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SEGAL ALGEBRA AS AN IDEAL IN ITS SECOND DUAL SPACE

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Abstract

For a locally compact group G, let $S(G)$ be a symmetric Segal algebra. We prove that $S(G)$ is an ideal in its second dual space if and only if G is compact, where the second dual is equipped with an Arens multiplication.

1. Introduction

Let A be an arbitrary Banach algebra. On the second dual $A^{\star\star}$ of A may be equipped two Banach algebra multiplication, known as first and second Arens multiplication [1,2,5] each of which is an extension of the original multiplication in A as canonically embedded in $A^{\star\star}$. From now on, we shall denote by $A^{\star\star}$, the algebra $A^{\star\star}$ equipped with the first Arens multiplication and consider A as subalgebra of $A^{\star\star}$.

Let G be a locally compact group, $L^1(G)$ the group algebra of G. K. P. Wong in [19] has proved that if G is a compact group, then $L^1(G)$ is an ideal in its second dual. For the converse of Wong's result S. Watanabe gave two different proofs [17,18] (the case G abelian had earlier been proved by P. Civin [4]). Other proofs were also provided by M. Grosser [9], D. L. Johnson [12], A. Ülger ([16], Prop. 4.8) and J. Duncan and A. Ülger ([6], Prop. 2.5). In [7] F. Ghahramani has extended this results to weighted group algebras.

In this note we find a necessary and sufficient condition for a symmetric Segal algebra to be an ideal in its second dual space. This generalizes the above-mentioned result for group algebras.

2. Preliminaries

Throughout, G will be a locally compact group and $d\mathrm{g}$ a fixed left Haar measure on G. In the Banach algebra $L^1(G)$ we have the left translation operator L_{g} and right translation operator R_{g} defined by

$$L_{\mathrm{g}}f(s) = f(\mathrm{g}^{-1}s), R_{\mathrm{g}}f(s) = \Delta(\mathrm{g}^{-1})f(s\mathrm{g}^{-1}),$$

where $\Delta(\mathrm{g})$ is the modular function of G. We recall also that, for any $f \in L^1(G)$, f^v and $\tilde{f}$ are defined by $f^v(\mathrm{g})=f(\mathrm{g}^{-1})$ and $\tilde{f}(\mathrm{g})=\Delta(\mathrm{g}^{-1})f(\mathrm{g}^{-1})$.

A linear subspace of $L^1(G)$ is said to be a Segal algebra, and denoted by $S(G)$, if it satisfies the following conditions (1)-(4), [14,15].

(1) $S(G)$ is dense in $L^1(G)$

(2) $S(G)$ is a Banach space under some norm $\| \,.\, \|_S$ and

$$\| f \|_1 \leq C \| f \|_S$$

for all $f \in S(G)$ and for some constant $C > 0$.

(3) $S(G)$ is left norm-invariant: $f \in S(G) \Rightarrow L_{\mathrm{g}}f \in S(G)$ and $\| L_{\mathrm{g}}f \|_S = \| f \|_S$ for all $f \in S(G)$ and all $\mathrm{g} \in G$.

(4) The mapping $\mathrm{g} \rightarrow L_{\mathrm{g}}f$ of G into $S(G)$ is continuous.

"Right-hand" versions of (3) and (4) are the following conditions.

(3′) $S(G)$ is right norm-invariant: $f \in S(G) \rightarrow R_{\mathrm{g}} f \in S(G)$ and $\| R_{\mathrm{g}}f \|_S = \| f \|_S$ for all $f \in S(G)$ and all $\mathrm{g} \in G$.

(4′) The mapping $\mathrm{g} \rightarrow R_{\mathrm{g}}f$ of G into $S(G)$ is continuous.

A Segal algebra is said to be symmetric if it satisfies (3′) and (4′). About Segal algebras, ample information can be found in H. Reiter's books [14, 15]. We now give some concrete examples of Segal algebras [14, 15].

(i) The continuous functions in $L^1(G)$ that vanish at infinity form a Segal algebra, the norm being defined by

$$\| f \|_S = \| f \|_1 + \| f \|_\infty$$

(ii) The algebra $L^1(G) \cap L^p(G)(1 < p < \infty)$, equipped with the norm

$$\| f \|_S = \| f \|_1 + \| f \|_1 + \| f \|_p$$

is a Segal algebra.

The examples (i) and (ii) are symmetric Segal algebras if and only if G is unimodular ([15], p. 24).

