

MODULAR SYMMETRY CLASSES OF TENSORS

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Abstract

We introduce the notion of modular symmetry classes of tensors and give a necessary and sufficient condition for a modular symmetry class of tensors associated with the full symmetric group to be non-zero. Then we use modular symmetry classes of tensors to study the polynomial representations of $GL(V)$, where V is a vector space over a field of characteristic p . At the end we introduce a non-degenerate bilinear form on a modular symmetry class. Some problems are also given.

1. Introduction

Let V be an n -dimensional complex vector space, and G be a permutation group on m elements. Let χ be any irreducible character of G . For any $g \in G$, define the operator

$$P_g : \bigotimes^m V \rightarrow \bigotimes^m V$$

by

$$P_g(v_1 \otimes \dots \otimes v_m) = v_g^{-1} \otimes \dots \otimes v_g^{-1}(m).$$

Then the symmetry class of tensors associated with G and χ is the image of the following symmetry operator

$$S_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) P_g.$$

The symmetry class of tensors associated to G and χ is denoted by $V_\chi(G)$. A tremendous number of articles have appeared on the past 40 years concerning symmetry classes of tensors. The two volumes book of Marvin Marcus [4] and recent book of Russell Merris [5] are valuable sources to this theme. Unfortunately there is no similar notion for the case of vector spaces over arbitrary fields in literature. The aim of this expository article is to introduce the notion of modular symmetry classes of tensors for the first time, and introduce various questions which might become the subject of further research.

Let \mathbf{A} be the field of algebraic numbers over \mathbf{Q} and p be a prime. Let R be a valuation subring of \mathbf{A} with the unique maximal ideal P in which $P \cap \mathbf{Z} = p\mathbf{Z}$. Assume that $F = R/P$. It is well known that F is an algebraically closed field of characteristic p . Let $\star : R \rightarrow F$ be the canonical map.

Now assume that G is a permutation group on m elements and let $Bl(G)$ denotes the set of all p -blocks of G over F . For any $B \in Bl(G)$ define the corresponding *Osim* idempotent

$$e_B^\star = \sum_{g \in G} \left[\frac{1}{|G|} \sum_{\chi \in Irr(B)} \chi(1)\chi(g) \right]^\star g^{-1^m},$$

where $Irr(B) = B \cap Irr(G)$ and $Irr(G)$ is the set of irreducible characters of G . It is well known that the set of all $e_B^\star s$, where $B \in Bl(G)$, is a complete set of primitive idempotents in $Z(FG)$.

Let V be an n -dimensional vector space over F . We define the action of G on the tensor product space $\otimes^m V$ as follows:

$$g(v_1 \otimes \dots \otimes v_m) = v_{g^{-1}(1)} \otimes \dots \otimes v_{g^{-1}(m)}.$$

Then $\otimes^m V$ becomes a FG -module. Define

$$V_B(G) = e_B^\star (\otimes^m V)$$

and call it the *modular symmetry class of tensors associated with G and B* . We know that

$$\bigotimes^m V = \sum_{B \in \text{Bl}(G)} V_B(G).$$

The tensor

$$e_B^*(v_1 \otimes \dots \otimes v_m)$$

is called a *decomposable symmetrized tensor* and denoted by $v_1 \star \dots \star v_m$.

If p does not divide $|G|$, then every p -block B contains a unique ordinary irreducible character of G . In this case the ordinary symmetry class of tensors and the modular one are essentially the same. In this paper we study the vanishing problem of modular symmetry classes of tensors, when $G = S_m$. We use modular symmetry classes of tensors to study the polynomial representations of $GL(V)$, where V is a vector space over a field of characteristic p . At the end of the article we introduce a non-degenerate bilinear form on the modular symmetry class. Some open problems are also given.

2. Preliminaries

The objective of this section is to give some elementary properties of indecomposable modules. Also we give a brief review of F -representations of the symmetric group S_m . For proofs and further studies, see [1, chap. 8], [3, chap. 3, 4, 10, 11 and 12] and [7, chap. 1, 2].

