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SOME RESULTS ON SPACE-LIKE LINE CONGRUENCES AND THEIR SPACE-LIKE PARAMETER RULED SURFACES

E. Özyılmaz

Abstract

Using dual vectors, space-like ruled surfaces was introduced in [2]. In this paper, we define space-like line congruence and their space-like parameter ruled surfaces $\mathbf{R_{11}(t)}$ and $\mathbf{R_{21}(t)}$ of R_1^3. Then, by choosing space-like parameter ruled surfaces as space-like principle ruled surfaces, we obtain some relations among the magnitudes of space-like ruled surfaces $\mathbf{R_1(t)}$, $\mathbf{R_{11}(t)}$.

1. Introduction

Let $A = a + \epsilon a_0$ be a dual number, $A \in ID = \{(a, a_0) \mid a, a_0 \in R\}$ and ID be a commutative ring with a unit element. We call the dual number $\epsilon = (0,1) \in ID$ dual unit and $\epsilon^2 = (0,0).(ID^3, +)$ is a module on the dual number ring. We call it ID-module, and dual vectors are the elements of this modul. We denote dual unit vector $\mathbf{A}$ as

$$\mathbf{A} = (\mathbf{a}, \mathbf{a_0}) = \mathbf{a} + \epsilon \mathbf{a_0} \quad , \quad \mathbf{aa = 1, aa_0 = 0}, \tag{1}$$

where $\mathbf{a}, \mathbf{a_0} \in \mathbf{IR^3}$. The dual vectors with unit length correspond to oriented lines of E^3 [1].

Theorem 1.1. *The oriented lines in IR^3 are in one-to-one correspondence with the points of the dual unit sphere in ID^3.*

The scalar product of two dual vectors $\mathbf{A} = (\mathbf{a}, \mathbf{a_0}) = \mathbf{a} + \epsilon\mathbf{a_0}$ *and* $\mathbf{B} = (\mathbf{b}, \mathbf{b_0}) = \mathbf{b} + \epsilon\mathbf{b_0}$ *is*

$$\mathbf{AB} = \mathbf{a}.\mathbf{b} + \epsilon(\mathbf{ba_0} + \mathbf{ab_0}) = \cos\varphi - \epsilon\varphi^{\star}\sin\varphi, \tag{2}$$

where, φ *is the real angle between dual unit vectors* $\mathbf{A}$ *and* $\mathbf{B}$ *and* $\varphi^{\star}$ *is the shortest distance between the lines.*

The Blaschke trihedron (A_1, A_2, A_3) *depends on the ruled surface striction of* $\mathbf{A_1}(\mathbf{t})$ *in dual space* D^3 *[1]. According to this, the first axis* $\mathbf{A_1}$ *of the trihedron is the generator which passes from the striction point of the ruled surface, the second axis* $\mathbf{A_2}$ *is the surface normal at this point and finally the third axis* $\mathbf{A_3}$ *is the tangent of the striction line at this point.*

2. Dual Lorentzian Space D_1^3

Let we consider vector space R_1^3 of R^3 provided with Lorentzian inner product of signature (+,+,-). For any vector $\mathbf{a}=(a_1, a_2, a_3)$ of R_1^3 ;

i) if $\langle a, a\rangle > 0$, $\mathbf{a}$ is said to be space-like,

ii) if $\langle a, a\rangle < 0$, $\mathbf{a}$ is said to be time-like,

iii) if $\langle a, a\rangle = 0$, $\mathbf{a}$ is said to be light-like (null)

The Lorentzian and hyperbolic sphere of radius 1 in R_1^3 are defined by

$$S_1^2 = \{a = (a_1, a_2, a_3) \in R_1^3, \langle a, a\rangle = 1\} \tag{3}$$

$$H_0^2 = \{a = (a_1, a_2, a_3) \in R_1^3, \langle a, a\rangle = -1\} \tag{4}$$

respectively.

