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A CLASS OF BANACH ALGEBRAS WHOSE DUALS HAVE THE SCHUR PROPERTY

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Abstract

Call a commutative Banach algebra A a γ-algebra if it contains a bounded group Γ such that $\overline{aco(\Gamma)}$ contains a multiple of the unit ball of A. In this paper, first by exhibiting several concrete examples, we show that the class of γ-algebras is quite rich. Then, for a γ-algebra A, we prove that $A^\star$ has the Schur property iff the Gelfand spectrum $\sum$ of A is scattered iff $A^\star = ap(A)$ iff $A^\star = \overline{Span(\sum)}$.
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1. Introduction

Let A be a commutative Banach algebra. We call A, for want of a better name, a γ-algebra if A contains a bounded group Γ (for multiplication) such that $\overline{aco(\Gamma)} \supseteq cA_1$, for some constant $c > 0$. Here $\overline{aco(\Gamma)}$ is the absolute convex hull of Γ and A_1 is the closed unit ball of A. The example that motivated us to introduce this class of Banach algebras is the algebra $A = C(K)$ of the continuous complex valued functions on a compact (Hausdorff) space K. For this algebra, the set $\Gamma = \{e^{ig} : g \epsilon C_R(K)\}$ is a bounded group for the multiplication and, by Russo-Day-Palmer Theorem $[B - D; p.208], \overline{aco(\Gamma)} = A_1$. In section 2 we give several examples of γ-algebras and study stability properties of these algebras under some Banach algebras construction procedures. In particular we show that the quotient algebras of Segal algebras [*Re*] by closed ideals with compact hulls are γ-algebras, and that the projective tensor product $A\hat{\otimes}B$ of two γ-algebras A and B is again a γ-algebra. In section 3 we study structural and geometric properties of γ-algebras. The main results are the following. We first prove that, in spite of the diversity of the examples, the only γ-algebras are the quotient algebras of the group algebra $\ell^1(\Gamma)$ for a discrete abelian group Γ. This result reduces the study of the γ-algebras to that of the quotient algebras of the discrete group algebra $\ell^1(\Gamma)$. The second main result establishes the following equivalences for a γ-algebra A with the spectrum $\sum$: The space $A^\star$ has the Schur property (i.e. weakly convergent sequences in $A^\star$ are norm convergent) iff $A^\star = ap(A)$ iff $\overline{Span(\sum)} = A^\star$ iff $\sum$ is scattered (i.e. it has no nonempty perfect subset). These results generalize and unify several known results or parts of them e.g. [P-S; Main Theorem], [D-Ü; Proposition 3.5], [L-P1;Theorem 1]. The main ingredients of the proofs are a result of Loomis [Lo; Theorem 4.] stating that a continuous function on an locally compact abelian group with scattered spectrum is almost periodic, and Theorem 2.14 of Lust-Piquard [L-P 2].

Notation and Preliminaries.

Our rotation and terminologies are quite standard. If X is a Banach space, we denote by $X^\star$ its dual and by X_1 its closed unit ball. For x in X and f in $X^\star$, we denote by $< x, f >$ the natural duality between X and $X^\star$. We always consider X as naturally embedded into its second dual $X^{\star\star}$. For any subset E of $X^\star$, we denote by $\overline{Span(E)}$ and $\overline{Span(E)^\star}$, respectively, the norm and weak* closures of $Span(E)$ in $X^\star$.

