

CYCLOIDAL NORMAL SUBGROUPS OF HECKE GROUPS OF FINITE INDEX

İ. Naci Cangül, Osman Bizim

Abstract

Cycloidal normal subgroups, that is, subgroups with only one cusp, of finite index of Hecke groups, which have been introduced by E. Hecke, and can be thought of as some kind of generalisation of the well-known modular group, are studied. It is shown that they correspond to cyclic quotients of Hecke groups and therefore have non-compact associated Riemann surfaces with a cusp. Finally, their total number have been formulated.

1. Introduction

Let G be a Fuchsian group of the first kind. i.e. let G be a subgroup of the group $\text{PSL}(2, \mathbf{R})$ of orientation preserving isometries of the upper half plane. Let N be a normal subgroup of G having finite index μ . We know that N has a signature $(g; m_1, \dots, m_r; t)$ as a Fuchsian group. Here g is a non-negative integer called the genus of the quotient surface on which N acts. m_1, \dots, m_r are called the periods of N and are the orders of the finite ordered elements. Here each m_k is an integer greater than 1. t is the parabolic class number of N which is the number of conjugacy classes of parabolic elements. In our case these are the elements of infinite order in N . When the quotient surface U/N is considered, the number t corresponds to so called cusps which can be thought as the holes on the surface going towards infinity. Finally we define the level n of N to be the least positive integer so that $T^n \in N$, where T is the parabolic generator having infinite order in the group G . It is known that these three numbers are connected to others by the relation

$$\mu = n.t. \tag{1}$$

Let Γ be the modular group of all linear fractional transformations

$$Y(z) = \frac{az + b}{cz + d}, \tag{2}$$

where $a, b, c, d \in \mathbb{Z}$ such that $ad-bc=1$. Γ is generated by

$$R(z) = -\frac{1}{z} \text{ and } S(z) = \frac{1}{z+1}. \tag{3}$$

Here $R^2 = S^3 = I$. We take $T=RS$. Then T is of infinite order in Γ .

Normal and non-normal subgroups of Γ have been studied widely, see, e.g. [5], [6], [7]. Our concern in this work is the ones with only one cusp, i.e. with $t=1$, which are called *cycloidal* by Petersson, [8].

Our aim in this paper is to classify the cycloidal normal subgroups of the Hecke groups $H(\lambda_q)$ which are just some kind of generalisation of the modular group Γ , generated by

$$R(z) = -\frac{1}{z} \text{ and } S(z) = \frac{1}{z + \lambda_q}, \tag{4}$$

where $\lambda_q = 2 \cos \pi/q$, $q \in \mathbb{N}$, $q \geq 3$. Note that for $q=3$, we have the modular group. Look at [1] and [2] for more information on Hecke groups.

Throughout this paper we denote by N a cycloidal normal subgroup of $H(\lambda_q)$ of finite index μ , having $t=1$ cusp and of level n . Therefore $\mu = n$.

In the modular group case, Millington, [4], studied them in terms of g , the genus, μ , the index, and, e_2 and e_3 , the numbers of inequivalent elliptic fixed point vertices of orders 2 and 3, respectively, in a fundamental region.

As $H(\lambda_q)$ has two generators R and S satisfying $R^2 = S^q = I$, it has the signature $(0; 2, q, \infty)$ as $T=RS$ is parabolic. Now let N have the signature $(g; m_1, \dots, m_r; 1)$. The relations between the two signatures are given in [1] and [9] as follows:

For $1 \leq i \leq k$, $m_i \mid 2$, and for $k+1 \leq i \leq r$, $m_i \mid q$. Note that in special cases k could be 0 or r . Here, g can be calculated by the Riemann-Hurwitz formula:

$$g = \frac{1}{2} \left[1 + \mu \frac{q-2}{2q} - \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right]. \tag{5}$$

Now let N be a normal subgroup of $H(\lambda_q)$ of finite index. We can form the quotient $H(\lambda_q)/N$. It acts on the Riemann surface U/N (see [3]). We say that U/N is the associated Riemann surface for the subgroup N . As $H(\lambda_q)$ and its normal subgroups of finite index have cusps, associated Riemann surfaces will be non-compact, as they are found to be so in the modular group case. By the genus g of N , we, in fact, mean the genus of the associated Riemann surface U/N .