By considering the Lorentzian inner product, we may write inner product of $\mathbf{A}$ and $\mathbf{B}$ as follows:

$$\mathbf{AB} = \mathbf{a}.\mathbf{b} + \epsilon(\mathbf{ba_0} + \mathbf{ab_0}) \tag{5}$$

We call it dual Lorentzian space which is defined and denote by ID_1^3.

Definition 2.1. *Let* $\mathbf{A} = (\mathbf{a}, \mathbf{a_0}) = \mathbf{a} + \epsilon\mathbf{a_0} \in ID_1^3$. *The dual vector* $\mathbf{A}$ *is said to be space-like if the vector* $\mathbf{a}$ *is space-like, time-like if the vector* $\mathbf{a}$ *is time-like, and light-like (or null) if the vector* $\mathbf{a}$ *is light-like.*

We also defined the time orientation as follows:

A time-like vector $\mathbf{A} = \mathbf{a} + \epsilon\mathbf{a_0}$ *is future pointing if the vector* $\mathbf{a}$ *is future pointing.*

Definition 2.2. *Let* $\mathbf{A}$, $\mathbf{B} \in ID_1^3$. *We define the Lorentzian cross product of* $\mathbf{A}$ *and* $\mathbf{B}$ *by*

$$A \wedge B = \begin{vmatrix} E_1 & E_2 & -E_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{6}$$

where, $\mathbf{A} = (A_1, A_2, A_3), \mathbf{B} = (B_1, B_2, B_3)$ *and* $E_1 \wedge E_2 = E_3, E_2 \wedge E_3 = -E_3, E_3, E_3 \wedge E_1 = -E_2$, *[2].*

3. Space-Like Congruence in Dual Lorentzian Space ID_1^3

A space-like line congruence in line space R_1^3 can be represented by a unit space like dual vector which is depending on two real parameters u and v as follows:

$$\mathbf{R}(u, v) = \mathbf{r}(u, v) + \epsilon\mathbf{r}^\star(u, v) \quad , \quad \mathbf{R^2} = 1 \tag{7}$$

The dual arc element of a space like ruled surface of space-like congruence can be given as

$$\begin{aligned} dS^2 &= d\mathbf{R}^2 = (\mathbf{R}_u du + \mathbf{R_v} dv)^2 \\ &= -Edu^2 + 2Fdudv + Gdv^2 \end{aligned} \tag{8}$$

Where

$$
\begin{array}{lll}
E=-R_u^2 & F=R_uR_v & G=R_v^2 \\
E=e+\epsilon e^\star & F=f+\epsilon f^\star & G=g+\epsilon g^\star \\
 & & \\
e=-r_u^2 & f=r_ur_v & g=r_v^2 \\
e^\star=2r_ur_u^\star & f^\star=r_ur_v^\star & g^\star=2r_vr_v^\star
\end{array}
\tag{9}
$$

The differential forms I and II of the space-like line congruence are

$$I=-edu^2+2fdudv+gdv^2$$

$$II=-e^\star du^2+2f^\star dudv+g^\star dv^2 \tag{10}$$

respectively. If we use the relations (8),(9) and (10), we have

$$dS^2=I+\epsilon II \tag{11}$$

and the dral of a space-like ruled surface of space like congruence can write as

$$\frac{1}{d}=\frac{II}{2I} \tag{12}$$

Definition 3.3. *Let $\frac{1}{d_1}$ and $\frac{1}{d_2}$ be extremum values of the dral. These values are called principle drals.*
The principle drals can calculated following relation:

$$\begin{vmatrix} -edu+fdv & -e^\star du+f^\star dv \\ fdu+gdv & f^\star du+g^\star dv \end{vmatrix} \tag{13}$$