Now let A be a Banach algebra. For f in $A^\star$ and a in A, we denote by $f.a$ the functional defined on A by $< f.a, x >=< f, ax >$. The functional f is said to be almost periodic on A if the set $H(f) = \{f.a : a \in A_1\}$ is relatively compact in $A^\star$. By $ap(A)$ we denote the space of the almost periodic functionals on A. This is a closed subspace of $A^\star$. Now let Γ be a discrete abelian group, G be its dual group, and $\ell^1(\Gamma)$ be its group algebra. For a in $\ell^1(\Gamma)$, we denote by $\hat{a}$ the Fourier transform of a. We consider each element f of G as a multiplicative linear functional on $\ell^1(\Gamma)$ acting through the formula $< a, f >= \hat{a}(f)$. For a subset E of G, we denote by $\overline{Span(E)}$ the norm closed subspace of $\ell^\infty(\Gamma)$ generated by E. By $AP(\Gamma)$ we denote the space of the almost periodic functions on Γ. As is well known, $AP(\Gamma) = \overline{Span(G)} = ap(\ell^1(\Gamma))$. For φ in $\ell^1(\Gamma)$, the spectrum of φ is denoted by $\sigma(\varphi)$. This is the set

$$\sigma(\varphi) = G \cap \overline{\{\varphi.a : a \in l^1(\Gamma)\}^{\star}}. \qquad [K; p.170]$$

For a closed subset E of G, by $k(E)$ we denote the kernel of E. This is largest closed ideal of $\ell^1(\Gamma)$ whose hull is E. Finally, by $k(E)^{\perp}$, we denote the annihilator of $k(E)$ in $\ell^{\infty}(\Gamma)$.

Examples of γ-Algebras.

As introduced in the preceding section, by a γ-algebra we mean a commutative Banach algebra A that contains a bounded group Γ such that $\overline{aco(\Gamma)} \supseteq cA_1$, for some constant $c > 0$. Such an algebra necessarily has a unit element. Beside the commutative $C^{\star}$-algebras, several well known Banach algebras are γ algebras. In this section we give several examples of γ-algebras and study some stability properties of them.

a) Group Algebra. Let G be a locally compact abelian group and $L^1(G)$ be its group algebra.

Proposition 2.1 *The algebra $L^1(G)$ is a γ-algebra iff the group G is discrete.*

Proof. Since a γ-algebra is unital, if $L^1(G)$ is a γ-algebra then G is discrete. Conversely, if G is dicrete then the Dirac measures $\delta_g (g \in G)$ are in $L^1(G)$ and $\Gamma = \{\delta_g : g \in G\}$ is a bounded group in $L^1(G)(\delta_f \star \delta_g = \delta_{f+g})$, and the closed absolute convex hull of Γ is the closed unit ball of $\ell^1(G) = L^1(G)$. □

As an immediate corollary of this proposition we have following result.

Corollary 2.2. *The Fourier algebra $A(G)$ is a γ-algebra iff the group G is compact.*

b) Segal Algebra. Let G be a locally compact abelian group. We recall that a Segal algebra $S(G)$ on G is a Banach algebra such that

i) $S(G)$ is a translation invariant dense subalgebra of $L^1(G)$,

ii) For $a \in S(G)$ and $g \in G, \| a_g \| = \| a \|$, where a_g is defined by $a_g(f) = a(f+g)$, and $\| \cdot \|$ is the norm of $S(G)$,

iii) For each $a \in S(G)$, the mapping $g \mapsto a_g$ is continuous from G into $S(G)$ at the unit element of G.

About Segal algebras, ample information can be found in Reiter's book [Re]. We now give some concrete examples of Segal algebras.

1) The algebra $S(G) = L^1(G) \cap L^p(G)(1 \leq p < \infty)$, equipped with the norm $\| a \| = \| a \|_1 + \| a \|_p$, is a Segal algebra.

2) The algebra $S(G) = L^1(G) \cap C_o(G)$, equipped with the norm $\| a \| = \| a \|_1 + \| a \|_\infty$, is a Segal algebra.

A Segal algebra $S(G)$ is a commutative semisimple regular Banach algebra whose Gelfand spectrum is $\hat{G}$, the dual group of G. For $a \in S(G)$, the Gelfand transform of a is just the Fourier transform of $a \in L^1(G)$. From these facts it follows that, for a closed subset K of $\hat{G}, I_K = \{a \in S(G) : \hat{a}(K) = 0\}$ is the largest, and the set $J_K = \overline{\{a \in S(G) : Supp(\hat{a}) \cap K = \theta\}}$ is the smallest, closed ideals of $S(G)$ whose hull is K. We shall also need the following two facts about Segal algebras [Re; p.128]:

($\star$) There is a constant $c > 0$ such that, for $a \in S(G), \| a \|_1 \leq c \| a \|$.