Let A be a finite dimensional F -algebra with identity and assume that

$$A = A_1 \oplus \dots \oplus A_r$$

is the decomposition of A , as an A -module under left multiplication, into the direct sum of non-zero indecomposable submodules. Then every A_i is called an *essential indecomposable A -module*. By the *Krull-Schmidt* theorem, the decomposition is unique and so there is no ambiguity in this definition. It is easy to see that a left ideal I in A is an essential indecomposable A -module if and only if $I = Ae$ for some primitive idempotent e . Hence for any $1 \leq i \leq r$, there is a primitive idempotent e_i such that $A_i = Ae_i$. The set $\{e_1, \dots, e_r\}$ is a complete set of primitive idempotents in A . It is well known that every Ae_i has only one maximal submodule, namely $J(A)e_i$, where $J(A)$ is

the Jacobson radical of A . So A -modules $Ae_i/J(A)e_i, 1 \leq i \leq r$, are irreducible and further $Ae_i/J(A)e_i \cong Ae_j/J(A)e_j$ if and only if $Ae_i \cong Ae_j$. The following theorem has a central role in the next sections.

Theorem 2.1. *Let M be an A -module which has a composition series. Then $Ae_j/J(A)e_i$ is a composition factor of M if and only if $e_i(M) \neq 0$.*

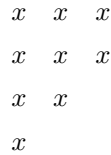
Now let $B \in Bl(G)$, and let M be an irreducible FG -module. We say that M belongs to the block B , if its corresponding Brauer character belongs to B . It is well known that this occurs if and only if $e_B^*(M) \neq 0$. Every irreducible FG -module belongs to a unique block.

Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be an (integer) partition of m , where, as usual $\lambda_1 \geq \lambda_2 \geq \dots \lambda_s \geq 1$. We call s the height of λ and denote it by $h(\lambda)$. Suppose that $\lambda = (\lambda_1, \dots, \lambda_s)$ and $\mu = (\mu_1, \dots, \mu_l)$ are two partitions of m . We say that λ is majorized by μ , if for any $1 \leq i \leq \min\{s, l\}$, the inequality

$$\lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i$$

holds. In this case we write $\lambda \leq \mu$. This is a partial ordering on the set of all partitions of m .

If λ is a partition of m , then the Ferrers diagram of λ is a left-justified of m nodes x , arranged in s rows and for any $1 \leq i \leq s$, the i -th row consists of λ_i nodes. For example if $\lambda = (3, 3, 2, 1)$, then he Ferrers diagram of λ is the following:



A Young tableau associated with λ , or a λ -tableau, arises from the Ferrers diagram of λ by replacing the nodes \times by the numbers $1, 2, \dots, m$. For example if $\lambda = (3, 3, 2, 1)$, then the following is a λ -tableau:

4 2 5
8 1 3
7 6
9

If t is a λ -tableau, then the number in the (i, j) -place is denoted by t_{ij} . The symmetric group S_m acts on the set of λ -tableaux if we set $(t^g)_{ij} = g(t_{ij})$ for $g \in S_m$. Two tableaux t and t' are said to be equivalent, if their corresponding rows contain the same integers. The equivalence class of t is denoted by $\{t\}$ and it is called a λ -tabloid. For any $g \in S_m$, define:

$$\{t\}^g = \{t^g\}.$$

So S_m acts on the set of λ -tabloids. Let M^λ be the free vector space generated by the set of λ -tabloids. Then M^λ is a FS_m -module and we have

$$\dim M^\lambda = \frac{m!}{\lambda_1! \dots \lambda_s!}$$

The Young subgroup associated with λ is defined as follows

$$S_\lambda = S_{\{1, \dots, \lambda_1\}} \times S_{\{\lambda_1+1, \dots, \lambda_1+\lambda_2\}} \times \dots$$

We have $S_\lambda \cong S_{\lambda_1} \times \dots \times S_{\lambda_s}$

Theorem 2.2. Let $H = S_\lambda$ and $\theta_1, \dots, \theta_l$ be a left transversal for H in S_m . If we consider $F[H\theta_1, \dots, H\theta_l]$ as a FS_m -module, then

$$M^\lambda \cong F[H\theta_1, \dots, H\theta_l].$$

Let t be a λ -tableau and C_t be the column stabilizer subgroup of t . Define:

$$\kappa_t = \sum_{g \in C_t} \epsilon(g)g$$

where ϵ is the alternating character of S_m . Now define the λ -polytabloid e_t as $e_t = \kappa_t\{t\}$. Let S_λ be the submodule of M^λ , generated by the set of all λ -polytabloids. This is called the Specht module associated with λ .

