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NORMAL SUBGROUPS AND ELEMENTS OF $H'(\lambda_q)$

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Abstract

In this study, we consider the normal subgroups of $H'(\lambda_q)$, where $H(\lambda_q)$ denotes the Hecke groups. After recalling some results from [2], particularly on the group structure and on the relations with the power subgroups of $H(\lambda_q)$, the even subgroup $H_e(\lambda_q)$ of $H(\lambda_q)$ is discussed. It is shown that $H'(\lambda_q)$ is a normal subgroup of $H_e(\lambda_q)$ with index q. For this reason each subgroup of $H'(\lambda_q)$ consists of only even elements. $H''(\lambda_q)$ is also considered and it is concluded that it is the normal subgroup of $H'(\lambda_q)$ generated by all commutators of the elements of $H'(\lambda_q)$. Using the Kurosh subgroup theorem, the group structure of normal subgroups of $H(\lambda_q)$ can be found to be free groups. Their ranks are given in terms of the index.

1. Introduction

Let Γ be the classical modular group. Let R and S be its generators of order 2 and 3, respectively, defined by

$$R(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z+1}, \tag{1}$$

A generalisation of Γ is known as Hecke groups denoted by $H(\lambda_q)$ and generated by two elements

$$R(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z+\lambda_q}, \tag{2}$$

of order 2 and q, respectively, where $\lambda_q = 2\cos\pi/q$, $q \in N$, $q \geq 3$.

In [2], the group structure of $H'(\lambda_q)$, the commutator subgroup of $H(\lambda_q)$, is obtained and some of its properties are given. In particular it is shown that $H'(\lambda_q)$ is a free group of rank q-1 and also normal with index 2q in $H(\lambda_q)$. This coincides with the result given for the modular group where q = 3, [4]. We define $H^m(\lambda_q)$ to be the subgroup of $H(\lambda_q)$ generated by the m-th powers of the elements of $H(\lambda_q)$. $H^m(\lambda_q)$ is called the m-th power subgroup of $H(\lambda_q)$. It is also normal in $H(\lambda_q)$.

Relations between the modular group Γ, its commutator subgroup Γ' and power subgroups of Γ are discussed in [5]. These are generalised to some Hecke groups in [2].

2. Elements of $H'(\lambda_q)$ and its Relation with the Even Subgroup

We now study the place of $H'(\lambda_q)$ amongst the normal subgroups of $H(\lambda_q)$. First we look at the elements of $H(\lambda_q)$. They form two classes:

$$(a) \qquad \begin{pmatrix} a & b\lambda_q \\ c\lambda_q & d \end{pmatrix} ad - bc\lambda_q^2 = 1, \tag{3}$$

$$(b) \qquad \begin{pmatrix} a\lambda_q & b \\ c & d\lambda_q \end{pmatrix} ad\lambda_q^2 - bc = 1, \tag{4}$$

where a, b, c, d are all polynomials of λ_q^2 with rational integer coefficients. The elements of type (a) are called even while those of type (b) are called odd. It is easy to see that

$$\begin{aligned} &odd \cdot odd = even \cdot even = even \\ &even \cdot odd = odd \cdot even = odd. \end{aligned} \tag{5}$$

When q is even, the even elements form a subgroup of $H(\lambda_q)$ of index 2 called the even subgroup, denoted by $H_e(\lambda_q)$:

$$H_e(\lambda_q) = \left\{ M = \begin{pmatrix} a & b\lambda_q \\ c\lambda_q & d \end{pmatrix} : \quad M \in H(\lambda_q) \right\}. \tag{6}$$

It is generated by $T = R \cdot S$ and $T \cdot U = RS^2R$ and in fact

$$H_e(\lambda_q) \cong < T > * < T \cdot U > \tag{7}$$

Being of index 2, $H_e(\lambda_q)$ is normal in $H(\lambda_q)$. The set of odd elements form the other coset of $H_e(\lambda_q)$ in $H(\lambda_q)$. Actually

$$H(\lambda_q) = H_e(\lambda_q) \cup R \cdot H_e(\lambda_q) \tag{8}$$

as $R \notin H_e(\lambda_q)$, (see [3]).
When q is odd, $H(\lambda_q)$ does not have an even subgroup.
We now see the connection between the even subgroup and $H'(\lambda_q)$ for even q:

Theorem 2.1. *Let q be even. Then the commutator subgroup $H'(\lambda_q)$ of $H(\lambda_q)$ is a normal subgroup of the even subgroup $H_e(\lambda_q)$ with index q.*

Proof. Recall that $H'(\lambda_q)$ is a normal subgroup of $H(\lambda_q)$ with index 2q. The even subgroup $H_e(\lambda_q)$, having index 2, is also normal in $H(\lambda_q)$. Therefore the required index is q.
Let us take two elements A, B of $H(\lambda_q)$. Note that whatever A and B are, their commutator $[A, B] = ABA^{-1}B^{-1}$ is always even. Hence for every pair of elements

$$[A, B] \in H_e(\lambda_q). \tag{9}$$

That is

$$H'(\lambda_q) \triangleleft H_e(\lambda_q). \tag{10}$$

□

Corollary 2.1. *A subgroup of $H'(\lambda q)$ consists of only even elements.*

3. Second Commutator Subgroup

Lemma 3.1. *Let G be a finitely generated group and H be the subgroup of G generated by all the commutators of the generators of G. Then G/H is abelian.*

Proof. Let aH, bH be two elements of G/H. Then as $b^{-1}a^{-1}ba \in H$ we have

$$aH \cdot bH = abH = ab(b^{-1}a^{-1}ba)H = baH = bH \cdot aH$$

which proves the lemma. □

Theorem 3.1. *Let $H'(\lambda_q)$ be generated by the q-1 elements $a_1, a_2, \ldots, a_{q-1}$. Let N be the normal subgroup of $H'(\lambda_q)$ generated by all the commutators $[a_i, a_j]$ of the generators. Then*

$$N = H''(\lambda_q).$$

Proof. By Lemma 3.1, $H'(\lambda_q)/N$ is abelian. But it is well-known that $H'(\lambda_q)/H''(\lambda_q)$ is the largest abelian quotient group of $H'(\lambda_q)$. As $N < H''(\lambda_q)$, the result follows. □

4. The Structure of Normal Subgroups of $H'(\lambda_q)$

Recall that $H'(\lambda_q)$ is a free group of rank q-1. By the Kurosh subgroup theorem, all of its subgroups will be free. Therefore we only need to find the rank of a normal subgroup. We use the following result:

Theorem 4.1. *Let H be a subgroup of finite index μ in $H'(\lambda_q)$. Then the rank r of H will also be finite and can be found by the formula*

$$r = 1 + \mu \cdot (q - 2). \tag{11}$$

Proof. We need two results. First we use the fact that the rank of a free group of genus g having t parabolic classes is equal to $2g+t-1$, (see [1]). We also need the Riemann-Hurwitz formula

$$\mu = \frac{\mu(H)}{\mu(H'(\lambda_q))}, \tag{12}$$

where $2\pi \cdot \mu(H)$ denotes the hyperbolic area of a fundamental region for H given by

$$\mu(H) = 2g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + t. \tag{13}$$

Here m_i are the orders of finite ordered elements in H. Using these two results, one can easily prove the theorem. □

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