Turk J Math
23 (1999) , 407 – 415.

ALTERNATIVE POLYNOMIAL AND HOLOMORPHIC DUNFORD-PETTIS PROPERTIES*

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Abstract

Alternatives to the Polynomial Dunford-Pettis property and the Holomorphic Dunford-Pettis property, called the PDP1 and HDP1 properties, respectively, are introduced. These are shown to be equivalent to the DP1 property, an alternative Dunford-Pettis property previously introduced by the author, thus mirroring the equivalence of the three original properties.

Introduction

In [4], R. Ryan proved that the Dunford-Pettis property, the Polynomial Dunford-Pettis property, and the Holomorphic Dunford-Pettis property are all equivalent. In [1], a property closely related to the Dunford-Pettis property, called the DP1 property, is introduced and defined as follows:

A Banach space X has the DP1 property if for any Banach space Y and any weakly compact linear operator $T : X \to Y$, if $x_n \to x$ weakly in X with $\|x_n\| = \|x\| = 1$ for all n, then $Tx_n \to Tx$ in norm in Y.

We will consider two alternative properties, the PDP1 property and the HDP1 property, in the same spirit as [4], and show that like the original properties, DP1, PDP1 and HDP1 are all equivalent. Some applications to Banach algebras are also given.

Notation and Background

Throughout the paper, X and Y will denote Banach spaces over the field of complex numbers. We identify X with its image in X^{**} under its canonical embedding in X^{**}. The Banach space of all bounded linear operators from X to Y will be denoted $\mathcal{L}(X;Y)$. Given $x_0 \in X$ and $r > 0$, the open and closed balls centered at x_0 with radius r will be denoted $\Delta(x_0, r)$ and $\overline{\Delta(x_0, r)}$, respectively. By the term 'operator', we will always

*This paper is a part of the author's doctoral dissertation at the University of California, Santa Barbara.

mean a bounded linear operator. Given $f \in X$ and $a \in X^*$ we often write $f(a)$ or $\langle f, a \rangle$ for the evaluation of a on f. The Banach space of all sequences in Y that are norm convergent to 0 will be denoted by $c_0(Y)$, and $c(Y)$ will denote the Banach space of all convergent sequences in Y. For each $n \in \mathbb{N}$, let $\pi_n : c(Y) \to Y$ be the nth coordinate map, i.e., $\pi_n(y_j) = y_n$. If $T : X \to c(Y)$, the composition $\pi_n T$ will be denoted by T_n. Let $\pi : c(Y) \to Y$ be the operator defined by $\pi(y_n) = \lim_n y_n$, and let $\iota : Y \to c(Y)$ be the operator defined by $\iota(y) = (y, y, \ldots)$.

Recall that a Banach space X is said to have the Dunford-Pettis Property (DP) if for any Banach space Y, every weakly compact operator from X to Y maps weakly convergent sequences to norm convergent sequences.

The PDP1 property

Recall the following standard result regarding weakly compact operators: An operator $T : X \to Y$ is weakly compact if and only if $T^{**}(X^{**}) \subseteq Y$. The following lemma gives a useful characterization of weakly compact operators $T : X \to c_0(Y)$.

Lemma ([4, Lemma 1.2]) $T : X \to c_0(Y)$ is weakly compact if and only if

(a) For all n, $T_n : X \to Y$ is weakly compact, and

(b) For all $g \in X^{**}$, one has $\lim_n \|T_n^{**} g\| = 0$.

We use this result to prove the following lemma which characterizes weakly compact operators $T : X \to c(Y)$.

Lemma 1.1 Let $T : X \to c(Y)$. Then T is weakly compact if and only if

(a) For all n, $T_n : X \to Y$ is weakly compact, and

(b) For all $g \in X^{**}$, one has $\lim_n \|(T_n^{**} - \pi^{**} T^{**}) g\| = 0$.

Proof. If T is weakly compact, then clearly T_n and $T - \iota\pi T : X \to c_0(Y)$ are weakly compact. Since $\pi_n(T - \iota\pi T) = T_n - \pi T$, the lemma above implies that $\|(T_n^{**} - \pi^{**} T^{**}) g\| \to 0$ for all $g \in X^{**}$.

For the converse, suppose that (a) and (b) hold. It follows from (a) that for all $g \in X^{**}$, we have $T_n^{**} g \in Y$, and so by (b) we have that $\pi^{**} T^{**} g \in Y$ as well. It follows that πT is weakly compact and hence $T_n - \pi T$ is weakly compact for all n. The previous lemma then implies that $T - \iota\pi T$ is weakly compact, and hence T is weakly compact as well. □

We now state and prove the main result.

