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SOME THEOREMS INVOLVING INEQUALITIES ON P-VALENT FUNCTIONS

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Abstract

In this paper, some theorems involving inequalities on p-valent functions (that is p-valenty close-to-convex functions, p-valently starlike functions, and p-valently convex functions) are given. Moreover, some applications in the theorems which are important for geometric function theory are also included.

Keywords: Analytic, p-valent, p-valently colse-to-convex functions, p-valently starlike functions, p-valently convex functions, open unit disk, and Jack's Lemma.

1. Introduction and Definitions

Let $T(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in N = \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $U = \{z : z \in \mathcal{C} \text{ and } \mid z \mid < 1\}$. A function $f(z) \in T(p)$ is said to be in the subclass $TK(p)$ of p-valently close-to-convex functions with respect to the origin in U if it satisfies the inequality (cf.[1-3]):

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > 0, \ \ (z \in U; p \in N). \tag{2}$$

On the other hand, a function $f(z) \in T(p)$ is said to be in the subclass $TS(p)$ of p-valently starlike functions with respect to the origin in U if it satisfies the inequality (cf.[1-3]):

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in U; p \in N). \tag{3}$$

Furthermore, a function $f(z) \in T(p)$ is said to be in the subclass $TC(p)$ of p-valently convex functions with respect to the origin in U if it satisfies the inequality (cf.[1-3]):

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in U; p \in N). \tag{4}$$

A function $f(z) \in T(p)$ is said to be in the subclass $V_q(p)$ if it satisfies the inequality:

$$\left|\frac{(p-q)!}{p!}.\frac{D_z^q f(z)}{z^{p-q}} - 1\right| < 1, \tag{5}$$

$$(z \in U; p > q; p \in N; q \in N_0 = N \cup \{0\}),$$

and, a function $f(z) \in T(p)$ is said to be in the subclass $W_q(p)$ if it satisfies the inequality:

$$\left|\frac{zD_z^{q+1} f(z)}{D_z^q f(z)} - (p-q)\right| < p - q, \tag{6}$$

$$(z \in U; p > q; p \in N; q \in N_0).$$

Here and throughout this paper, D_z^q denotes the q th-order ordinary differential operator. For a function $f(z) \in T(p)$,

$$D_z^q f(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q}, \quad (p > q; p \in N; q \in N_0). \tag{7}$$

To establish our results, we need the following Lemma given by Jack [4] (also, by Miller and Mocanu [5]).

Lemma *Let $w(z)$ be analytic in U with $w(0) = 0$. If $\mid w(z) \mid$ attains its maximum value on the circle $\mid z \mid = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = c w(z_0), \tag{8}$$

where c is a real number and $c \geq 1$.

2. Some Theorems Involving Inequalities on P-valent Functions

We first prove the following theorem.

Theorem 1. *If $f(z) \in T(p)$ satisfies the inequality*

$$Re\left\{q + \frac{zD_z^{q+1}f(z)}{D_z^q f(z)} - p\right\} < \frac{1}{2}, \quad (z \in U; p > q; p \in N; q \in N_0), \tag{9}$$

then $f(z) \in V_q(p)$.

Proof. Let the function $f(z) \in T(p)$. Then defining the function $w(z)$ by

$$\frac{(p-q)!}{p!} . \frac{D_z^q f(z)}{z^{p-q}} = 1 + w(z), \quad (z \in U; p > q; p \in N; q \in N_0), \tag{10}$$

we have that $w(z)$ is analytic in U and $w(0) = 0$. It follows from (10) that

$$\frac{(p-q)!}{p!} . \frac{zD_z^{q+1} f(z)}{z^{p-q}} = zw'(z) + (p-q)[1 + w(z)]. \tag{11}$$

Then, we have from (10) and (11) that

$$F(z) = q + \frac{zD_z^{q+1}f(z)}{D_z^q f(z)} - p = \frac{zw'(z)}{1 + w(z)}. \tag{12}$$

Now, suppose that there exists a point $z_0 \in U$ such that

$$max_{|z| \leq |z_0|} \mid w(z) \mid = \mid w(z_0) \mid = 1. \tag{13}$$

Then, using the Lemma and letting $w(z_0) = e^{i\theta} (w(z_0) \neq -1)$ in the equation (12), we have

$$Re\{F(z_0)\} = Re\left\{\frac{z_0 w'(z_0)}{1+w(z_0)}\right\} = Re\left\{\frac{cw(z_0)}{1+w(z_0)}\right\}$$

$$= cRe\left\{\frac{e^{i\theta}}{1+e^{i\theta}}\right\} = \frac{c}{2} \geq \frac{1}{2}, \tag{14}$$

which contradicts the hypothesis (9). Therefore, we conclude that $\mid w(z) \mid< 1$ for all $z \in U$, and the definition (10) immediately yields the inequality

$$\left|\frac{(p-q)!}{p!}.\frac{D_z^q f(z)}{z^{p-q}} - 1\right| < 1. \quad (z \in U; p > q; p \in N; q \in N_0), \tag{15}$$

that is, that $f(z) \in V_q(p)$. Thus, the proof is completed.