Now we define a bilinear form of M^λ such that for any two λ -tableaux t and t' ,

$$\langle \{t\}, \{t'\} \rangle = \delta_{\{t\}, \{t'\}}.$$

With respect to this bilinear form, let S^{λ^\perp} be the orthogonal complement of S^λ . Define the quotient module

$$D^\lambda = \frac{S^\lambda}{S^{\lambda^\perp}}.$$

A partition λ is p -regular if there is no r for which $\lambda_r = \lambda_{r+1} = \dots = \lambda_{r+p}$.

It is known that every irreducible F -representation of S_m is of the form D^λ , where λ is a p -regular partition. See [3, chap. 11].

Definition 2.3. Let λ and μ be partitions of m . We say that T is a generalized μ -tableau of type λ if

$$T : \{(i, j) \mid 1 \leq i \leq h(\mu), 1 \leq j \leq \mu_i\} \rightarrow \{1, 2, \dots, m\}$$

is a function such that, for any $1 \leq i \leq m$, we have $|T^{-1}(i)| = \lambda_i$. This generalized tableau is called semistandard if, for each $1 \leq i \leq h(\mu)$, $j_i < j_2$ implies $T(i, j_1) \leq T(i, j_2)$, and for each $1 \leq j \leq \mu_1$, $i_1 < i_2$ implies $T(i_1, j) < T(i_2, j)$. In other words T is semistandard, if every row of T is non-descending and every column of T is ascending. The number of such semistandard tableaux is denoted by $K_{\mu\lambda}$ and called the Kostka number. We have $K_{\mu\lambda} \neq 0$ if and only if $\lambda \trianglelefteq \mu$.

Theorem 2.4. If μ is a p -regular partition, then

$$\dim \text{Hom}_{FS_m}(S^\mu, M^\lambda) = K_{\mu\lambda}.$$

Using the above theorem, we obtain the composition factors of M^λ .

Corollary 2.5. *Composition factors of M^λ are those D^μ 's for which μ is p -regular and $\lambda \trianglelefteq \mu$.*

Later on we denote by the notation $(D^\mu; M^\lambda)$, the multiplicity of D^μ as a composition factor of M^λ . This quantity can be obtained from the decomposition matrix of S_m .

3. The vanishing problem of $V_B(S_m)$

Let Γ_n^m be the set of all sequences $\omega = (\omega_1, \dots, \omega_m)$ with $1 \leq \omega_i \leq n$. We know that $|\Gamma_n^m| = n^m$. Define the action of S_m on Γ_n^m as follows

$$\omega^g = (\omega_g^{-1}(1), \dots, \omega_g^{-1}(m)).$$

Let $F[\Gamma_n^m]$ be the free vector space generated by Γ_n^m . Then $F[\Gamma_n^m]$ is a FS_m -module and we have

$$F[\Gamma_n^m] \cong \bigotimes^m V.$$

Let Ω be an orbit of Γ_n^m under the action of S_m . It is clear that $F[\Omega]$ is a submodule of $F[\Gamma_n^m]$. We have

$$F[\Gamma_n^m] = \sum_{\Omega} F[\Omega],$$

where the summation is over all orbits of Γ_n^m .

Let $\omega \in \Gamma_n^m$ and $1 \leq t \leq n$. Then $m_t(\omega)$ denotes the multiplicity of t in ω . Hence $0 \leq m_t(\omega) \leq m$ and $\sum_t m_t(\omega) = m$. We set

$$m(\omega) = (m_1(\omega), \dots, m_n(\omega)).$$

Then $m(\omega)$ is an improper partition of m with length n . Removing zero of $m(\omega)$ and rearranging remaining terms in the decreasing order, we obtain a partition of m , say λ . We call this λ the corresponding partition of ω . If ω belongs to the orbit Ω , then the corresponding partition of Ω is just λ . This definition has no ambiguity, because two sequences ω and γ belong to the same orbit of Γ_n^m , if and only if $m(\omega) = m(\gamma)$.

