand the function on S which corresponds to the smallest rational cycle Z, we define:

Definition 4.1 *[9] A line of the tangent cone $\mathcal{C}_{S,\xi}$ is called exceptional tangent of S at ξ if it corresponds to a singular point of $\bar{S}$ or a singular point of $Proj \mid \mathcal{C}_{S,\xi} \mid$.*

Definition 4.2 *[14] A function f on S is called generic if it is defined by a non-singular function F defined in a neighbourhood of ξ in $\mathbf{C}^N$ so that the tangent hyperplan at ξ to he hypersurface $(F = 0)$ is not tangent to the tangent cone $C_{S,\xi}$ and doesn't contain any exceptional tangent of S at ξ.*

Let h be a function on S defined by a hyperplane H_1. Assume that h is a generic function on S. This implies that the zero locus of $h \circ \sigma$ intersects the components of $Proj \mid \mathcal{C}_{S,\xi} \mid$ transversely at the non-singular points of $Proj \mid \mathcal{C}_{S,\xi} \mid$ and of $\bar{S}$. By [4], the compact part of the total transform of such a function h by a resolution π of S is exactly the fundamental cycle of the resolution (*i.e.* we have $(\pi^* h) = Z + h''$ where h'' is the strict transform by π of h).

Now to costruct the elements of $\mathcal{E}^+$, choose a component E_{io} of E. Let $Z_0 = Z + E_{io}$. Consider the sequence of the positive cycles

$Z_0 = Z + E_{io}, \ldots, Z_{i+1} = Z_i + E_{m(i)}$ if there exists $m(i)$ such that $(Z_i.E_{m(i)}) > 0$ and $Z_{i+1} = Z_i$ otherwise. This proces is finite (see [8] or [12], proposition 1.2).

Lemma 4.3 *Let Y be a positive cycle supported on E such that there is a component E_i in Y for which $(Y.E_i) > 0$. Then there exists a cycle $Z_1 > Y$ such that Z_1 is in $\mathcal{E}^+$ and verify $Z_1 \geq Y + E_i$.*

Proof. Let $Z = \sum_{i=1}^n a_i E_i$ be the fundamental cycle and $Y + E_i = \sum_{i=1}^n m_i E_i$ a positive cycle supported on E. For all i, there exists $n_i \in \mathbf{N}^*$ such that $n_i a_i \geq m_i$. Let $s = sup_{i=1,\ldots,n} n_i$. We have then $sZ \geq Y + E_i$. Moreover $sZ \in \mathcal{E}^+$. So this gives the existence of a positive cycle Z_1 in $\mathcal{E}^+$ which verify $Z_1 \geq Y + E_i$.

Let us consider $A_i = \{Z_1 \in \mathcal{E}^+ \mid Z_1 \geq Z + E_i\}$. Denote $lnf A_i = \bar{Z}(E_i)$. By theorem 2.3, $\bar{Z}(E_i) \in \mathcal{E}^+$. By [16], if E_i is contained in a Tyurina component B_i of E, then $\bar{Z}(E_i) = Z + \Delta Z_i$ where ΔZ_i is a linear combination of the irreducible components of the Tyurina component B_i. In that case, H. Pinkham ([12], proposition of section 14) gives precisely ΔZ_i. We give this result in the following theorem: □

Theorem 4.4 *(Pinkham's theorem) Let E_i^j be an irreducible component of E which is contained in a Tyurina component B_j of E. Then the smallest $\bar{Z}(E_i^j)$ in $\mathcal{E}^+$ such that $\bar{Z}(E_i^j) \geq Z + E_i^j$ is equal to $Z + Z(B_j)$ where $Z(B_j)$ is the fundamental cycle associated with B_j.*

We shall speak later about ΔZ_i in the case where E_i is a non-Tyurina component. First we prove theorem 4.4. The aim of our proof is to see the relation between the Tyurina components of the exceptional divisor and the functions on the surface S.

Proof. Let f be a function in the maximal ideal $\mathcal{M}$ of $\mathcal{O}_{S,\chi}$. Let $(\pi^* f) = Y + f''$ where f'' is the strict transform by π of f where Y is the compact part of the divisor $(\pi^* f)$. Let us denote E_i^j an irreducible component E_i of E which belongs to a Tyurina component $B_j (1 \leq j \leq k)$. As in Section 2, let $A_i = \{D \mid D \geq Z + E_i^j\}$. Assume that Y is an element of A_i. Let us denote by Z_1 (resp. Y_1) the positive cycle supported on the $ProjC_{S,\xi}$ such that $\bar{\pi}^*(Z_1) = Z$ (resp. $\bar{\pi}^*(Y_1) = Y$) (see proof of proposition 3.6). W have $Y_1 \geq Z_1$. Since the case $Y_1 > Z_1$ will be an obvious consequence of the case $Y_1 = Z_1$, we assume that $Y_1 = Z_1$. The total transform by π of f can be written in the form $(\pi^* f) = (\bar{\pi}^* Z_1)$ where f' is the strict transform by σ of f. This give $(\pi^* f) = Z + F + f''$ where F is a positive cycle supported on the components of the divisor $\bar{\pi}^{-1}(f' \cap ProjC_{S,\xi})$. We notice that $Y = Z + F$.