Theorem 1.2 Assume that $X_1, X_2, \ldots, X_k$ are Banach spaces with the DP1 property. Let $(x_n^i)_{n=1}^\infty$ for $1 \le i \le k$ be weakly convergent norm-one sequences in X_i with norm-one limits $x_i \in X_i$, and set $X = X_1 \widehat{\otimes} X_2 \widehat{\otimes} \cdots \widehat{\otimes} Xk$. For any Banach space Y, if $T \in \mathcal{L}(X; Y)$ is weakly compact then the sequence

$$(T(x_n^1 \otimes x_n^2 \otimes \cdots \otimes x_n^k))_{n=1}^\infty$$

is norm convergent to $T(x_1 \otimes x_2 \otimes \cdots \otimes x_k)$ in Y.

Proof. Suppose that X_i has DP1 for each $1 \le i \le k$. We use a technique similar to that used in [3] and [4], proceeding by induction on k, the case of $k = 1$ holding by definition of DP1. Assume the statement is true for k; suppose that $X_1, X_2, \ldots, X_{k+1}$ are Banach spaces having DP1; let $X = X_1 \widehat{\otimes} X_2 \widehat{\otimes} \cdots \widehat{\otimes} Xk + 1$, and $W = X_1 \widehat{\otimes} X_2 \widehat{\otimes} \cdots \widehat{\otimes} Xk$; suppose $T \in \mathcal{L}(X; Y)$ is weakly compact, and that for all $1 \le i \le k+1$, (x_n^i) is a weakly convergent norm-one sequence in X_i with norm-one limit $x_i \in X_i$.

It is easy to see that for fixed $z_i \in X_i$, $1 \le i \le k$, the operator

$$x \mapsto T(z_1 \otimes z_2 \otimes \cdots \otimes z_k \otimes x)$$

which maps X_{k+1} to Y is weakly compact since T is weakly compact, and so since X_{k+1} has DP1, the sequence

$$(T(z_1 \otimes \cdots \otimes z_k \otimes x_n^{k+1}))_n$$

is norm convergent in Y to $T(z_1 \otimes \cdots \otimes z_k \otimes x_{k+1})$. We can thus define an operator $t : W \to c(Y)$ by setting

$$t(z_1 \otimes \cdots \otimes z_k) = (T(z_1 \otimes \cdots \otimes z_k \otimes x_n^{k+1}))_n,$$

and extending linearly. Assuming t is weakly compact, it then follows from the induction hypothesis that the sequence

$$(t(x_i^1 \otimes x_i^2 \otimes \cdots \otimes x_i^k))_{i=1}^\infty = ((T(x_i^1 \otimes \cdots \otimes x_i^k \otimes x_n^{k+1})_n)_{i=1}^\infty$$

is norm convergent to $(T(x_1 \otimes \cdots \otimes x_k \otimes x_n^{k+1}))_{n=1}^{\infty} \in c(Y)$. It follows easily that the sequence of "diagonal" elements

$$(T(x_n^1 \otimes x_n^2 \otimes \cdots \otimes x_n^{k+1}))_{n=1}^{\infty}$$

of the above sequence is norm convergent to $T(x_1 \otimes \cdots \otimes x_{k+1})$ in Y, which then completes the proof.

Now, since $t_n(z_1 \otimes \cdots \otimes z_k) = T(z_1 \otimes \cdots \otimes z_k \otimes x_n^{k+1})$, and T is weakly compact, it is easy to see that t_n is weakly compact.

Given $S \in X^*$, and $x \in X_{k+1}$, define $S' : X_{k+1} \to W^*$ by

$$S'(x)(z_1 \otimes z_2 \otimes \cdots \otimes z_k) = S(z_1 \otimes \cdots \otimes z_k \otimes x).$$

For any $\phi \in W^{**}$, define $\tilde{\phi} : X_{k+1} \to X^{**}$ by

$$\langle S, \tilde{\phi}(x) \rangle = \langle S'(x), \phi \rangle, \quad \text{for all} \quad x \in X_{k+1}, \; S \in W^*.$$

For all $z_i \in X_i$, $1 \leq i \leq k$, and $\psi \in Y^*$ we have

$$\begin{aligned} T^*(\psi)'(x_n^{k+1})(z_1 \otimes \cdots \otimes z_k) &= T^*(\psi)(z_1 \otimes \cdots \otimes z_k \otimes x_n^{k+1}) \\ &= \psi(T(z_1 \otimes \cdots \otimes z_k \otimes x_n^{k+1})) \\ &= \psi(t_n(z_1 \otimes \cdots \otimes z_k)) \\ &= t_n^*(\psi)(z_1 \otimes \cdots \otimes z_k) \end{aligned}$$

so that $T^*(\psi)'(x_n^{k+1}) = t_n^*(\psi)$.