Thus, we may write mean dral and Gaussian dral as follows, respectively:

$$h=\frac{1}{2}(\frac{1}{d_1}+\frac{1}{d_2}) \tag{14}$$

$$k = \frac{1}{d_1 d_2} \tag{15}$$

Definition 3.4. *The space-like ruled surfaces which are obtained by the relation (13) of the space like congruence are called the space-like principle ruled surfaces.*

Definition 3.5. *The space-like ruled surfaces u=constant and v=constant of a space-like line congruence are called the space-like parameter ruled surfaces.*

4. The Relations Among The Magnitudes of the Space Like Ruled Surfaces $\mathbf{R_1}, \mathbf{R_{11}}$ and $\mathbf{R_{21}}$

Let we consider a space-like ruled surface **R=R**(t) of the space-like congruence **R=R**(u,v), Where, u and v are functions of t.

Let we write the space-like parameter ruled surfaces as

$$\mathbf{R_{11}} = \mathbf{R_{11}}(u, v_0) \text{and} \mathbf{R_{21}} = \mathbf{R_{21}}(u_0, v) \tag{16}$$

The space-like ruled surfaces $\mathbf{R_1}, \mathbf{R_{11}}$ and $\mathbf{R_{21}}$ have common space-like line which is defined by the following relation:

$$\mathbf{R_0} = \mathbf{R}(u_0, v_0) = \mathbf{R_1}(u_0, v_0) = \mathbf{R_{21}}(u_0, v_0) \tag{17}$$

Blaschke trihedrons of these space-like ruled surfaces are in the following form:

$$(\mathbf{R_0} = \mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}) \quad , (\mathbf{R_0} = \mathbf{R_{11}}, \mathbf{R_{12}}, \mathbf{R_{13}}) \quad , (\mathbf{R_0} = \mathbf{R_{21}}, \mathbf{R_{22}}, \mathbf{R_{23}}) \tag{18}$$

Where $\mathbf{R_2}, \mathbf{R_{12}}$ and $\mathbf{R_{23}}$ are time-like, $\mathbf{R_{13}}, \mathbf{R_3}, \mathbf{R_{22}}$ and $\mathbf{R_0}$ are space-like. Thus we may write

$$\begin{aligned}
&\mathbf{R_1^2} = \mathbf{R_3^2} = \mathbf{1}, \mathbf{R_2^2} = -\mathbf{1}\mathbf{R_3} \wedge \mathbf{R_1} = \mathbf{R_2}, \mathbf{R_2} \wedge \mathbf{R_3} = -\mathbf{R_1}, \mathbf{R_1} \wedge \mathbf{R_2} = -\mathbf{R_3} \\
&\mathbf{R_{11}^2} = \mathbf{R_{13}^2} = \mathbf{1}, \mathbf{R_{12}^2} = -\mathbf{1}\mathbf{R_{13}} \wedge \mathbf{R_{11}} = +\mathbf{R_{12}}, \mathbf{R_{12}} \wedge \mathbf{R_{13}} = -\mathbf{R_{11}}, \mathbf{R_{11}} \wedge \mathbf{R_{12}} = -\mathbf{R_{13}} \\
&\mathbf{R_{21}^2} = \mathbf{R_{22}^2} = \mathbf{1}, \mathbf{R_{23}^2} = -\mathbf{1}\mathbf{R_{23}} \wedge \mathbf{R_{21}} = -\mathbf{R_{22}}, \mathbf{R_{22}} \wedge \mathbf{R_{23}} = -\mathbf{R_{21}}, \mathbf{R_{21}} \wedge \mathbf{R_{22}} = \mathbf{R_{23}}
\end{aligned} \tag{19}$$

On the other hand, if we choose the space-like paremeter ruled surfaces as space like principle ruled surfaces, we may write f=0 and $f^{\star}=0$. Thus,

$$F=0, \quad \mathbf{R_u}.\mathbf{R_v}=0 \tag{20}$$

can write.