Let ξ_j be the singular point of $\bar{S}$ obtained by the contraction of the Tyurina component B_j of E. If f' doesn't passe through E_j in $\bar{S}$ then B_j is not contained in F (see remark 3.7-(4)). So we exclude this case because Y is not contained in A_i If f' passe through ξ_j in $\bar{S}$ then B_j is contained in F. Then we can write $Y \geq Z + Z(B_j)$. (Notice that, if $Y_1 > Z_1$ above, we have $Y \geq Z + Z(B_j)$). Now in order to prove $lnfA_i = Z + Z(B_j)$, we will show that $Z + Z(B_j)$ is an element of A_i. This is equivalent to show that $Z + Z(B_j)$ is a rational cycle. So it is sufficient to prove $(Z + Z(B_j)).E_i \leq 0$ for all irreducible components E_i of $E, (1 \leq i \leq n)$). We prove it in the following two cases:

(1) If E_i is contained in a Tyurina component of E, we have $(Z.E_i) = 0$ and $(Z(B_j).E_i) \leq 0$, so $(Z + Z(B_j)).E_i \leq 0$.

(2) If E_i is a non-Tyurina component of E we have two cases: If $E_i \cap B_j = \theta$, we have $(Z(B_j).E_i) = 0$; this gives $(Z + Z(B_j)).E_i \leq 0$ since $(Z.E_i) < 0$. If $E_i \cap B_j \neq \theta$, we have $(Z(B_j).E_i) > 0$. This gives $(Z(B_j).E_i) = a^j_m$ where a^i_m is the multiplicity of the component E^j_m in $Z(B_j)$ attached to E_i. By [10], theorem 4.6), this multiplicity is equal to one. Then we obtain $(Z + Z(B_j)).E_i \leq 0$.

Therefore $Z + Z(B_j)$ is a rational cycle. Since $Z + Z(B_j) \geq Z + E^j_i$, it is an element of A_i. By definition this gives $lnfA_i = (Z + Z(B_j))$.

Hence a rational cycle D is an element of A_i if and only if there exists a function g in $\mathcal{M}$ such that $(\pi^* g) = D + g''$ where g'' is the strict transform by π of g and the strict transform g' by σ in $\bar{S}$ of g passe through the singular point ξ_j.

In particular, D is the smallest element of A_i if and only if the strict transform g' by σ of g intersects $Proj \mid C_{S,\xi} \mid$ at the nonsingular points of $Proj \mid C_{S,\xi} \mid$ and of $\bar{S}$ except ξ_1 and the branch of g' passing through ξ_j is a generic function on $\bar{S}$. □

Remark 4.5 *If E^j_l and E^j_m are two irreducibles components in the Tyurina component*

B_j of E then we have $\bar{Z}(E_l^j) = \bar{Z}(E_m^j)$.

4.6 Now let us consider the set $A_i = \{Z_1 \in \mathcal{E}^+ \mid Z_1 \geq Z + E_i\}$ when E_i is a non-Tyurina component of E. In order to give precisely the smallest cycle $\bar{Z}(E_i) = Z + \Delta Z_i$ of this set, we introduce some notations:

Let us denote by B a Tyurina component of E and by E_1 an irreducible component of E which is contained in B. Let $B^0 = B$ and let $Z(B^0)$ be the fundamental cycle of B^0. If $(Z(B^0).E_1) < 0$ then we put $B^l = B^0$ for $l \geq 1$. If $(Z(B^0).E_1) = 0$ then we denote by B^1 the $(Z(B^0).E_1) < 0$ then we put $B^l = B^0$ for $B^l = B^0$ for $l \geq 1$. If $(Z(B^0).E_1) = 0$ then we denote by B^1 the Tyurina component of $Z(B^0)$ which contains E_1. We have $B^0 \supset B^1$. By induction, we define the sequence $B^0 = B, B^1, \ldots, B^p$ such that B^l is a Tyurina component of $B^{l-1}E_1$, is contained in B^l and E_1 is a non-Tyurina component of $Z(B^p), 1 \leq l \leq p$. As in [10], we define:

Definition 4.7 *[10] We call $B^0 = B, B^1, \ldots, B^p$ desingularization sequence of E_1 and p desingularization depth of E_1.*

Theorem 4.8 *Let E be a non-Tyurina component of E. Let us denote by $B_1, \ldots, B_q$ the Tyurina components of E attached to E and by $F_1, \ldots, F_q$ the irreducible components of $B_1, \ldots, B_q$ respectively such that $(F_t \cap E) \neq \theta$ for all $t, (1 \leq t \leq q)$. Then the smallest cycle $\bar{Z}(E)$ in $\mathcal{E}^+$ which is greater than $Z + E$, is*

$$\bar{Z}(E) = Z + E + \sum_{t=1}^{q}(\sum_{t=0}^{p} Z(B_t^l))$$

Here $l = 0, \ldots, p$ is the desingularization depth of $F+$ in $B_t, (1 \leq t \leq q)$.

In particular, if E is not attached to any Tyurina component then we have $\bar{Z}(E) = Z + E$.

This result has been proved during the proofs and it will appear in a forthcoming paper.

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