(iii) Let G be an abelian group with character group $\hat{G}$, For $1 < p < \infty, A_p(G)$ denotes the set of all $f \in L^1(G)$ whose Fourier transforms $\hat{f}$ are in $L^p(\hat{G})$. A_p(G) is a (symmetric) Segal algebra with the norm

$$\| f \|_{A_p} = \| f \|_1 + \| \hat{f} \|_p$$

Any Segal algebra $S(G)$ is a left Banach $L^1(G)$ - convolution module, that is, if $h \in L^1(G)$ and $f \in S(G)$, then $h\star\ f \in S(G)$ and

$$\| h \star f \|_S \leq \| h \|_1 + \| f \|_S \quad f \in S(G), h \in L^1(G).$$

In particular, $S(G)$ is a Banach subalgebra of $L^1(G)$ under $\| \, . \, \|_S$. If S(G) is symmetric, then S(G) is also a right Banach $L^1(G)$ - convolution module. Since $L^1(G)$ has a bounded (two-sided) approximate identity, it follows from the Cohen-Hewitt factorization theorem [11,32.22] that, if $S(G)$ is a symmetric Segal algebra, then

$$S(G) = L^1(G) \star S(G) = S(G) \star L^1(G).$$

On the other hand, we see that if $S(G)$ is a symmetric Segal algebra, then $L^1(G)$ is a Banach $S(G)$ - convolution bimodule. Using Cohen-Hewitt factorization theorem again one can see that $S(G)$ cannot have bounded (in the Segal norm) approximate identity (left or right) unless $S(G) = L^1(G)$. However, a symmetric Segal algebra has approximate (two-sided) identity that have L^1-norm one ([15], p.34). Later on, we shall consider symmetric Segal algebras only.

Next, we racall definitions of some functions spaces which we shall use in this note.

Let $C(G)$ be the space of bounded continous complex-valued functions on G with sup-norm and $C_0(G)$ the subspace of $C(G)$ consisting of functions vanishing at infinity. By $C_{lu}(G)$, $C_{ru}(G)$ and $C_u(G)$ we denote in order, the subspaces of $C(G)$ consisting of the left, right and both left and right uniformly continuous functions on G. It is well known ([11], 32. 45) that

$$C_{lu}(G) = L^1(G) \star L^\infty(G), C_{ru}(G) = L^\infty(G) \star L^1(G)^v,$$

where $L^1(G)^v = \{f^v | f \in L^1(G)\}$. By $WAP\ (G)$ we denote the subspace of $C(G)$ consisting of the weakly almost periodic functions on G. It is well known ([3], p. 42, Theorem 3.11) that $WAP(G)$ is a (norm) closed linear subspace of $C_u(G)$. Burckel ([3], p. 68, Theorem 4.10) proved that $C(G) = WAP(G)$ if and only if G is compact. In [8] Granirer provided the following improvement of this result: $C_u(G) = WAP(G)$ if and only if G is compact.

3. The main result

The main result of this note is the following theorem.

Theorem. *A symmetric Segal algebra S(G) is a right (resp. left) ideal in its second dual algebra if and only if G is compact.*

For the proof of the theorem we need some premilinary results. If X is a Banach space, we denote by $X^\star$ its dual and by $X_{(1)}$ its closed unit ball. For x in X and φ in $X^\star$, we denote by $< \varphi, x >$ the natural duality between X and $X^\star$. Now, let $S(G)$ be a Segal algebra. It follows from (1) and (2) that, $L^\infty(G)$ can in natural way be embedded in $S(G)^\star$, that is if $\varphi \in L^\infty(G)$, then $\varphi \in S(G)^\star$ and

$$\parallel \varphi \parallel_{S^\star} \leq C \parallel \varphi \parallel_\infty .$$

Moreover, if f$\in$ $S(G)$ then we have

$$< \varphi, f >= \int_G \varphi(\mathrm{g}) f(\mathrm{g}) d\mathrm{g}, \quad \varphi \in L^\infty(G).$$

As we have seen before if $S(G)$ is a symmetric Segal algebra, then $S(G)$ is a Banach $L^1(G)$ - convolution bimodule. It follows that $S(G)^\star$ is a Banach $L^1(G)$ -bimodule under the adjoint action defined by

$$\begin{aligned} < h \circ \varphi, f > &= < \varphi, f \star h >, \\ < \varphi \circ h, f > &= < \varphi, h \star f >, \end{aligned}$$

where $h \in L^1(G)$, $f \in S(G)$, $\varphi \in S(G)^\star$. It is easily verified that

$$\begin{aligned} (h \star f) \circ \varphi &= h \circ (f \circ \varphi) \\ \varphi \circ (f \star h) &= (\varphi \circ f) \circ h, \end{aligned}$$

$h \in L^1(G)$, $f \in S(G)$, $\varphi \in S(G)^\star$.

An easy calculation will show that, for φ in $L^\infty(G)$ and f in $L^1(G)$, $f \circ \varphi = \varphi \star f^v$ and $\varphi \circ f = \tilde{f} \star \varphi$.