($\star\star$) $S(G)$ is an ideal in $L^1(G)$ and, for $a \in S(G)$ and $b \in L^1(G), \| a \star b \| \leq \| a \| . \| b \|_1$.

Proposition 2.3. *Let I be a closed ideal of a Segal algebra $S(G)$ with compact hull. Then the quotient algebra $S(G)/I$ is a γ-algebra.*

Proof. Let K be the hull of I. Choose a function $u \in S(G)$ such that $\hat{u} = 1$ on a neighborhood of K. Then, the element $u + I$ is the unit element of the algebra $A = S(G)/I$. Put $\Gamma = \{u_g + I : g \in G\}$. As $u_f \star u_g = u_{f+g}$ and $\| u_g + I \| = \| u + I \| \leq \| u \|, \Gamma$ is a bounded group in A. To prove that $\overline{aco(\Gamma)}$ contains a multiple of A_1, it is enough to show that, for $\varphi \in I^\perp, \| \varphi \| \leq c Sup_{g \in G} | < \varphi, u_g > |$, for some positive constant c. To this end, observe that, since for each $a \in S(G), a \star u - a \in J_K \subseteq I$, we have $< \varphi, a > = < \varphi, u \star a >$. From this equality, by ($\star\star$), we get that $\varphi = \varphi \star u$ and that

$$| < \varphi, a > | \leq \| \varphi \| \cdot \| u \star a \| \leq \| \varphi \| \cdot \| u \| \cdot \| a \|_1 .$$

This inequality shows that φ is bounded on $S(G)$ for the norm of $L^1(G)$. Hence, $S(G)$ being dense in $L^1(G)$, φ can be extended in a unique way to $L^1(G)$. We denote by $\overline{\varphi}$

this unique extension of φ. Since, by $(\star)$, $l^{\infty}(G)$ embeds continuously into $S(G)^{\star}$, there is a constant $c > 0$ such that $\| \varphi \| \leq c \| \overline{\varphi} \|_{\infty}$. Now, since $\varphi = \varphi \star u, \varphi$ is a continuous function on G so that

$$\| \overline{\varphi} \|_{\infty} = \| \varphi \|_{\infty} = Sup_{g \in G} |(\varphi \star u)(-g)| = Sup_{g \in G} | < \varphi, u_g > |$$

From this we conclude that $\| \varphi \| \leq cSup_{g \in G} | < \varphi, u_g > |$, and so $\overline{aco(\Gamma)} \supseteq cA_1$. □

This proposition has several immediate corollaries.

Corollary 2.4. *Let I be a closed ideal of $L^1(G)$ with compact hull. Then the algebra $L^1(G)/I$ is a γ-algebra.* □

Corollary 2.5. *Let $A(G)$ be the Fourier algebra of G, K be a compact subset of G, and I_K be the largest closed ideal of $A(G)$ whose hull is K. Then the algebra $A(K) = A(G)/I_K$ is a γ-algebra.* □

c) Projective tensor products and quotients of γ-algebras.

If A and B are two commutative Banach algebras then their projective tensor product $A\hat{\otimes}B$ is a commutative Banach algebra for the multiplication which is the linear extension of the multiplication $a \otimes b.c \otimes d = ac \otimes bd$ on the simple tensors [B-D; Chapter VI].

Proposition 2.6 The projective tensor product $A\hat{\otimes}B$ of two γ-algebras A and B is also a γ-algebra.