Next we prove the following theorem. □

Theorem 2. *If $f(z) \in T(p)$ satisfies the inequality*

$$Re\left\{1 + z\left(\frac{D_z^{q+2} f(z)}{D_z^{q+1} f(z)} - \frac{D_z^{q+1} f(z)}{D_z^q f(z)}\right)\right\} < \frac{1}{2}, \quad (z \in U; p > q; p \in N; q \in N_0), \tag{16}$$

then $f(z) \in W_q(p)$.

Proof. Let the function $f(z) \in T(p)$. Now consider the function $w(z)$ defined by

$$\frac{zD_z^{q+1} f(z)}{D_z^q f(z)} = (p-q)[1 + w(z)], \quad (z \in U; p > q; p \in N; q \in N_0). \tag{17}$$

Cleary the function $w(z)$ is analytic in U and $w(0) = 0$. It follows from the definition of $w(z)$ that

$$1 + \frac{zD_z^{q+2} f(z)}{D_z^{q+1} f(z)} = (p-q)\left(1 + w(z) + zw'(z)\frac{D_z^q f(z)}{D_z^{q+1} f(z)}\right)$$

$$= (p-q)[1 + w(z)] + \frac{zw'(z)}{1+w(z)}. \tag{18}$$

Now suppose that there exists a point $z_0 \in U$ such that

$$max_{|z| \leq |z_0|} \mid w(z) \mid = \mid w(z_0) \mid = 1. \tag{19}$$

Then the Lemma gives $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = cw(z_0)(c \geq 1)$. Therefore, we obtain from (17)-(20) that

$$Re\left\{1 + z\left(\frac{D_z^{q+2} f(z)}{D_z^{q+1} f(z)} - \frac{D_z^{q+1} f(z)}{D_z^{q} f(z)}\right)|_{z=z_0}\right\} = Re\left\{\frac{z_0 w'(z_0)}{1 + w(z_0)}\right\}$$
$$= Re\left\{\frac{cw(z_0)}{1 + w(z_0)}\right\} = cRe\left\{\frac{e^{i\theta}}{1 + e^{i\theta}}\right\} = \frac{c}{2} \geq \frac{1}{2}, \tag{20}$$

which contradicts the condition (16). Hence, we conclude that $\mid w(z) \mid < 1$ for all $z \in U$, that is, that $f(z) \in W_q(p)$. This completes the proof of Theorem 2.

By taking $q = 0$ in Theorems 1 and 2, we have the following corollaries. □

Corollary 1. *If $f(z) \in T(p)$ satisfies the inequality*

$$Re\left\{\frac{zf'(z)}{f(z)} - p\right\} < \frac{1}{2}, \quad (z \in U; p \in N), \tag{21}$$

then

$$\left|\frac{f(z)}{z^p} - 1\right| < 1, \ (z \in U; p \in N). \tag{22}$$

Corollary 2. *If $f(z) \in T(p)$ satisfies the inequality*

$$Re\left\{1 + z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right\} < \frac{1}{2}, \ (z \in U; p \in N), \tag{23}$$

then $f(z) \in TS(p)$ and

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p, \ (z \in U; p \in N). \tag{24}$$

By taking $q = 1$ in Theorems 1 and 2, we have the following.

Corollary 3. *If $f(z) \in T(p)$ satisfies the inequality*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}-p\right\}<\frac{1}{2}, \quad (z \in U; p \in N), \tag{25}$$

then $f(z) \in TK(p)$ and

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|<p, \quad (z \in U; p \in N). \tag{26}$$

Corollary 4. *If $f(z) \in T(p)$ satisfies the inequality*

$$Re\left\{1+z\left(\frac{f'''(z)}{f''(z)}-\frac{f''(z)}{f'(z)}\right)\right\}<\frac{1}{2}, \quad (z \in U; p \in N), \tag{27}$$

then $f(z) \in TC(p)$ and

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|<p-1, \quad (z \in U; p \in N-\{1\}). \tag{28}$$

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