By $S(G) \circ S(G)^\star$, $S(G)^\star \circ S(G)$ and $S(G)^\star$ $S(G)$ we denote, respectively, the sets $\{f \circ \varphi | f \in S(G), \varphi \in S(G)^\star\}$, $\{\varphi \circ f | \varphi \in S(G),^\star f \in S(G)\}$ and $\{f \star h | f, h \in S(G)\}$. Put $S(G)^v = \{f^v | f \in S(G)\}$.

Lemma 1.a) *$S(G) \circ$ $S(G)^\star$ is in C_{ru} (G) and moreover*

$$\overline{S(G) \circ S(G)^\star}^{\|\cdot\|_\infty} = C_{ru}(G)$$

l S(G)$^\star \circ$ S(G) is in C_{lu} (G) and moreover

$$\overline{S(G)^\star \circ S(G)^\star}^{\|\cdot\|_\infty} = C_{lu}(G)$$

Here "$-\| \,.\, \|_\infty$" denotes the sup-norm closure.

Proof. a). Assume that $f \in S(G)$ and $\varphi \in S(G)^\star$. First we observe that $f \circ \varphi$ is in $L^\infty(G)$. From the equality $< f \circ \varphi, h > = < \varphi, h \star f >$, where $h \in S(G)$, we have □

$$| < f \circ \varphi, h > | \leq \| \varphi \|_{S^\star} \| h \star f \|_S \leq \| \varphi \|_{S^\star} \| f \|_S \| h \|_1 .$$

This inequality shows that $f \circ \varphi$ is bounded on $S(G)$ for the norm of $L^1(G)$. Hence, $S(G)$ being dense in $L^1(G)$, $f \circ \varphi$ can be extended in a unique way to $L^\infty(G)$. Thus $f \circ \varphi$ can be considered as an element of $L^\infty(G)$. On the other hand, since $S(G) = L^1(G) \star S(G)$, f can be represented as $f = h \star k$, where $h \in L^1(G)$, $k \in S(G)$. Now we have

$$f \circ \varphi = (h \star k) \circ \varphi = h \circ (k \circ \varphi).$$

Since $k \circ \varphi \in L^\infty(G)$, this gives

$$f \circ \varphi = (k \circ \varphi) \star h^v.$$

From this we deduce that $f \circ \varphi \in C_{ru}(G)$ for all $f \in S(G)$ and $\varphi \in S(G)^\star$. Thus we have the following:

$$\overline{S(G) \circ S(G)^\star}^{\|\cdot\|_\infty} \subset C_{\ell u}(G).$$

To prove the opposite inclusion it is enough to show that

$$\subset C_{\ell u}(G) L^\infty(G) \star L^1(G)^v \subset \overline{S(G) \circ L^\infty(G)}^{\|\cdot\|_\infty} = \overline{L^\infty(G) \star S(G)^v}^{\|\cdot\|_\infty}$$

Since $S(G)$ is dense in $L^1(G)$, it remains to observe that if a sequence (f_n) in S(G) converges (in the L^1-norm) to some $f \in L^1(G)$, then $\varphi \star f_n^v \to \varphi \star f^v$ uniformly for all $\varphi \in L^\infty(G)$.

The proff of b) is similar. ■

Now, let A be an arbitrary Banach algebra and a is an element of A. By $R_a : A \to A$ (resp. $L_a : A \to A$) we denote the right multiplication operator (left multiplication operator) defined by $R_a(b) = ba (L_a(b) = ab)$. The following lemma was proved in [5,7].

Lemma 2. *A is a right (resp. left) ideal of $A^{\star\star}$ if and only if, for each $a \in A$, the right multiplication operator R_a (the left multiplication operator L_a) is weakly compact.*

Now we can prove the main result of this note.

Proof of Theorem Assume that $S(G)$ is a right ideal in its second dual algebra. Then by Lemma 2 the set $\ell \star f \parallel \ell \parallel_S \leq 1$ is relatively weakly compact in $S(G)$ for all $f \in S(G)$. Assume first that f is of the form: $f = h \star k$ where, $h, k \in S(G)$. Without loss of generality we can suppose that $\parallel h \parallel_S \leq 1$. Since $L_{\mathrm{g}} f = L_{\mathrm{g}}(h \star k) = L_{\mathrm{g}} h \star k$ and since $\parallel L_{\mathrm{g}} h \parallel_S = \parallel h \parallel_S \leq 1$ we have

$$\{L_{\mathrm{g}} f | g \in G\} \subset \{\ell \star k| \parallel \ell \parallel_S \leq 1\}$$