Lemma 3.1. *Let $\lambda = (a_1^{r_1}, a_2^{r_2}, \dots, a_l^{r_l})$ be a partition of m of length s . Then the number of orbits of Γ_n^m , with corresponding partition λ , is $f(n, \lambda)$, where:*

a) *If $s \leq n$ then*

$$f(n, \lambda) = \frac{n!}{(n-s)!r_1!r_2!\dots r_l!}.$$

b) *Otherwise $f(n, \lambda) = 0$*

Now let Ω be an orbit of Γ_n^m and let λ be the corresponding partition of Ω .

Theorem 3.2. $M^\lambda \cong F[\Omega]$.

Proof. Choose the sequence $\omega \in \Omega$ such that $S_\lambda = \text{Stab}(\omega)$. Now let $H = S_\lambda$. Let $\theta_1, \dots, \theta_k$ be a set of right transversals for H in S_m . Using 2.2, it is enough to show that

$$F[\Omega] \cong F[H\theta_1, \dots, H\theta_k].$$

But $\Omega \cong S_m/H$ as S_m -sets, from which it follows that $F[\Omega] \cong F[S_m/H] \cong F[H\theta_1, \dots, H\theta_k] \cong M^\lambda$. This completes the proof. \square

Corollary 3.3.

$$\bigotimes^m V \cong \sum_{\lambda} f(n, \lambda) M^\lambda.$$

Corresponding partitions of orbits of Γ_n^m are those λ 's such that $h(\lambda) \leq n$. So, if $m \leq n$, then every partition of m corresponds to some orbit.

Now we obtain the composition factors of $\bigotimes^m V$, using 2.5 and 3.3.

Theorem 3.4. *The composition factors of $\bigotimes^m V$ are those D^μ 's such that μ is a p -regular partition of m and $h(\mu) \leq n$. The multiplicity of D^μ as a composition factor of $\bigotimes^m V$ is equal to*

$$m_\mu = \sum_{\lambda \leq \mu} f(n, \lambda)(D^\mu; M^\lambda).$$

Now suppose $C_m = Z(FSm)$. We know that the set

$$\{e_B^* \mid B \in Bl(S_m)\}$$

is a complete set of primitive idempotents of C_m . Also $\otimes^m V$ is a C_m -module. According to 2.1, $V_B(S_m) \neq 0$ if and only if $C_m e_B^*/J(C_m)e_B^*$ is a composition factor of $\otimes^m V$. So $V_B(S_m) \neq 0$ if and only if $C_m e_B^*/J(C_m)e_B^*$ is a composition factor of D^μ for some p -regular partition μ with $h(\mu) \leq n$. It is well known that this occurs if and only if $D^\mu \in B$.

Theorem 3.5. We have $V_B(S_m) \neq 0$ if and only if D^μ belongs to B for some p -regular μ with $h(\mu) \leq n$.

Corollary 3.6. If $m \leq n$, then $V_B(S_m) \neq 0$.

Note The above corollary is also true even if $G \neq S_m$, for, let $m \leq n$ and B be any p -block of $G \leq S_m$. Then $\alpha = (1, 2, \dots, m) \in \Gamma_n^m$. We claim that $e_B^*(e_{\alpha(1)} \otimes \dots \otimes e_{\alpha(m)}) \neq 0$. Because if $e_B^*(e_{\alpha(1)} \otimes \dots \otimes e_{\alpha(m)}) = 0$, then

$$\sum_{g \in G} \left[\frac{1}{|G|} \sum_{\chi \in Irr(B)} \chi(1)\chi(g) \right]^* e_{\alpha(g^{-1}(1))} \otimes \dots \otimes e_{\alpha(g^{-1}(m))} = 0.$$

By the linear independence of the vectors $e_{\alpha(g^{-1}(1))} \otimes \dots \otimes e_{\alpha(g^{-1}(m))}$, we obtain

$$\left[\frac{1}{|G|} \sum_{\chi \in Irr(B)} \chi(1)\chi(g) \right]^* = 0,$$

for all $g \in G$. But this implies $e_B^* = 0$ which is impossible.