It follows that for all $\phi \in W^{**}$, and $\psi \in Y^*$, we have

$$\langle \psi, t_n^{**}(\phi) \rangle = \langle t_n^*(\psi), \phi \rangle = \langle T^*(\psi), \tilde{\phi}(x_n^{k+1}) \rangle = \langle \psi, T^{**}(\tilde{\phi}(x_n^{k+1})) \rangle,$$

and hence for all n we have

$$t_n^{**}(\phi) = T^{**}(\tilde{\phi}(x_n^{k+1})) \in Y^{**}.$$

As T is weakly compact, the map $T^{**}\tilde{\phi} : X_{k+1} \to Y^{**}$ is weakly compact. Hence, since X_{k+1} has DP1,

$$\|T^{**}(\tilde{\phi}(x_n^{k+1})) - T^{**}(\tilde{\phi}(x_{k+1}))\| = \|t_n^{**}(\phi) - T^{**}(\tilde{\phi}(x_{k+1}))\| \to 0.$$

Thus to complete the proof, we need only show that

$$T^{**}(\tilde{\phi}(x_{k+1})) = \pi^{**}t^{**}(\phi),$$

and apply Lemma 1.1.

For any $\psi \in Y^*$, we have

$$\langle \psi, T^{**}(\tilde{\phi}(x_{k+1}))\rangle = \langle T^*(\psi), \tilde{\phi}(x_{k+1})\rangle = \langle (T^*(\psi))'(x_{k+1}), \phi\rangle,$$

but for all $z_1, z_2, \dots, z_k \in X$, we have

$$\begin{aligned}(T^*(\psi))'(x_{k+1})(z_1 \otimes \cdots \otimes z_k) &= (T^*(\psi))(z_1 \otimes \cdots \otimes z_k \otimes x_{k+1}) \\ &= \psi(T(z_1 \otimes \cdots \otimes z_k \otimes x_{k+1})) \\ &= \psi(\pi(t(z_1 \otimes \cdots \otimes z_k)))\end{aligned}$$

so that

$$\langle \psi, T^{**}(\tilde{\phi}(x_{k+1}))\rangle = \langle (T^*(\psi))'(x_{k+1}), \phi\rangle = \langle \psi\pi t, \phi\rangle = \langle \psi, \pi^{**}t^{**}(\phi)\rangle.$$

Thus $T^{**}(\tilde{\phi}(x_{k+1})) = \pi^{**}t^{**}(\phi)$, completing the proof. □

We now review some concepts from the references. Given Banach spaces X_j, $1 \le j \le k$, and Y, a k-linear operator $T : X_1 \times X_2 \times \cdots \times X_k \to Y$ is a k-linear mapping which is continuous in all variables x_i simultaneously. The set of all k-linear operators forms a Banach space with the usual vector space operations and the norm

$$\|T\| = \sup\{\|T(x_1, x_2, \dots, x_k)\| : \sup_i \|x_i\| \le 1\}.$$

We will denote this Banach space by $\mathcal{L}^{(k)}(X_1, X_2, \dots, X_k; Y)$. If all $X_i = X$, for some Banach space X, we write $\mathcal{L}^{(k)}(X^k; Y)$ for $\mathcal{L}^{(k)}(X, \dots, X; Y)$. Letting $W = X_1 \widehat{\otimes} X_2 \widehat{\otimes} \cdots \widehat{\otimes} Xjk$, where $\widehat{\otimes}$ denotes the projective tensor product, we make careful note that there is a canonical isometric isomorphism of $\mathcal{L}^{(k)}(X_1, X_2, \dots, X_k; Y)$ with $\mathcal{L}(W; Y)$ associating the k-linear operator $T \in \mathcal{L}^{(k)}(X_1, X_2, \dots, X_k; Y)$ with the operator $\widehat{T} \in \mathcal{L}(W; Y)$ where $\widehat{T}(z_1 \otimes z_2 \otimes \cdots \otimes z_k) = T(z_1, z_2, \dots, z_k)$ for $z_i \in X_i$, $1 \le i \le k$.