Proof. Let $\Gamma_1 \subseteq A$ $(\Gamma_2 \subseteq B)$ be a bounded group such that $\overline{aco(\Gamma_1)} \supseteq c_1 A_1$ $(\overline{aco(\Gamma_2)} \supseteq c_2 B_1)$ for some constant $c_1 > 0 (c > 0)$. Then $\Gamma = \{a \otimes b : a \in \Gamma_1, b \in \Gamma_2\}$ is a bounded group in $A\hat{\otimes}B$, and by the very definition of the projective tensor norm, $\overline{aco(\Gamma)}$ contains $c(A\hat{\otimes}B)_1$, where $c = min\{c_1, c_2\}$. □

Proposition 2.7. Let A be a γ-algebra and B be an arbitrary Banach algebra. If there is a continuous onto homomorphism $h : A \to B$, then the algebra B is also a γ-algebra.

Proof. Let Γ_1 be a bounded group in A such that $\overline{aco(\Gamma_1)} \supseteq c_1 A_1$, for some constant $c_1 > 0$. As h is onto, by the Open Mapping Theorem, there is a constant $c_2 > 0$ such that $h(A_1) \supseteq c_2 B_1$. Let $\Gamma_2 = h(\Gamma_1)$. Then Γ_2 is a bounded group in B and $\overline{aco(\Gamma_2)} \supseteq h\overline{(aco(\Gamma_1))} \supseteq c_1 h(A_1) \supseteq c_1 h(A_1) \supseteq c_1 c_2 B_1$. This proves that B is also a γ-algebra. □

3. Structure of γ-Algebras.

The above results show that the class of γ-algebras is quite rich and the algebras constituting it are quite various. The first main result of this chapter shows that, in spite of the diversity of γ-algebras, γ-algebras are exactly the quotient algebras of the discrete group algebra $\ell^1(\Gamma)$ for some discrete group Γ.

Theorem 3.1. *A commutative Banach algebra A is a γ-algebras iff it is isomorphic to a quotient algebra of the group algebra $\ell^1(\Gamma)$ for some discrete group Γ.*

Proof. Let Γ be a bounded group in A such that $\overline{aco(\Gamma)} \supseteq cA_1$, for some constant $c > 0$. Consider the mapping $\omega : \ell^1(\Gamma) \to A$, defined by $\omega(a) = \sum_{\gamma \epsilon \Gamma} a(\gamma)\gamma^{-1}$. This is not really the Fourier transform but exactly as in the case of the Fourier transform one can easily see that ω is an algebra homomorphism. The group Γ being bounded, ω is bounded. Moreover, for $a = \delta_{\gamma^{-1}}$ (Dirac measure at γ^{-1}), $\omega(a) = \gamma$ so that $\omega(\ell^1(\Gamma)_1) \supseteq aco(\Gamma)$. It follows that $\overline{\omega(\ell^1(\Gamma)_1)} \supseteq cA_1$. From this inclusion, as in the most standard proof of the open mapping theorem (see e.g. [H-S; p.215]), we deduce that ω is onto. If follows that A is isomorphic to a quotient algebra of the algebra $\ell^1(\Gamma)$. For the reverse implication it is enough to apply Propositions 2.1 and 2.7. □

For the proof of the second main result of this section we need the following three lemmas

Lemma 3.2. *Let Γ be a discrete abelian group and E be a closed subset of the dual group G of Γ. Then the following assertions are equivalent.*

a) The set E is scattered.

b) $k(E)^\perp \subseteq AP(\Gamma)$.

c) $k(E)^{\perp} = \overline{Span(E)}$.