4. Polynomial representations of $GL(V)$

Let $G_n = GL(V)$ and $T : V \rightarrow V$ be a linear operator. We define an operator

$$K^m(T) : \bigotimes^m V \rightarrow \bigotimes^m V$$

as follows:

$$K^m(T)(v_1 \otimes \dots \otimes v_m) = Tv_1 \otimes \dots \otimes Tv_m.$$

This is called the Kronecker power of T . The map $K^m : T \mapsto K^m(T)$ is a representation of the group G_n over $\otimes^m V$, so we can consider $\otimes^m V$ as a FG_n -module. It is easy to see that

$$gK^m(T)(v_1 \otimes \dots \otimes v_m) = K^m(T)g(v_1 \otimes \dots \otimes v_m)$$

for any $g \in S_m$. Hence the modular symmetry class $V_B(G)$ is a FG_n -submodule of $\otimes^m V$. By restriction of $K^m(T)$ to $V_B(G)$ we obtain a linear operator

$$K_B^G(T) : V_B(G) \rightarrow V_B(G).$$

This is the *induced operator* associated with G and B . Now $K_B^G : T \mapsto K_B^G(T)$ is a representation of G_n and we have

$$K^m = \sum_{B \in Bl(G)} K_B^G.$$

In the special case, if $G = S_m$, then we use K_B^m instead of $K_B^{S_m}$.

Definition 4.1. Let $B \in Bl(S_m)$. Then \tilde{B} is the set of all $D^\lambda \in B$ in which λ is p -regular and $h(\lambda) \leq n$.

As we saw in section 2, we may write

$$FS_m = U_1 \oplus \dots \oplus U_r,$$

a direct sum of principal indecomposable modules of FS_m , numbered so that $U_1 = (FS_m)e_1, \dots, U_r = (FS_m)e_r$ are a full set of non-isomorphic modules among these summands. For any $1 \leq i \leq r$, there is a unique p -regular partition λ , such that

$$\frac{(FS_m)e_i}{J(FS_m)e_i} \cong D^\lambda,$$

so we can write $e_i = e_\lambda$. Every e_λ is a primitive idempotent of FS_m and we have

$$e_B^* = \sum_{D^\lambda \in B} e_\lambda.$$

So we have

$$V_B(S_m) = \sum_{D^\lambda \in B} V_B^\lambda,$$

where $V_B^\lambda = e_\lambda(\otimes^m V)$. But $V_B^\lambda \neq 0$ if and only if D^λ is a composition factor of $\otimes^m V$. This occurs if and only if λ is p -regular and $h(\lambda) \leq n$. So

$$V_B(S_m) = \sum_{D^\lambda \in \tilde{B}} V_B^\lambda.$$

Also every V_B^λ is a FG_n -submodule of $\otimes^m V$. For the proof of the following lemma, see [1, chap. 8].

Lemma 4.2. *Let F be a splitting field for the finite dimensional algebra A and let M be an A -module. For any primitive idempotent $e \in A$ the multiplicity of $Ae/J(A)e$, as a composition factor of M , is equal to $\dim_F eM$.*

Using the above lemma, we obtain

$$\dim V_B^\lambda = \sum_{\mu \trianglelefteq \lambda} f(n, \mu)(D^\lambda; M^\mu).$$

Corollary 4.3.

$$\dim V_B^m(S_m) = \sum_{D^\lambda \in \tilde{B}} \sum_{\mu \trianglelefteq \lambda} f(n, \mu)(D^\lambda; M^\mu).$$

It is well known that D^λ and D^μ are in the same p -block of S_m if and only if λ and μ have the same p -core. Hence one can state the above result in a more combinatorial form.

Corollary 4.4. *Let $E_B^m(\lambda)$ be the restriction of K^m to V_B^λ . Then*

$$K_B^m = \sum_{D^\lambda \in \tilde{B}} E_B^m(\lambda);$$

in addition the degree of $E_B^m(\lambda)$ is equal to the multiplicity of D^λ in $\otimes^m V$.

Corollary 4.5. *If the representation K_B^m is irreducible, then \tilde{B} has only one element.*

Let U be any vector space over F and let $\varphi : G_n \rightarrow GL(U)$ be a polynomial representation of G_n . By a similar argument as in [4, part II], φ is equal to a direct sum of homogeneous polynomial representations. Suppose $\theta : G_n \rightarrow GL(U)$ is a homogeneous polynomial representation of degree m and let \mathbf{B}_m be the bisymmetric algebra, generated by the $K^m(T)$'s, $T \in G_n$. Then we have

$$\theta = hK^m$$

for some algebra homomorphism $h : \mathbf{B}_m \rightarrow L(U, U)$. So corollary 4.4 shows that the study of representations $E_B^m(\lambda)$ is equivalent to the study of polynomial representations of G_n .