The dual arc elements of the space-like ruled surfaces $\mathbf{R_1}, \mathbf{R_{11}}$ and $\mathbf{R_{21}}$ can be given respectively as

$$dS=Pdt \quad, dS_1=P_1 du \quad, dS_2=P_2 dv \tag{21}$$

where

$$P_1=\sqrt{\mid \mathbf{R_u^2} \mid}=\sqrt{-\mathbf{R_u^2}}=\sqrt{E}P_2=\sqrt{\mathbf{R_v^2}}=\sqrt{\mathbf{G}} \quad P=\sqrt{\mathbf{R_1'^2}} \tag{22}$$

$$\mathbf{R_1'}=\mathbf{R}_u\frac{du}{dt}+\mathbf{R_v}\frac{dv}{dt}$$

Using (21) and (22), we have

$$dS_1=\sqrt{E}du \text{and} dS_2=\sqrt{G}dv \tag{23}$$

The derivative formulas of these Blaschke trihedrons, defined by (18), are

$$\begin{aligned}
&\mathbf{R_1'}=\mathbf{PR_2}\mathbf{R_2'}=\mathbf{PR_1}+\mathbf{QR_3}\mathbf{R_3'}=\mathbf{QR_2}\\
&\mathbf{R_{11}'}=\mathbf{P_1R_{12}}\mathbf{R_{12}'}=\mathbf{P_1R_{12}}+\mathbf{Q_1R_{13}}\mathbf{R_{13}'}=\mathbf{Q_1R_{12}}\\
&\mathbf{R_{21}'}=\mathbf{P_2R_{22}}\mathbf{R_{22}'}=-\mathbf{P_2R_{21}}-\mathbf{Q_2R_{23}}\mathbf{R_{23}'}=-\mathbf{Q_2R_{22}}
\end{aligned} \tag{24}$$

Thus, the Blaschke vectors of the Blaschke trihedrons can be given by the following relations:

$$\mathbf{B}=-\mathbf{QR_0}+\mathbf{PR_3} \quad, \mathbf{B_1}=-\mathbf{Q_1R_0}+\mathbf{P_1R_{13}} \quad, \mathbf{B_2}=-\mathbf{Q_2R_0}-\mathbf{P_2R_{23}} \tag{25}$$

respectively.

The dual unit vectors $\mathbf{R_2}, \mathbf{R_{12}}$ and $\mathbf{R_{22}}$ which are the second edges of the trihedrons (18), can be written as

$$\mathbf{R_2} = \frac{\mathbf{R_1'}}{\mathbf{P}} = \frac{\mathbf{1}}{\mathbf{P}}(\mathbf{R_u}\frac{\mathbf{du}}{\mathbf{dt}} + \mathbf{R_v}\frac{\mathbf{dv}}{\mathbf{dt}}) \tag{26}$$

$$\mathbf{R_{12}} = \frac{\mathbf{R_{11}'}}{\mathbf{P_1}} = \frac{\mathbf{R_u}}{\mathbf{P_1}} = \frac{\mathbf{R_u}}{\sqrt{\mathbf{E}}} \tag{27}$$

$$\mathbf{R_{22}} = \frac{\mathbf{R_{21}'}}{\mathbf{P_2}} = \frac{\mathbf{R_v}}{\mathbf{P_2}} = \frac{\mathbf{R_v}}{\sqrt{\mathbf{G}}} \tag{28}$$

Using the relation

$$\mathbf{R} = \frac{\mathbf{R_u x R_v}}{\| \mathbf{R_u} \times \mathbf{R_v} \|} \tag{29}$$

, we can express the relation (29) for the common space-like line $\mathbf{R_0}$ as

$$\mathbf{R_0} = \frac{\mathbf{R}_u(u_0, v_0)x\mathbf{R}_v(u_0, v_0)}{\| \mathbf{R}_u(u_0, v_0)x\mathbf{R}_v(u_0, v_0) \|} = \mathbf{R}(u_0, v_0) \tag{30}$$