A k-linear operator $T \in \mathcal{L}^{(k)}(X_1, X_2, \dots, Xk; Y)$ is called weakly compact if for any k bounded sequences (z_n^i) in X_i, $1 \le i \le k$, the sequence

$$(T(z_n^1, z_n^2, \dots, z_n^k))_{n=1}^{\infty}$$

has a weakly convergent subsequence. As shown in [4, p. 374], a k-linear operator $T \in \mathcal{L}^{(k)}(X_1, X_2, \ldots, X_k; Y)$ is weakly compact if and only if the associated operator $\widehat{T}$ is weakly compact. Applying these facts to the previous result, we obtain the following corollary.

Corollary 1.3 X has DP1 if and only if for every Banach space Y and any $k \in \mathbb{N}$, if $T \in \mathcal{L}^{(k)}(X^k; Y)$ is weakly compact, and $(x_n^i)_{n=1}^\infty$ for $1 \leq i \leq k$ are weakly convergent norm-one sequences with norm-one limits x_i, then the sequence $(T(x_n^1, x_n^2, \ldots, x_n^k))_{n=1}^\infty$ is norm convergent to $T(x_1, \ldots, x_k)$ in Y.

Continuing with the review, let X be a Banach space. A map $p : X \to Y$ is called a k-homogeneous polynomial, $(k \geq 1)$, if there exists a k-linear operator $T \in \mathcal{L}^{(k)}(X^k; Y)$, called a generator of p, such that for all $x \in X$,

$$p(x) = T(x, \ldots, x).$$

The space of all k-homogeneous polynomials from X into Y becomes a Banach space with the norm

$$\|p\| = \sup\{\|p(x)\| : \|x\| \leq 1\}.$$

A map $p : X \to Y$ is called a polynomial if there exists $n \in \mathbb{N}$ and k-homogeneous polynomials p_k, with $0 \leq k \leq n$, such that $p = p_0 + p_1 + \cdots + p_n$. (By definition, a 0-homogeneous polynomial is any constant map from X to Y.) A polynomial $p : X \to Y$ is said to be weakly compact if for every bounded sequence (x_n) in X, the sequence $(p(x_n))$ has a weakly convergent subsequence in Y.

A Banach space X is said to have the Polynomial Dunford-Pettis property (PDP) if for any Banach space Y, every weakly compact polynomial from X into Y maps weakly Cauchy sequences onto norm convergent sequences. We will say that X has PDP1 if for any Y, and for every weakly compact polynomial $p : X \to Y$, if (x_n) is a sequence in X such that $x_n \to x$ weakly and $\|x\| = 1 = \|x_n\|$ for all n, then the sequence $(p(x_n))$ is norm convergent in Y. In [4], a question posed by Pełczyński is answered, namely it is shown that X has DP if and only if X has PDP. The next result mirrors the equivalence of DP and PDP.

Corollary 1.4 Let X be a Banach space. Then X has DP1 if and only if X has PDP1.

Proof. For the forward implication, let $p : X \to Y$ be a weakly compact polynomial, where $p = p_0 + p_1 + \cdots p_n$, for k-homogeneous polynomials p_k, $0 \leq k \leq N$, and let

$x_n \to x$ weakly with $\|x_n\| = \|x\| = 1$. As p is weakly compact, p_k is weakly compact and it follows that p_k has a weakly compact generator [3, Proposition 3]. Hence by Corollary 1.3, for each k, the sequence $(p_k(x_n))$ is norm convergent in Y, and hence $(p(x_n))$ is norm convergent in Y, whence X has PDP1.

The backward implication follows immediately from the fact that every operator is a 1-homogeneous polynomial. □

From the theorem, we also obtain an alternative proof of (one direction of) [1, Theorem 3.1].

Corollary 1.5 Let A be a C^*-algebra. Then A has DP1 if and only if whenever $a_n \to a$ weakly in A, with $\|a_n\| = \|a\| = 1$ for all n, then $a_n \to a$ σ-strong* in A^{**}.