5. A bilinear form

For any $B \in Bl(G)$ define a function $c_B : G \rightarrow F$ as follows

$$c_B(g) = \left[\frac{1}{|G|} \sum_{\chi \in Irr(B)} \chi(1)\chi(g) \right]^*.$$

Using the relations $\sum_{B \in Bl(G)} e_B^* = 1$ and $e_B^* e_{B'}^* = \delta_{B, B'} e_B^*$ we can prove

$$\sum_{g \in G} c_B(g) c_B(xg^{-1}) = c_B(x)$$

for any $x \in G$.

Let $X = [a_{ij}]$ be an $m \times m$ matrix on F . We define

$$d_G^B(X) = \sum_{g \in G} c_B(g) \prod_{i=1}^m \alpha_{i, g(i)}.$$

We call the function d_G^B the *generalized matrix function* corresponding to B and G . If $G = S_m$, then we write d^B instead of $d_{S_m}^B$.

Let \langle, \rangle be a nondegenerate symmetric bilinear form on V . Then we have a symmetric form on $\otimes^m V$ defined as follows:

$$\langle x_1 \otimes \dots \otimes x_m, y_1 \otimes \dots \otimes y_m \rangle = \prod_{i=1}^m \langle x_i, y_i \rangle .$$

This is well defined by the universality property of the tensor space. Also this form is non-degenerate, for, let $E = \{e_1, \dots, e_n\}$ be a basis of V and M be the matrix representation of the form on V with respect to E . Then $E^* = \{e_\alpha^\otimes \mid \alpha \in \Gamma_n^m\}$ is a basis of $\otimes^m V$ and $M \otimes \dots \otimes M$, (m times), is the matrix representation of the induced form. Since $\det M \neq 0$, so $\det (M \otimes \dots \otimes M) \neq 0$. This proves that the induced form is non-degenerate. Now $V_B(G)$ is a subspace of $\otimes^m V$, so we have an induced form on $V_B(G)$.

Lemma 5.1. *We have*

$$\langle x_1 \star \dots \star x_m, y_1 \star \dots \star y_m \rangle = \sum_{g \in G} c_B(g) d_G^B[\langle x_{g(i)}, y_j \rangle].$$

Proof. An easy computation. □

Lemma 5. 2. *Let x_1, \dots, x_m be an orthonormal set of vectors in V . Then*

$$\langle x_1 \star \dots \star x_m, x_1 \star \dots \star x_m \rangle = \sum_{g \in G} c_B(g)^2.$$

Proof. We have

$$[\langle x_{g(i)}, y_j \rangle] = [\delta_{g(i),j}],$$

so

$$d_G^B[\langle x_{g(i)}, y_j \rangle] = c_B(g).$$

Hence according to the lemma 5.1, we have

$$\langle x_1 \star \dots \star x_m, x_1 \star \dots \star x_m \rangle = \sum_{g \in G} c_B(g) c_B(g) = \sum_{g \in G} c_B(g)^2.$$

Now let $E = \{e_1, \dots, e_n\}$ be an orthonormal basis of V . Let Γ_n^m be defined as in section 3. For any $\alpha \in \Gamma_n^m$, let e_α^m be the tensor $e_{\alpha(1)} \star \dots \star e_{\alpha(m)}$. We have

$$\begin{aligned} \langle e_\alpha^*, e_\alpha^* \rangle &= \langle e_{\alpha(1)} \star \dots \star e_{\alpha(m)}, e_{\alpha(1)} \star \dots \star e_{\alpha(m)} \rangle \\ &= \sum_{g \in G} c_B(g) d_G^B[\langle e_{\alpha(g(i))}, e_{\alpha(j)} \rangle]. \end{aligned}$$

But $\langle e_{\alpha(g(i))}, e_{\alpha(j)} \rangle = \delta_{\alpha(g(i)), \alpha(j)}$, so

$$\begin{aligned} d_G^B[\langle e_{\alpha(g(i))}, e_{\alpha(j)} \rangle] &= \sum_{h \in G} c_B(h) \prod_{i=1}^m \delta_{\alpha(g(i)), \alpha(h(i))} \\ &= \sum_{h \in G} c_B(h) \delta_{\alpha g, \alpha h} \\ &= \sum_{h \in G \alpha g} c_B(h), \end{aligned}$$

hence we obtain

$$\langle e_\alpha^*, e_\alpha^* \rangle = \sum_{g \in G} \sum_{x \in G \alpha} c_B(g) c_B(xg).$$