In our study, we will take the space-like parameter ruled surfaces as the space-like principle ruled surfaces. Thus, if we consider (20), (27) and (28), we have

$$\mathbf{R_{12}}.\mathbf{R_{22}} = 0 \tag{31}$$

From the relation below

$$\mathbf{R_{12}} \times \mathbf{R_{22}} = \frac{\mathbf{R_u} \times \mathbf{R_v}}{\sqrt{\mathbf{EG}}} = \mathbf{R_0} \tag{32}$$

and from (26), (27) and (28), we obtain

$$\mathbf{R_2} = \frac{\mathbf{P_1}}{\mathbf{P}}\frac{\mathbf{du}}{\mathbf{dt}}\mathbf{R_{12}} + \frac{\mathbf{P_2}}{\mathbf{P}}\frac{\mathbf{dv}}{\mathbf{dt}}\mathbf{R_{22}} \tag{33}$$

Thus, using the dual angle between the unit dual vectors $\mathbf{R_2}$ and $\mathbf{R_{12}}$, we may write

$$\mathbf{R_2} = ch\Phi\mathrm{R}_{12} + sh\Phi\mathrm{R}_{22} \tag{34}$$

where

$$ch\Phi = \frac{dS_1}{dS} = \sqrt{E}\frac{du}{dS} \quad , sh\Phi = \frac{dS_2}{dS} = \sqrt{G}\frac{dv}{dS} \tag{35}$$

$$tg\Phi = \frac{dS_2}{dS_1} = \sqrt{\frac{G}{E}}\frac{dv}{du} \tag{36}$$

and from (19) and (34)

$$-\mathbf{R_3} = \mathbf{R_0} \times \mathbf{R_2} = \mathbf{R_0}\times(ch\Phi\mathbf{R_{12}} + sh\Phi\mathbf{R_{22}}) \tag{37}$$

$$-\mathbf{R_3} = ch\Phi\mathbf{R_{13}} - sh\Psi\mathbf{R_{23}} \tag{38}$$

are obtained. Then, by the relations (19) and (32), we have

$$\mathbf{R_{12}} \times \mathbf{R_{13}} = -\mathbf{R_0}, \quad \mathbf{R_{22}} \times \mathbf{R_{23}} = -\mathbf{R_0} \quad , \mathbf{R_{12}} \times \mathbf{R_{22}} = \mathbf{R_0}$$

$$\mathbf{R_{12}} \times (\mathbf{R_{13}} + \mathbf{R_{22}}) = \mathbf{0} \rightarrow \mathbf{R_{13}} + \mathbf{R_{22}} = \mathbf{MR_{12}} \tag{39}$$

$$\mathbf{R_{22}} \times (\mathbf{R_{23}} - \mathbf{R_{12}}) = \mathbf{0} \rightarrow \mathbf{R_{23}} - \mathbf{R_{12}} = \mathbf{NR_{22}} \tag{40}$$

where M and N are dual scalars. Taking dot product of the both sides of (39) and (40) by the unit dual vectors $\mathbf{R_{12}}$ and $\mathbf{R_{22}}$, respectively, and considering the relations (18) and (31), we have M=0 and N=0. Then, if we insert the values M and N into (39) and (40), we have

$$\mathbf{R_{12}} = -\mathbf{R_{22}} \tag{41}$$

$$\mathbf{R_{23}} = \mathbf{R_{12}}$$

Finally, we obtain following teorem:

Theorem 4.2. *The edges of Blascke trihedrons of the space-like parameter ruled surfaces coincide with each other under the condition that their directions and orders are not the same.*

Result 1. The Blaschke vectors $\mathbf{B}$, $\mathbf{B_1}$ and $\mathbf{B_2}$ can be expressed following form by the vectors $\mathbf{R_{12}}$, $\mathbf{R_{22}}$ and $\mathbf{R_0}$.

