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AUGMENTED NOETHERIAN GRADED MODULES

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Abstract

Let G be a multiplicative group with identity e, and let R be an associative G-graded ring with unity 1. In this paper we study the augmented graded Noetherian modules and try to give some relationships between the Noetherian modules in the category $R - Agr$ and the Noetherian modules in the category $R_e - gr$.

Keywords and phrases: Graded Noetherian modules, graded rings, Augmented graded modules.

Introduction

Let G be a multiplicative group with identity e, and let R be an associative G-graded ring with unity 1. Let R_e be the identity component of R, and let $R_e - gr$ be the category of all graded R_e-modules and their graded R_e-maps. In [5,6], we defined the concepts of augmented graded rings and augmented graded modules and we studied some of their properties.

In this paper we use these concepts to study the augmented graded Noetherian modules. Some of the material in this paper are related to the work done by C. Nastasescu and F. Van Oystaeyen in [1, 2, 3, 4].

In Section 1, we give some definitions of graded rings and graded modules which are necessary in this paper.

In Section 2, we discuss some facts concerning the augmented graded Noetherian modules, and we give some relationships between the Noetherian modules in the category $R - Agr$ and the Noetherian modules in the category $R_e - gr$.

1. Preliminaries

In this section we give some definitions of graded rings and graded modules which are necessary in this paper. For more details one can look in [5, 6].

Let G be a group with identity e, and R be a G-graded ring. We consider $R-gr$ to be the category of all left graded R-modules and their graded R-maps. It is well known that R-gr is a Grothendieck category [3]. Also, we consider $\text{supp}(R,G) = \{g \in G : R_g \neq 0\}$.

Definition 1.1. *A G-graded ring $R = \underset{g\in G}{\oplus} R_g$ is said to be an augmented G-graded ring if it satisfies the following conditions :*

1. If R_e is the identity component of the graduation (R,G), then $R_e = \underset{g\in G}{\oplus} R_{e-g}$, where R_{e-g} is an additive subgroup of R_e and $R_{-g}R_{e-h} \subseteq R_{e-gh}$ for all $g,h \in G$. (R_e is a G-graded ring.)

2. For each $g \in G$ there exists $r_g \in R_g$ such that
$R_g = \underset{h\in G}{\oplus} R_{e-h}r_g$, and in which we assume $r_e = 1$.

3. If $g,h,\in G$ and r_g, r_h are both non-zero, then $r_g r_h = r_{gh}$ and $(xr_g)(yr_h) = xyr_{gh}$ for all $x,y \in R_e$.

Definition 1.2. *Let R be an augmented G-graded ring. A graded R-module $M = \underset{g\in G}{\oplus} M_g$ is said to be an augmented G-graded R-module if it satisfies the following conditions:*

1. $M_g = \underset{h\in G}{\oplus} M_{g-h}$ where M_{g-h} is an R_{e-e}-submodule of M_g and
$R_{-\sigma}M_{g-h} \subseteq M_{g-\sigma h}$ for all σ, g and $h \in G$.

2. $R_{g-h}M_{\sigma-\tau} \subseteq M_{g\sigma-h\tau}$ for all g,h,σ and $\tau \in G$.

Let M,N be augmented G-graded R-modules. A map $f : M \to N$ is said to be an augmented G-graded R-map if $f(M_{g-h}) \subseteq N_{g-h}$ for all $g,h \in G$. We consider R-Agr to be the category of all augmented G-graded R-modules and their augmented G-graded R-maps.

Definition 1.3. *Let M be an augmented G-graded R-module, and X be an R-submodule of M. Then X is said to be an augmented G-graded R-submodule of M if it satisfies the following conditions:*

1. $X = \bigoplus_{g \in G} X_g$ *where* $X_g = X \cap M_g$, *i.e.,* X *is a* G*-graded* R*-submodule of* M.

2. $X_g = \bigoplus_{\sigma \in G} X_{g-\sigma}$ *where* $X_{g-\sigma} = X_g \cap M_{g-\sigma}$.