It follows that, the set $\{L_{\mathrm{g}} f | g \in G\}$ is relatively weakly compact for all f in $S(G) \star S(G)$. Since g$\rightarrow L_{\mathrm{g}}$ is a (continuous) representation of G on $S(G)$, by Eberlein theorem ([3], p.36, Theorem 3.1), the function $g \rightarrow < \varphi L_{\mathrm{g}} f >$ is in $WAP(G)$ for all $\varphi \in S(G)^{\star}$ and $f \in S(G) \star S(G)$. Moreover, since $S(G)$ has an approximate identity, for any $f \in S(G)$, there is a sequence (f_n) in $S(G) \star S(G)$ such that $f_n \rightarrow f$ in the Segal norm. This implies that $< \varphi, L_{\mathrm{g}} f_n > \rightarrow \varphi, L_{\mathrm{g}} f >$ uniformly. From this it follows that the function $g \rightarrow < \varphi, L_{\mathrm{g}} f >$ is in $WAP(G)$ for all $\varphi \in S(G)^{\star}$ and $f \in S(G)$. Now, we claim that $< \varphi, L_{\mathrm{g}} f > = (f \circ \varphi)$ (g). In fact, for given any $h \in S(G)$ we can write

$$\int_G h(\mathrm{g}) < \varphi, L_{\mathrm{g}} f > d\mathrm{g} = < \varphi, \int_G h(\mathrm{g}) L_{\mathrm{g}} f d\mathrm{g} >$$

$$= < \varphi, h \star f > = < f \circ \varphi, h > = \int_G h(\mathrm{g})(f \circ \varphi)(\mathrm{g}) d\mathrm{g}.$$

Since $S(G)$ is dense in $L^1(G)$ and since the functions $g \rightarrow < \varphi, L_{\mathrm{g}} f >$ and g $\rightarrow (f \circ \varphi)(\mathrm{g})$ are hoth continuous, the last equality clearly implies that

$$< \varphi, L_{\mathrm{g}} f > = (f \circ \varphi)(\mathrm{g}).$$

Further, by Lemma 1 we have

$$\overline{C_{ru}(G) = S(G) \circ S(G)^{\star}}^{\parallel \cdot \parallel_\infty} \subset WAP(G),$$

and consequently $C_u(G) = WAP(G)$. However, this equality is possible only if G is compact [8].

Now assume that G is a compact group. We shall prove that the right multiplication operator $R_f : h \to h \star f$ is compact on $S(G)$ for all $f \in S(G)$. Before begining the proof, we recall the following fact which is an immediate consequense of the Peter-Weyl theory: If T is a (consuniuous) representation of the compact group G on some Banach space X, then X is a closed linear span of finite dimensional invariant subspaces of T ([13], p.91, Corollary 1).

Now, since $\mathrm{g} \to L_{\mathrm{g}}$ is a (continuous) representation of G on $S(G)$, by virtue of the above-mentioned fact, is a closed linear span of finite dimensional invariant subspaces of $L_{\mathrm{g}}(\mathrm{g} \in G)$. Let J be an invariant (closed) subspace of L_{g} and let $f \in J$. By the very definition of vector-valued integral we have

$$h \star f = \int_G h(\mathrm{g}) L_{\mathrm{g}} f d\mathrm{g} \in J, \qquad h \in S(G)$$

Hence J is a (closed) left ideal of $S(G)$. Thus we see that, actually, $S(G)$ is a closed linear span of finite dimensional left ideals. This means that, for given any $f \in S(G)$ and $\epsilon > 0$, there exist finite dimensional left ideals $J_1, \ldots, J_n$ and $f_1 \in J_1, \ldots, f_n \in J_n$ such that

$$\| f - \sum_{i=1}^{n} f_i \| < \epsilon$$

This implies that

$$\| R_f - \sum_{i=1}^{n} R_{fi} \| < \epsilon$$

Since $J_i (i = 1, \ldots, n)$ are finite dimensional left ideals, $R_{f_i} (i = 1, \ldots, n)$ are finite rank operators. On the other hand, the preceding inequality show that R_f can be approximate (in the operator norm) by finite rank operators. From this we conclude that R_f is a compact operator.

"Right-version" of this arguments shows that $S(G)$ is a left ideal in its second dual if and only if G is compact. The proof is complete.

As an immediate corollary of the Theorem we have the following results.

Corollary 3. *a) The algebra* $L^1(G) \cap C_0(G)$ *equipped with the norm* $\| f \| = \| f \|_1 + \| f \|_\infty$ *is a right (resp. left) ideal in its second dual space if and only if G is compact.*
b) The algebra $L^1(G) \cap L^p(G)(1<p<\infty)$ *equipped with the norm*

$\| f \| = \| f \|_1 + \| f \|_p$ *is a right (resp. left) ideal in its second dual space if and only if G is compact.*

Notice that, if G is compact, then $L^1(G) \cap L^p(G)(1<p<\infty)$ *is iqual to* $L^p(G)$ *which is a reflexive.*

Corollary 4. *The algebra* $A_p(G)$ *is an ideal in its second dual if and only if G is compact.*

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