It is clear that

$$\begin{aligned} \mathbf{B} &= -\mathbf{Q}\mathbf{R_0} + \mathbf{P}(-\mathbf{ch}\Phi\mathbf{R_{22}} - \mathbf{sh}\Phi\mathbf{R_{12}}) \\ \mathbf{B_1} &= -\mathbf{Q_1}\mathbf{R_0} - \mathbf{P_1}\mathbf{R_{22}} \\ \mathbf{B_2} &= -\mathbf{Q_2}\mathbf{R_0} - \mathbf{P_2}\mathbf{R_{12}} \end{aligned} \tag{42}$$

can be written easily from (38) and (41)

Result 2. If the trihedron $(\mathbf{R_0}, \mathbf{R_{12}} = \mathbf{R_{23}}, \mathbf{R_{13}} = -\mathbf{R_{22}})$ moves on the striction curves of the space-like parameter ruled surface $\mathbf{R_{11}}$, it changes as a function of dual arc S_1 of v=constant ruled surface. If the $(\mathbf{R_0}, \mathbf{R_{12}} = \mathbf{R_{23}}, \mathbf{R_{13}} = -\mathbf{R_{22}})$ moves on the striction curves of the space-likeparameter ruled surface $\mathbf{R_{11}}$, it changes as a function of dual arc S_2 of u=constant ruled surface. Thus, the edges of this trihedron are depend on two parameters

Theorem 4.3. *If we consider Blaschke trihedrons and their derivative formulaes of the space-like ruled surface which are determined by (18) and (24), we have*

$$\begin{aligned} &\frac{\partial \mathbf{R_0}}{\partial u} = \mathbf{B_1} \times \mathbf{R_0}, \frac{\partial \mathbf{R_{23}}}{\partial u} = \mathbf{B_1} \times \mathbf{R_{23}}, \frac{\partial \mathbf{R_{22}}}{\partial u} = \mathbf{B_1} \times \mathbf{R_{22}}, \\ &\frac{\partial \mathbf{R_0}}{\partial v} = \mathbf{B_2} \times \mathbf{R_0}, \frac{\partial \mathbf{R_{13}}}{\partial v} = \mathbf{B_2} \times \mathbf{R_{13}}, \frac{\partial \mathbf{R_{12}}}{\partial v} = \mathbf{B_2} \times \mathbf{R_{12}}, \end{aligned} \tag{43}$$

Proof. If we write derivative formulas of the edges of Blaschke trihedrons by the Blaschke vectors, we have

$$\begin{aligned} &\frac{\partial \mathbf{R_0}}{\partial u} = \mathbf{B_1} \times \mathbf{R_0}, \frac{\partial \mathbf{R_{12}}}{\partial u} = \mathbf{B_1} \times \mathbf{R_{12}}, \frac{\partial \mathbf{R_{13}}}{\partial u} = \mathbf{B_1} \times \mathbf{R_{13}}, \\ &\frac{\partial \mathbf{R_0}}{\partial v} = \mathbf{B_2} \times \mathbf{R_0}, \frac{\partial \mathbf{R_{22}}}{\partial v} = \mathbf{B_2} \times \mathbf{R_{22}}, \frac{\partial \mathbf{R_{23}}}{\partial v} = \mathbf{B_2} \times \mathbf{R_{23}}, \end{aligned} \tag{44}$$

If we insert (41) into (44), we get (43). □

Theorem 4.4. *If we use space-like paremeter ruled surfaces, we can write below form:*

$$\begin{aligned} \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_1} &= -\mathbf{R}_{22}\frac{\partial \mathbf{R}_{12}}{\partial S_1} = -\frac{(\sqrt{E})_v}{\sqrt{EG}} \\ \mathbf{R}_{22}\frac{\partial \mathbf{R}_{12}}{\partial S_2} &= -\mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_2} = \frac{(\sqrt{G})_u}{\sqrt{EG}} \end{aligned} \tag{45}$$