Proof. We prove only the necessity of the condition. Suppose A has DP1 and let $a_n \to a$ weakly in A with $\|a_n\| = \|a\| = 1$ for all n. Let $f \in A^*$. The map $\Phi : A \times A \to \mathbb{C}$ defined by $\Phi(x, y) = f(xy)$ is obviously bounded, 2-linear, and weakly compact. Hence, the sequences $(\Phi(a_n, a_n^*))$ and $(\Phi(a_n^*, a_n))$ are both convergent to $\Phi(a, a^*)$ and $\Phi(a^*, a)$, respectively, so that $f(a_n^* a_n + a_n a_n^*) \to f(a^*a + aa^*)$, as desired. □

The HDP1 property

A map $f : X \to Y$ is called holomorphic if for every $\xi \in X$ there exists a sequence p_k of k-homogeneous polynomials, $k = 0, 1, \ldots$, (depending on ξ), and a constant $r > 0$ such that the series

$$\sum_{k=0}^{\infty} p_k(x - \xi)$$

converges uniformly to $f(x)$ for all $x \in \Delta(\xi, r)$. The sequence (p_k) is then unique at each ξ by [2, Proposition 4.2], and this series is called the Taylor series of f at ξ.

As defined in [4], a holomorphic map $f : X \to Y$ is said to be weakly compact if for each $\xi \in X$ there exists $s > 0$ such that f maps $\Delta(\xi, s)$ into a relatively weakly compact subset of Y. When f is holomorphic and weakly compact, $r(f; \xi)$ will denote the supremum of all $r \in \mathbb{R}$ such that the Taylor series of f at ξ converges uniformly for $z \in \overline{\Delta(\xi, s)}$ for all $0 < s < r$, and such that f maps $\Delta(\xi, r)$ into a relatively weakly compact subset of Y.

A Banach space X is said to have the Holomorphic Dunford-Pettis Property (HDP), as defined in [4], if for every Banach space Y and every weakly compact holomorphic map $f : X \to Y$, if (x_n) is a weakly Cauchy sequence in X and there exists $x \in X$ and $\rho > 0$ such that $\|x_n - x\| \leq \rho < r(f; x)$ for all n, then the sequence $(f(x_n))$ is norm convergent in Y. We will say that X has HDP1 if for any Y, and any weakly compact holomorphic map $f : X \to Y$, if $x_n \to x$ weakly such that there exists $\xi \in X$ and a constant $s > 0$ with $\|x_n - \xi\| = \|x - \xi\| = s < r(f; \xi)$ for all n, then the sequence $(f(x_n))$ is norm convergent in Y.

As shown in [4], DP and HDP are equivalent. The next result mirrors that equivalence.

Theorem 1.6 X has DP1 if and only if X has HDP1.

Proof. Suppose X has DP1. Let Y be a Banach space, let $f : X \to Y$ be a weakly compact holomorphic map, and let $x_n \to x$ weakly such that there exists $\xi \in X$ and a constant $s > 0$ with $\|x_n - \xi\| = \|x - \xi\| = s < r(f; \xi)$ for all n. Let

$$f(z) = \sum_{k=0}^{\infty} p_k(z - \xi)$$

be the Taylor series of f at ξ. Let $\epsilon > 0$, and choose $N > 0$ such that for all $z \in \overline{\Delta(\xi, s)}$,

$$\Big\| \sum_{k=N+1}^{\infty} p_k(z - \xi) \Big\| < \epsilon.$$

Since X has DP1, X has PDP1 by Corollary 1.4, so it follows that for all $0 \leq k \leq N$, the sequence $(p_k(x_n - \xi))$ is norm convergent. Hence there exists $M > 0$ such that for all $0 \leq k \leq N$, and $n, m \geq M$, we have

$$\|p_k(x_n - \xi) - p_k(x_m - \xi)\| < \frac{\epsilon}{N}.$$

It follows easily that $\|f(x_n) - f(x_m)\| < 3\epsilon$, so that the sequence $(f(x_n))$ is norm convergent in Y, whence X has HDP1.

The converse follows immediately from the fact that operators are holomorphic maps [2, Proposition 5.1]. □

Now, let A be a unital Banach algebra. It is easily seen that the exponential map

$$x \mapsto e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in A$$

is holomorphic. In particular, if A is reflexive then this mapping is also weakly compact, while if A is a subalgebra of $\mathcal{B}(H)$, the operators on a Hilbert space H, then for any $\xi \in H$, the map $x \mapsto e^x\xi$ is holomorphic and weakly compact. Using these facts, we obtain the following corollary.

Corollary .7 Let A be a unital Banach algebra with DP1 and let $a_n \to a$ weakly in A with $\|a_n\| = \|a\| = 1$.

(a) If A is reflexive, then $e^{a_n} \to e^a$ in norm.

(b) If $A \subseteq \mathcal{B}(H)$ is a subalgebra of the operators on a Hilbert space H, then $e^{a_n} \to e^a$ in the strong operator topology of $\mathcal{B}(H)$.

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Received 02.02.1998