2. Augmented G-graded R-Noetherian modules

In this section we study some facts concerning the augmented graded Noetherian modules, and we give some relationships between the Noetherian modules in the category $R - Agr$ and the Noetherian modules in the category $R_e - gr$.

Definition 2.1. *We say M is Noetherian in $R - Agr$ if M satisfies the ascending chain condition on augmented G-graded R-submodules of M. Similarly, one can define Noetherian $R - gr$ and Noetherian in $R_e - gr$.*

Lemma 2.2. *(Proposition 2.5 of [6].) Let $M \in R - Agr$ and X be a G-graded R_e-submodule of M_g. Then RX is an augmented G-graded R-submodule of M.*

Proposition 2.3. *Let M be a Noetherian in $R - Agr$. Then M_g is Noetherian in $R_e - gr$ for all $g \in G$.*

Proof. Let $g \in G$ and $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$ be a chain of G-graded R_e-submodules of M_g. By Lemma 2.2, $RX_1 \subseteq RX_2 \subseteq RX_3 \ldots$ is a chain of augmented G-graded R-submodules of M, and hence there exists $n \in N$ such that
$RX_n = RX_{n+1} = \ldots So, (RX_n)_g = (RX_{n+1})_g \ldots$ and then $X_n = X_{n+1} = \ldots$ because $(RX_i)_g = R_eX_i = X_i$ for all i. Therefore, M_g is Noetherian in $Re - gr$. □

Proposition 2.4. *Let G be a finite group and M be a Noetherian module in $R - Agr$. Then M is a Noetherian module in $R_e - gr$.*

Proof. Assume $G = \{g_1, g_2, \ldots, g_n\}$ and M is Noetherian module in $R - Agr$. Then clearly $M \in R_e - gr$ with graduation $M_{(g)} = \bigoplus_{s \in G} M_{s-g}$. By Proposition 2.3, M_{g_i} is

Noetherian module in $R_e - gr$ for all $g_i \in G$, and hence $M = \overset{n}{\underset{i=1}{\oplus}} M_{g_i} = \underset{g \in G}{\oplus} M_{(g)}$ is a Noetherian module in $R_e - gr$.

Let R be an augmented G-graded ring. Let $M \in R_e - gr$ and $\bar{M} = \underset{g \in G}{\oplus} M.g$. For $m \in M, \sigma \in G$ and $x_{e-h} \in R_{e-h}$ let $x_{e-h}r_g \in R_{e-h}r_g$ and $m\sigma \in \bar{M}$. Define $(x_{e-h}r_g)m\sigma = x_{e-h}mg\sigma$. Then one can easily extend this product to a multiplication of R on $\bar{M}$. □

Lemma 2.5. *Suppose supp$(R, G) = G$ and $M \in R_e - gr$. Let X be an augmented G-graded R-submodule of $\bar{M}$. Then $X = \bar{A}$ where A is the G-graded R_e-submodule of M generated by $\{m_g \in M_g : g \in G$ and $m_g.s \in X$ for some $s \in G\}$.*

Proof. Let $g \in G$ and $x \in X_g$. Then $x = m.g$ for some $m \in M$. Since $X_g = \bigoplus_{h \in G} X_{g-h}, m_h.g \in X_g$ for all $h \in G$ and hence $m_h \in A$ for all $h \in G$. So, $x \in (\bar{A})_g$, i.e., $X_g \subseteq (\bar{A})_g$ for all $g \in G$.

To show $\bar{A} \subseteq X$, it is enough to show that if $m_h.s \in \bar{A}$ where $s, h \in G$ and $m_h.t \in X$ for some $t \in G$ then $m_h.s \in X$.