Proof. From (22) and by taking derivatives of $(\sqrt{E})^2 = -\mathbf{R}_u^2, (\sqrt{G})^2 = \mathbf{R}_v^2$, we have

$$\sqrt{E}(\sqrt{E})_v = -\mathbf{R}_u\mathbf{R}_{uv}, \sqrt{G}(\sqrt{G})_u = \mathbf{R}_v\mathbf{R}_{vu} \tag{46}$$

and taking derivatives of (27) and (28)

$$\frac{\partial \mathbf{R}_{12}}{\partial v} = \frac{\mathbf{R}_{uv}\sqrt{E} - (\sqrt{E})_v\mathbf{R}_u}{E}, \quad \frac{\partial \mathbf{R}_{22}}{\partial u} = \frac{\mathbf{R}_{uv}\sqrt{G} - (\sqrt{G})_u\mathbf{R}_v}{G} \tag{47}$$

are obtained respectively. Then, from the relations above (27), (28), (46) and (20).

$$\begin{aligned} \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial u} &= \frac{\mathbf{R}_u}{\sqrt{E}}\frac{\partial \mathbf{R}_{22}}{\partial u} = \frac{\mathbf{R}_u\mathbf{R}_{uv}}{\sqrt{EG}} = -\frac{(\sqrt{E})_v}{\sqrt{G}} \\ \mathbf{R}_{22}\frac{\partial \mathbf{R}_{12}}{\partial v} &= \frac{\mathbf{R}_v}{\sqrt{G}}\frac{\partial \mathbf{R}_{12}}{\partial v} = \frac{\mathbf{R}_v\mathbf{R}_{uv}}{\sqrt{EG}} = \frac{(\sqrt{G})_u}{\sqrt{E}} \end{aligned} \tag{48}$$

are faund. If we use (23) in (48), we have

$$\begin{aligned} \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_1} &= R_{12}\frac{\partial \mathbf{R}_{22}}{\partial u}\frac{1}{\sqrt{E}} = -\frac{(\sqrt{E})_v}{\sqrt{EG}} \\ \mathbf{R}_{22}\frac{\partial \mathbf{R}_{12}}{\partial S_2} &= R_{22}\frac{\partial \mathbf{R}_{22}}{\partial v}\frac{1}{\sqrt{G}} = \frac{(\sqrt{G})_u}{\sqrt{EG}} \end{aligned} \tag{49}$$

$$\begin{aligned} \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_1} &= -R_{22}\frac{\partial \mathbf{R}_{12}}{\partial S_1} \\ \mathbf{R}_{22}\frac{\partial \mathbf{R}_{22}}{\partial S_2} &= -R_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_2} \end{aligned} \tag{50}$$

□

Result 3. There is following relation between $\mathbf{R_{12}}$ and $\mathbf{R_{22}}$:

$$\mathbf{R}_{12}.d\mathbf{R}_{22} = -d\mathbf{R}_{12}.\mathbf{R}_{22} = -\frac{(\sqrt{E})_v}{\sqrt{G}}du - \frac{(\sqrt{G})_u}{\sqrt{E}}dv \tag{51}$$

Proof. If we differentiate (31) and consider (23) and (45), we obtain

$$\begin{aligned}
&-\mathbf{R}_{12}.d\mathbf{R}_{22} = \mathbf{R}_{12}d\mathbf{R}_{22} = \mathbf{R}_{12}(\frac{\partial \mathbf{R}_{22}}{\partial S_1}dS_1 + \frac{\partial \mathbf{R}_{22}}{\partial S_2}dS_2) \\
&= \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_1}\sqrt{E}du + \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_2}\sqrt{G}dv \\
&= -\frac{(\sqrt{E})_v}{\sqrt{EG}}\sqrt{E}du - \frac{(\sqrt{G})_u}{\sqrt{EG}}\sqrt{G}dv
\end{aligned} \tag{52}$$