2. Preliminaries.

Let N be as above with generators r and s . Then r either goes to identity or to a product of transpositions and s maps to a product of m -cycles, $m|q$, by the result quoted above. As N is cycloidal T will be represented by a μ -cycle in the permutation representation of N . It follows that m is also a divisor of μ . $H(\lambda_q)/N$ is isomorphic to $(1,m,\mu)$ or to $(2,m,\mu)$. Therefore $H(\lambda_q)/N$ must be a finite cyclic group of order μ . Hence the problem of finding the cycloidal normal subgroups of $H(\lambda_q)/N$ is equal to the one of finding the finite cyclic quotients of $H(\lambda_q)$. As $H(\lambda_q)$ is finitely generated, there are only finitely many ephimorphisms of $H(\lambda_q)$ onto finite cyclic groups. We now give a result characterising the cyclic groups as triangle groups:

Lemma 2.1. *Every cyclic group acting on the sphere can be reduced to one of the following representations:*

$$C_n \cong (1, n, n) \text{ or } C_{2m} \cong (2, m, 2m) \tag{6}$$

for any $n \in \mathbb{N}$ and for odd $m \in \mathbb{N}$.

Proof. Let first $n \in \mathbb{N}$. If the generators α and β are chosen such that $\alpha = \beta^n = I$, then $(\alpha\beta)^n = I$, and therefore the former one represents a cyclic group C_n .

Secondly, if the generators α and β are such that $\alpha^2 = \beta^m = I$ for odd m , then α alone generates C_2 and β generates C_m . We want to show that α and β together generates C_{2m} . Note that

$$C_{2m} \cong C_2 \times C_m \cong \langle \alpha, \beta \mid \alpha^2 = \beta^m = 1, \alpha\beta = \beta\alpha \rangle \quad (7)$$

is the direct product representation of $C_2 \times C_m$. Now

$$\begin{aligned} (\alpha\beta)^{2m} &= (\alpha\beta)(\alpha\beta)\dots(\alpha\beta) \\ &= (\beta\alpha)(\alpha\beta)\dots(\alpha\beta) \\ &= \beta^2 \dots \beta^2 \\ &= \beta^{2m} \\ &= I. \end{aligned} \quad (8)$$

This completes the proof.

Apart from these, there are some cyclic groups acting on a torus which are images of the three infinite triangle groups (2,3,6), (2,4,4) and (2,6,3).

We now begin by considering the cycloidal subgroups of the three important Hecke groups where $q=4, 5$ and 6 : □

3. Cycloidal normal subgroups of $H(\sqrt{2})$.

Theorem 3.1. *$H(\sqrt{2})$ has five cycloidal normal subgroups of finite index.*

Proof. Let N be a normal cycloidal subgroup of $H(\sqrt{2}) \cong \langle R, S \mid R^2 = S^4 = I \rangle$ of finite index μ . We have seen that $H(\sqrt{2})/N$ is isomorphic to a cyclic group of order μ . Therefore we only have to consider the cyclic quotients of $H(\sqrt{2})$ to find such N . By the discussion at section 1, $H(\sqrt{2})$ can only be mapped onto C_1, C_2 and C_4 , homomorphically. If we map it onto the trivial group C_1 , then N is the whole group $H(\sqrt{2})$. If it is mapped onto C_2 , then there are two possibilities:

$$\begin{aligned} R &\rightarrow (1)(2) & R &\rightarrow (12) \\ S &\rightarrow (12) & \text{or} & S \rightarrow (1)(2) \\ T &\rightarrow (12) & T &\rightarrow (12) \end{aligned} \quad (9)$$

In the first case since $R^2 = S^4 = I$ in $H(\sqrt{2})$, we obtain two 2's from the permutation representation of R and a 2 from S in the signature of N. These are all finite periods as R and S are the only elliptic generators of $H(\sqrt{2})$. As T is represented with one cycle, there is also an ∞ in this signature. The genus can be found to be 0 by the Riemann-Hurwitz formula.

Note that whenever we map $H(\sqrt{2})$ to $C_n \cong (1, n, n)$, the index must be n and hence $g=0$.

The signatures of the two subgroups are $(0 ; 2, 2, 2, \infty)$ and $(0 ; 4, 4, \infty)$, respectively, which are isomorphic to the abstract groups $C_2 \star C_2 \star C_2$ and $C_4 \star C_4$.

Finally if we map $H(\sqrt{2})$ onto C_4 we have

$$\begin{aligned} R &\rightarrow (1)(2)(3)(4) \\ S &\rightarrow (1234) \\ T &\rightarrow (1234) \end{aligned} \tag{10}$$

and we obtain $N \cong (0 ; 2, 2, 2, 2, \infty) \cong C_2 \star C_2 \star C_2 \star C_2$. Recall that $(2, 4, 4)$, $(2, 3, 6)$ and $(2, 6, 3)$ are surface groups which are infinite triangle groups. They can also be mapped to finite triangle groups. As we are only concerned with the cyclic groups, we are only interested in the cyclic images of these three infinite groups. As $(2, 3, 6)$ has already been listed in Lemma 2.1 with $m=3$, and $(2, 6, 3)$ gives normal subgroups with two cusps, the only exception is $(2, 4, 4)$. In this case $(2, 4, 4)$ has a cyclic image C_4 , giving a normal subgroup $N \cong (1; \infty)$. These are all required subgroups.