Since supp$(R, G) = G, r_{st-1} \neq 0$ and hence

$r_{st^{-1}} m_h t = m_h s \in X$ as desired. □

Proposition 2.6. *Suppose supp$(R, G) = G$. Then M is Noetherian module in $R_e - gr$ iff $\bar{M}$ is Noetherian module in $R - Agr$.*

Proof. Let M be a G-graded R_e-Noetherian module. Let $X_1 \subseteq X_2 \subseteq \ldots$ be a chain of augmented G-graded R-submodules of $\bar{M}$. By Lemma 2.5, there exists a G-graded R_e-submodule A_i of M such that $\bar{A}_i = X_i$ for all $i \in \mathbb{N}$. Since $A_1 \subseteq A_2 \subseteq \ldots$ is a chain of G-graded R_e-submodules of M, there exists $n \in \mathbb{N}$ such that $A_n = A_{n+1} = \ldots$ So, $\bar{A}_n = \bar{A}_{n+1} = \ldots$ and hence $X_n = X_{n+1} = \ldots$ The other part follows directly from Proposition 2.3.

Let $H \leq G$. Then $R^{(H)} = \sum_{g \in G} \sum_{h \in H} R_{g-h}$ is a subring of R and $1 \in R^{(H)}$. Also, if $M \in R - Agr$ and $s \in G$ then $M^{(Hs)} = \sum_{g \in G} \sum_{h \in H} M_{g-hs}$ is an $R^{(H)}$-submodule of M. With these notations we have the following :

1. $R^{(H)}$ is an augmented G-graded ring with $R_g^{(H)} = \bigoplus_{h \in H} R_{g-h'}$

$$R_{e-g}^{(H)} = \left\{ \begin{array}{ll} R_{e-g} & g \in H \\ & \qquad \text{and} r_g^{(H)} = r_g. \\ 0 & \text{otherwise} \end{array} \right\}$$

2. $M^{(Hs)} \in R^{(H)} - Agr$ with $M_g^{(Hs)} = \sum_{h \in H} M_{g-hs}$ and

$$M_{g-t}^{(Hs)} = \left\{ \begin{array}{ll} M_{g-t} & t \in Hs \\ & \qquad , \text{where} s \in G. \\ 0 & \text{otherwise} \end{array} \right\}$$

□

Remark 2.7. *If X is an augmented G-graded $R^{(H)}$-submodule of $M^{(Hs)}$, then RX is an augmented G-graded R-submodule of M.*

Lemma 2.8. *Let X be an augmented G-graded $R^{(H)}$-submodule of $M^{(Hs)}$. Then $RX \cap M^{(Hs)} = X$.*

Proof. Clearly $X \subseteq RX \cap M^{(Hs)}$.

Consider $\langle RX \cap M^{(Hs)} \rangle_{g-\ell} = \sum_{t \in G} \sum_{h \in H} R_{gt^{-1}} - \ell s^{-1} h^{-1} X_{t-hs} \cap M_{g-\ell}$.

If $\ell \neq h_1 s$ for some $h_1 \in H$ then $\langle RX \cap M^{(Hs)} \rangle_{g-\ell} = 0$ and hence $\langle RX \cap M^{(Hs)} \rangle_{g-\ell} \subseteq X$. Also, if $\ell = h_1 s$ for some $h_1 \in H$ then $R_{gt^{-1} - \ell s^{-1} h^{-1}} \subseteq R^{(H)}$. and hence $\langle RX \cap M^{(Hs)} \rangle_{g-\ell} \subseteq X$. Therefore, $\langle RX \cap M^{(Hs)} \rangle_{g-\ell} \subseteq X$ for all $g \in G$ and $\ell \in Hs$, i.e., $X = RX \cap M^{(Hs)}$. □

Proposition 2.9. *Let M be a Noetherian module in $R - Agr$ and $H \leq G$. Then $M^{(Hs)}$ is Noetherian in $R^{(H)} - Agr$ for all $s \in G$.*

Proof. Let $X_1 \subseteq X_2 \subseteq \ldots$ be a chain of augmented G-graded $R^{(H)}$-submodules of $M^{(Hs)}$. By Remark 2.7, $RX_1 \subseteq RX_2 \subseteq \ldots$ is a chain of augmented G-graded R-submodules of M. So, there exists $n \in \mathbb{N}$ such that $RX_n = RX_{n+1} = \ldots$, and hence

$$RX_n \cap M^{(Hs)} = RX_{n+1} \cap M^{(Hs)} = \ldots \text{ By Lemma 2.8, } X_n = X_{n+1} = \ldots$$

□

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