Unfortunately there exist isotropic vectors in $\otimes^m V$, even if the form on V is not isotropic. For example, let $m = 2$ and let u_1 and u_2 be two non-zero orthogonal vectors in V . Then

$$(-1)^{\frac{1}{2}} u_1 \otimes u_2 + u_2 \otimes u_1 \neq 0.$$

But it is easy to see that

$$\langle (-1)^{\frac{1}{2}} u_1 \otimes u_2 + u_2 \otimes u_1, (-1)^{\frac{1}{2}} u_1 \otimes u_2 + u_2 \otimes u_1 \rangle = 0.$$

□

6. Problems

In this final section some open problems concerning modular symmetry classes of tensors are given. In the first problem we consider the dimension of $V_B(G)$. In the case of ordinary symmetry classes of tensors, several formulas for dimensions of symmetry classes are obtained; see [4], [5] and [6]. For example, if V is an n -dimensional complex vector space and χ is an irreducible character of G , then

$$\dim V_\chi(G) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g)n^{c(g)},$$

where $c(g)$ is the number of disjoint cycles, including cycles of length one, in the decomposition of g . For a second example, let $G = S_m$ and suppose λ is a partition of m . Define

$$Q^\lambda(n) = \prod_{1 \leq i \leq h(\lambda), 1 \leq j \leq \lambda_i} (n - i + j).$$

Then it is proved that

$$\dim V_\chi(S_m) = \frac{\lambda(1)^2}{m!} Q^\lambda(n).$$

For the general case see [6]. In this article we were able to obtain the dimension of $V_B(S_m)$ in terms of quantities $(D^\lambda; m^\mu)$. The following problem is more general and it seems that for solving it, we must begin with special cases.

6.1. Problem

Find a formula for $\dim V_B(G)$.

We studied the vanishing problem of $V_B(G)$ in the case $G = S_m$. In the general case we have the following problem.

6.2. Problem

When is $V_B(G) \neq 0$?

The third problem is related to the notion of decomposable symmetrized tensors. Let e_B^* be the corresponding Osima idempotent of B . We called the tensor

$$e_B^*(v_1 \otimes \dots \otimes v_m)$$

a decomposable symmetrized tensor and denoted it by $v_1 \star \dots \star v_m$.

6.3. Problem

When is $v_1 \star \dots \star v_m \neq 0$?

Now suppose we have a nondegenerate symmetric bilinear form on V . As in section 5, we can induce this form on $V_B(G)$ and obtain a new nondegenerate symmetric form. Let $E = \{e_1, \dots, e_n\}$ be an orthonormal basis of V .

6.4. Problem

Find a subset $S \subseteq \Gamma_n^m$, such that $\{e_\alpha^* \mid \alpha \in S\}$ is an orthogonal basis of $V_B(G)$.

We defined the induced operators $K_B^G(T)$ in the modular case as well as ordinary symmetry classes in section 4. The next problem concerns the elementary divisors of $K_B^G(T)$.

6.5. Problem

Compute the elementary divisors of $K_B^G(T)$ in terms of elementary divisors of T .

In fact K_B^G is a representation of the general linear group $GL(V)$ over $V_B(G)$. So we can ask:

6.6. Problem

When is the representation K_B^G irreducible?

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References

- [1] C. W. Curtins, and I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience Publishers, John Wiley and Sons Inc, 1962.
- [2] I. M. Isaacs, *Character theory of finite groups*, Academic Press, 1976.
- [3] G. D. James, *The representation theory of the symmetric group*, Lecture notes in math. Springer-Verlag, 1978.
- [4] M. Marcus, *Finite dimensional multilinear algebra, Part I and II*, Marcel Dekker, 1973.
- [5] R. Merris, *Multilinear algebra*, Gordon Breach, 1997.
- [6] R. Merris, Recent advances in symmetry classes of tensors, *Linear and Multilinear Algebra*, 1979, Vol 7, pp. 317-328.
- [7] B. M. Puttaswamaiah and J. D. Dixon, *Modular representations of finite groups*, Academic Press, 1977.

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