Proof. a)→ b). Assume that the set E is scattered. Let φ be an element of $k(E)^{\perp}$. We have to show that the function φ is almost periodic on Γ. Let us first see that the spectrum of φ is contained in E. To see this, let f be an element in $\sigma(\varphi)$. Then, by definition of $\sigma(\varphi), f = weak^{\star} - lim\varphi.a_{\alpha}$ for some net $(a_{\alpha})_{\alpha\epsilon J}$ in A. If $f \notin E$, then by regularity of the algebra $\ell^1(\Gamma)$, there exists an element a in $\ell^1(\Gamma)$ such that $< f, a >\neq 0$ and $\hat{a}(E) = 0$. Hence a is in the ideal $k(E)$ and, since $< f, a >\neq 0$, $f.a =< f, a > f \neq 0$. As $k(E)$ is an ideal, for any $b \in \ell^1(\Gamma), a \star b \in k(E)$ so that $< \varphi, a \star b >=< \varphi.a, b >= 0$. This implies that $\varphi.a = 0$. However, as $f.a = weak^{\star} - lim\varphi.a_{\alpha}.a = 0$, this contradicts the fact that $f.a \neq 0$. Hence $\sigma(\varphi) \subseteq E$, and the set E being scattered, by Loomis' Theorem [Lo; Theorem 4], φ is almost periodic on Γ. This proves the inclusion $k(E)^{\perp} \subseteq AP(\Gamma)$.

b)→ c). Assume b) holds. The inclusion $\overline{Span(E)} \subseteq k(E)^{\perp}$ being clear, we prove the reverse inclusion only. To prove this, let φ be an element of $k(E)^{\perp}$. As $\sigma(\varphi) \subseteq E$, and φ is almost periodic on $\Gamma, \varphi \in \overline{Span(E)}$, see e.g. [B; p. 110, Theorem 2.2.3]. This proves the inclusion $k(E)^{\perp} \subseteq \overline{Span(E)}$.

c)→ a). To prove this implication, the implication c)→ b) being clear, we shall prove that b)→ a). So assume that b) holds. To get a contradiction assume that the set E is not scattered. Then E contains a nonempty perfect set K. The set K being perfect, by Theorem 10 in [La; p.52], there exists a nontrivial regular continuous Borel measure μ on G supported by K. Let $\varphi : \Gamma \to C, \varphi(\lambda) = \int_G \overline{< f, \lambda >} d\mu(f)$, be the Fourier-Stieltjes transform of μ. It is clear that the function φ is bounded on Γ so that it is in $\ell^{\infty}(\Gamma)$. For $a \in K(E)$,

$$< \varphi, a >= \sum_{\lambda\in\Gamma} \varphi(\lambda)a(\lambda) = \int_G \sum_{\lambda\in\Gamma} a(\lambda)\overline{< f, \lambda >} d\mu(f) =$$

$$\int_G \hat{a}(f)d\mu(f) = 0$$

since the support of μ is contained in E. This shows that $\varphi \in k(E)^{\perp}$. Hence by b), $\varphi \in AP(\Gamma)$. Let m be the (unique) invariant mean on $AP(\Gamma)$ [Bu; p.15, Corollory 1.26]. Then $m(\varphi) = \mu(e)$, where e is the unit element of the group G. Define now a new Borel measure by $\tilde{\mu}(B) = \overline{\mu(B^{-1})}$ on the Borel σ-algebra of G. Then, as in [Bu; p.73, Corollary 4.13], we have

$$m(|\varphi|^2) = (\mu \star \tilde{\mu}(e) = \sum_{f \in G} |\mu(f)|^2 = 0$$

since μ is continuous. However this is not possible unless φ is identically zero [Bu; p.15, Corollary 1.23], which is not the case since μ is not trivial. This contradiction completes the proof. □

Lemma 3.3. *Let A be an arbitrary Banach algebra. If the space $A^\star$ has the Schur property, then $ap(A) = A^\star$.*

Proof. Suppose that $A^\star$ has the Schur property. Let f be an element of $A^\star$. We have to show that the set $H(f) = \{f.a : a \in A_1\}$ is relatively compact in $A^\star$. This is clearly equivalent to showing that the operator $T : A \to A^\star$, defined by $T(a) = f.a$, is compact. To prove this, let $(a_n)_{n \in N}$ be a sequence in A_1. Since $A^\star$ has the Schur property, A does not contain an isomorphic copy of ℓ^1[D; Theorem 3]. Hence, by Rosenthal's ℓ^1-Theorem [R], the sequence $(a_n)_{n \in N}$ has a subsequence, denoted again $(a_n)_{n \in N}$, which is weakly Cauchly. It follows that the sequence $(T(a_n))_{n \in N}$ is weakly Cauchy in $A^\star$. As $A^\star$ has the Schur property, it is weakly sequentially complete. Consequently the sequence $(T(a_n))_{n \in N}$ converges weakly, so in norm in $A^\star$. This proves that each f in $A^\star$ is almost periodic on A so that $A^\star = ap(A)$. □

For the proof of the next lemma we refer the reader to [D-Ü; Corollary 4.3].