Note that here, unlike the case of modular group, we do not have the commutator subgroup $H'(\sqrt{2})$ as a cycloidal normal subgroup of $H(\sqrt{2})$. This is because, in $H(\sqrt{2})/H'(\sqrt{2})$, we have $(RS)^4=I$ and $\mu=8$, i.e. the parabolic $T=RS$ is represented by two 4-cycles in the permutation representation. That is, $H'(\sqrt{2})$ is a normal subgroup having two parabolic classes. Hence it cannot be cycloidal. \square

4. Cycloidal normal subgroups of $H(\lambda_5)$.

$H(\lambda_5)$ case shows some differences from the case $H(\sqrt{2})$ as 5 is odd.

Theorem 4.1. *$H(\lambda_5)$ has four cycloidal normal subgroups of finite index.*

Proof. By the discussion at section 1, $H(\lambda_5)$ can be mapped onto the cyclic groups C_1, C_2, C_5 and C_{10} , as 5 is odd. By Lemma 2.1, these are the only cyclic images of $H(\lambda_5)$. Mapping onto C_1 gives $N \cong H(\lambda_5)$. Now let us map onto $C_2 \cong (2, 1, 2)$. Then

$$\begin{aligned} R &\rightarrow (12) \\ S &\rightarrow (1)(2) \\ T &\rightarrow (12) \end{aligned} \tag{11}$$

giving $N \cong (0 ; 5, 5, \infty) \cong C_5 \star C_5$, which is known as $H^2(\lambda_5)$.

Thirdly, mapping onto C_5 , we obtain $N \cong (0 ; 2^{(5)}, \infty)$. Here $2^{(5)}$ denotes that there are five periods all equal to 2. This is $H^5(\lambda_5)$ defined similarly to $H^2(\lambda_5)$.

Finally if we map onto $C_{10} \cong (2, 5, 10)$, we obtain $(2 ; \infty)$ which is the commutator subgroup $H'(\lambda_5)$. □

5. Cycloidal normal subgroups of $H(\sqrt{3})$

Although the two groups $H(\sqrt{2})$ and $H(\sqrt{3})$ have many similarities, the properties depending on q are quite different as 4 is a prime square and 6 is composite.

Theorem 5.1. *$H(\sqrt{3})$ has six cycloidal normal subgroups of finite index.*

Proof. To obtain the required subgroups, we can map onto C_1, C_2, C_3 and C_6 only. As before, mapping onto C_1 , we get $H(\sqrt{3})$. $H(\sqrt{3})$ can be mapped onto C_2 in two ways and the obtained subgroups are $(0 ; 2, 2, 3, \infty)$ that is isomorphic to $C_2 \star C_2 \star C_3$, and $(0 ; 6, 6, \infty)$ that is isomorphic to $C_6 \star C_6$.

Next, mapping onto C_3 , we obtain $H^3(\sqrt{3})$ with signature $(0 ; 2^{(4)}, \infty)$. Finally if we map onto C_6 , we obtain two normal cycloidal subgroups with signatures $(1 ; 2, 2, \infty)$ and $(0 ; 2^{(6)}, \infty)$. These are all required subgroups.

We now want to generalize all these to any $q, q \geq 3, q \in \mathbb{N}$: □

6. The general case

Let N be a cycloidal normal subgroup of $H(\sqrt{2})$ of finite index μ . Considering Lemma 2.1, one can see the difference between the odd and even q cases. This is because when q is odd the $(2, m, 2m)$ -groups, where $m|q$, are cyclic of order $2m$ and therefore the obtained subgroup is cycloidal. By Lemma 2.1 this does not happen when q is even.

First, we let q be odd. By Lemma 2.1, it is only possible to map $H(\sqrt{2})$ onto C_n or C_{2n} , where $n|q$ and $1 \leq n \leq q$. We now consider possibilities:

Mapping onto the trivial group C_1 , we get the group $H(\lambda_q)$ itself as a cycloidal normal subgroup.