Thus, we have the result as below. □

Result 4. There is following relation between $\mathbf{R_{12}}$ and $\mathbf{R_{22}}$:

$$\frac{\partial}{\partial v}(\mathbf{R}_{12}.\frac{\partial \mathbf{R}_{22}}{\partial u}) - \frac{\partial}{\partial u}(\mathbf{R}_{12}.\frac{\partial \mathbf{R}_{22}}{\partial v}) = -\frac{\partial}{\partial v}(\frac{(\sqrt{E})_v}{\sqrt{G}}) + \frac{\partial}{\partial u}(\frac{(\sqrt{G})_u}{\sqrt{E}}) \tag{53}$$

Proof. Taking derivative of (31) with respect to the parameters v and u

$$\mathbf{R}_{12}.\frac{\partial \mathbf{R}_{22}}{\partial v} = -\mathbf{R}_{22}.\frac{\partial \mathbf{R}_{12}}{\partial v} = -\frac{(\sqrt{G})_u}{\sqrt{E}}, R_{12}\frac{\partial R_{22}}{\partial u} = -\frac{(\sqrt{E})_v}{\sqrt{G}} \tag{54}$$

is written. Then, if we consider (48), (53) is obtain. □

Theorem 4.5. *There are following relations for the magnitudes* Q_1, Q_1 *and* Q *of the space-like ruled surfaces* $\mathbf{R_{11}}$, $\mathbf{R_{21}}$ *and* $\mathbf{R_1}$, *respectively.*

$$Q_1 = -\frac{(\sqrt{E})_v}{\sqrt{G}} \quad , \quad Q_2 = -\frac{(\sqrt{G})_u}{\sqrt{E}} \tag{55}$$

$$Q^2 = -sh^2\Phi(\Phi' - Q_2)^2 + ch^2\Phi(\Phi' - Q_1)^2 \tag{56}$$

Proof. If we consider relations (43) and (23), we have

$$\frac{\partial \mathbf{R}_{22}}{\partial S_1} = \frac{\partial \mathbf{R}_{22}}{\sqrt{E}} = \frac{1}{\sqrt{E}} \mathbf{B}_1 \times \mathbf{R}_{22}$$

Then, using the relation above and (45), (32) and (25), we may write

$$-\frac{(\sqrt{E})_v}{\sqrt{EG}} = \mathbf{R}_{12}.\frac{\partial \mathbf{R}_{22}}{\partial S_1} + \mathbf{R}_{12}\frac{\mathbf{B}_1 \times \mathbf{R_{12}}}{\sqrt{E}} = -\frac{\mathbf{R}_{12} \times \mathbf{R}_{22}}{\sqrt{E}}.\mathbf{B}_1 = -\frac{\mathbf{R}_0.\mathbf{B}_1}{\sqrt{E}} = \frac{Q_1}{\sqrt{E}}$$

By the same way and from the relations (45), (23), (44), (32) and (25), we have

$$\frac{(\sqrt{G})_u}{\sqrt{EG}} = -\mathbf{R}_{12}.\frac{\partial \mathbf{R}_{22}}{\partial S_2} = -\frac{1}{\sqrt{G}}\mathbf{R}_{12}.\frac{\mathbf{R}_{22}}{\partial v} = -\frac{\mathbf{R}_{12}.(\mathbf{B}_2 \times \mathbf{R}_{22})}{\sqrt{G}} = \frac{\mathbf{R}_0.\mathbf{B}_2}{\sqrt{G}} = -\frac{Q_2}{\sqrt{G}}$$

Finally, if we take derivative of the relation (38) by using the derivative formulas (24) and consider (41) and (31), it can be reached (55) and (56). □

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E. ÖZYILMAZ
Ege Üniversitesi, Fen Fakültesi,
Matematik Bölümü 35100,
Bornova, İzmir-TURKEY

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