Lemma 3.4. *Let A be any Banach algebra and I be a closed ideal of it. Then $ap(A/I) = ap(A) \cap I^\perp$.*

The second main result of this section is the following result.

Theorem 3.5. *Let A be a γ-algebra, and let $\sum$ be its Gelfand spectrum. Then the following assertions are equivalent.*

a) The space $\sum$ *is scattered.*

b) The space $A^\star$ *has the Schur property.*

c) $ap(A) = A^\star$.

d)$\overline{Span(\sum)} = A^\star$.

Proof. Since A is a γ-algebra, by Theorem 3.1, A is isomorphic to the quotient algebra $\ell^1(\Gamma)/I$ for some discrete abelian group Γ and a closed ideal I of $\ell^1(\Gamma)$. So we can assume that $A = \ell^1(\Gamma)/I$.

a)→ b). Assume that $\sum$ is scattered. This means that the hull of I is scattered. Then by [L-P1;Theorem 1] or [L-P2;Theorem 2.14], $A^\star$ has the Schur property.

b)→ c). This implication follows from Lemma 3.3 above.

c)→ d). Assume that $ap(A) = A^\star$. Since $A = \ell^1(\Gamma)/I$ and, by Lemma 3.4, $ap(\ell^1(\Gamma)/I) = ap(\ell^1(\Gamma)) \cap I^\perp = AP(\Gamma) \cap I^\perp = A^\star = I^\perp$, we conclude that $I^\perp \subseteq AP(\Gamma)$. Let E be the hull of I. As $I \subseteq k(E), k(E)^\perp \subseteq I^\perp \subseteq AP(\Gamma)$. Hence, by Lemma 3.2, E is scattered, so a set of synthesis, and $I = k(E)$. Hence, by Lemma 3.2 again, $I^\perp = k(E)^\perp = \overline{Span(E)}$. This proves that $\overline{Span(\sum)} = A^\star$.

d)→ a). Assume that the equality $\overline{Span(\sum)} = A^\star$ holds. As always one has the inclusion $\overline{Span(\sum)} \subseteq ap(A)$, we conclude that $ap(A) = A^\star$. This implies that $I^\perp \subseteq AP(\Gamma)$. From this, as above, we conclude that $\sum$ is scattered. □

Corollary 3.6. *Let X be a weak*- closed subspace of $\ell^\infty(\Gamma)$ which is also an $\ell^1(\Gamma)$-module. Then X has the Schur property iff $X \subseteq AP(\Gamma)$.*

Proof. Since X is weak*-closed and an $\ell^1(\Gamma)$-module, $X = I^\perp$ for some closed ideal I of $\ell^1(\Gamma)$. Let $E = hull(I)$. Then $I \subseteq k(E)$ so that $k(E)^\perp \subseteq I^\perp = X$. Now suppose that $X \subseteq AP(\Gamma)$. Then, by Lemma 3.2, the set E is scattered, so a set of synthesis and $I = k(E)$. Hence, by the preceding theorem, X has the Schur property. Conversely, assume that X has the Schur property. Then $k(E)^\perp$ has the Schur property. Hence, by Lemma 3.2, E is scattered, $X = k(E)^\perp$ and $X \subseteq AP(\Gamma)$. □

From examples in section 2 and he preceding theorem, the following corollaries are immediate. For related results we refer the reader to the papers of Lust-Piquard [L-P1 and L-P2], and Granirer [G].