We also can map $H(\lambda_q)$ onto $C_2 \cong (2, 1, 2)$. Then

$$\begin{aligned} R &\rightarrow (12) \\ S &\rightarrow (1)(2) \\ T &\rightarrow (12). \end{aligned} \tag{12}$$

Therefore $N \cong (0 ; q, q, \infty)$ which is a free product of two cyclic groups of order q , and is known to be $H^2(\lambda_q)$, generated by the squares of the elements of $H(\lambda_q)$.

Now we map $H(\lambda_q)$ onto C_q . Then R goes to the identity and S must go to a q -cycle. Then $T=RS$ goes to a q -cycle as well. Hence the obtained subgroup has signature $(0 ; 2^{(q)}, \infty)$ which is the free product of q cyclic groups of orders 2.

If we map $H(\lambda_q)$ onto $C_{2q} \cong (2, q, 2q)$, then

$$\begin{aligned} R &\rightarrow (12)(34) \dots (2q - 12q) \\ S &\rightarrow (135 \dots 2q - 1)(246 \dots 2q) \\ T &\rightarrow (14589 \dots 2q - 1 \ 2 \ 3 \ 6 \ 7 \dots 2q) \end{aligned} \tag{13}$$

which gives a subgroup with signature $(\frac{q-1}{2}; \infty)$ by the Riemann-Hurwitz formula. This last one is the commutator subgroup $H'(\lambda_q)$ of $H(\lambda_q)$, isomorphic to the free group of rank $q-1$.

We now have left two cases to consider. Firstly, mapping onto C_n , where $n|q$, gives a subgroup with signature $(0; 2^{(2n)} q/n, \infty)$ which is the free product of n cyclic groups of order 2 and one cyclic group of order q/n .

Secondly and finally, we can map $H(\lambda_q)$ onto C_{2n} , where $n|q$. Then $N \cong (\frac{n-1}{2}; q/n, q/n, \infty)$, isomorphic to the free product of the free group of rank $n-1$ together with two cyclic groups of order q/n each.

By Lemma 1 these are all the cycloidal normal subgroups of $H(\lambda_q)$ for odd q .

Secondly let $q \geq 4$ be even. Then we can only map $H(\lambda_q)$ onto C_n homomorphically where $1 \leq n \leq q$ and $n|q$, by Lemma 2.1. Now by mapping onto C_1 we obtain $H(\lambda_q)$ itself again. If we map onto C_q , this time the obtained subgroup has signature $(0; 2^{(2q)}, \infty)$ which is a free product of q cyclic groups of order 2. Finally if n is a proper divisor of q , then $N \cong (0; 2^{(2n)}, q/n, \infty)$ is isomorphic to the free product of n cyclic groups of order 2 and one cyclic group of order q/n .

7. The number of cycloidal normal subgroups of $H(\lambda_q)$ of finite index.

Now that we have found all cycloidal normal subgroups of $H(\lambda_q)$, we can find their total number in cases of odd and even q :

First let q be odd. By Lemma 2.1, $H(\lambda_q) \cong (0; 2, q, \infty)$ can either be mapped onto $(1, n, n)$ or $(2, n, 2n)$, homomorphically, where n is a divisor of q . We noted above that these are all possible cyclic images of $H(\lambda_q)$. As the groups we are considering are all normal, we obtain only one subgroup for each such image. If $\sigma(q)$ denotes the number of divisors of q given by

$$\begin{aligned} \sigma(q) &= \prod_{i=1}^k (1 + \alpha_i) \\ &= \prod_{p|q} (1 + \alpha_p) \end{aligned} \tag{14}$$

for $q = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is the prime power decomposition of q , then we obtain

Theorem 7.1. *Let q be odd. Then $H(\lambda_q)$ has exactly $2\sigma(q)$ cycloidal normal subgroups of finite index with quotient groups C_n and C_{2n} with $n|q$.*

Secondly we consider the case q is even. Again by Lemma 2.1 and the same argument above, we obtain the following result:

Theorem 2. *Let $q > 4$ be even. Then $H(\lambda_q)$ has exactly $\sigma(q)$ cycloidal normal subgroups of finite index with quotient groups C_n , $n|q$, while when $q=4$, this number is equal to $\sigma(q)+1$.*

Some small indexed normal cycloidal subgroups of Hecke groups are obtained in different ways in [1] and [2]. For further details of notation and group structure of these subgroups, see chapters 4, 5, 8, 9 and 10 of [1].

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CANGÜL, BİZİM

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İ. Naci CANGÜL, Osman BİZİM
University of Uludağ, Faculty of Science
Department of Mathematics,
Görükle 15069, Bursa-TURKEY
email: cangul@uludag.edu.tr
email: obizim@uludag.edu.tr

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