Corollary 3.7. *Let G be a locally compact abelian group, $S(G)$ be a Segal algebra on it and I be a closed ideal of $S(G)$ with compact hull. Then the space $(S(G)/I)^\star$ has the Schur property iff the hull of I is scattered.* $\square$

Corollary 3.8. *Let A and B be two γ-algebras. Then the space $(A\hat{\otimes}B)^\star$ has the Schur property iff the spectrums $\sum_A$ and $\sum_B$ of A and B are both scattered. Moreover, in this case, $(A\hat{\otimes}B)^\star =$ the closure in $K(A, B^\star)$ of the space $Span\{f \otimes g : f \epsilon \sum_A, g \epsilon \sum_B\}$. Here $K(A, B^\star)$ is the space of the compact linear operators $T : A \to B^\star$, and $f \otimes g : A \to B^\star$ is defined by $< f \otimes g, a >= f(a)g$.* $\square$

Corollary 3.9. *Let G be a locally compact abelian group, E be a compact subset of it, and $A(E) = A(G)/k(E)$. Then the space $A(E)^\star$ has the Schur property iff the set E is scattered.* $\square$

In general it is not easy to prove that the spectrum of a given Banach algebra A is scattered. In the case of γ-algebras we have the following result.

Theorem 3.10. *Let A and B be two γ-algebras, and assume that we have a continuous one-to one homomorphism $h : A \to B$ whose range is dense in B. Then the space $A^\star$ has the Schur property iff the space $B^\star$ has the Schur property.*

Proof. We first observe that, by Theorem 3.1, every γ-algebra is (Silov) regular. Now assume that the spectrum $\sum_A$ of A is scattered. Then the set $h^\star(\sum_B)$ being a compact subset of $\sum_A$, is also scattered. As $h^\star$ is a homeomorphism from $\sum_B$ onto $h^\star(\sum_B)$, we conclude that $\sum_B$ is also scattered. Now assume that $\sum_B$ has the Schur property. Let us see that $h^\star(\sum_B) = \sum_A$. If $f \in \sum_A \setminus h^\star(\sum_B)$, by regularity of A, there exists an element $a \in A$ such that $< a, f >= 1$ and $< h(a), g >= 0$ for each $g \in \sum_B$. As the space $\sum_B$ is scattered, the algebra B is semisimple by Theorem 3.1. Hence $h(a) = 0$. As h is one-to-one, $a = 0$. This contradiction proves that $h^\star(\sum_B) = \sum_A$, and $\sum_A$ is scattered. $\square$

From this theorem we easily deduce the following result.

Corollary 3.11. *Let A be a γ-algebra, which is algebraically isomorphic to a dense subalgebra of $C(K)$ for some compact space K. Then $A^\star$ has the Schur property iff K is scattered.* □

We end the paper with a few questions.

Questions 1) Let A be a commutative semisimple regular Banach algebra. If $A^\star$ has the Schur property, is it then true that the spectrum of A is scattered?

2) Let Γ be a discrete abelian group. In view of Lemma 3.3, the following question arises naturally. Let I be a closed ideal of $\ell^1(\Gamma)$. What should the hull of I be for the inclusion $I^\perp \subseteq WAP(\Gamma)$ to hold? Here $WAP(\Gamma)$ is the space of the weakly almost periodic functions on Γ.

3) Let G be a locally compact (abelian or not) group, $A_p(G)$ be its Figa-Talamanca-Herz algebra [H] and I be a closed ideal of it with compact hull. Is the algebra $A_p(G)/I$ a γ-algebra?

4) Let X be a closed subspace of $\ell^\infty(\Gamma)$ which is also an $\ell^\infty(\Gamma)$- module. Suppose that X has the Schur property. Is then true that

i) X is weak*- closed in $\ell^\infty(\Gamma)$?

ii) $X \subseteq AP(\Gamma)$?

iii) $X = \overline{Span(E)}$ for some closed scattered subset E of $\hat{\Gamma}$?

5l) How to prove Theorem 3.10 without using Theorem 3.5?

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