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Planar Linkages and Algebraic Sets

Henry C. King

Dedicated to Rob Kirby on the occasion of his 60th birthday

1. Linkages

An *abstract linkage* is a one dimensional simplicial complex L with a positive number $\ell(\overline{vw})$ assigned to each edge $\overline{vw}$. A *planar realization* of an abstract linkage (L,ℓ) is a mapping φ from the vertices of L to $\mathbb{C}$ so that $|\varphi(v)-\varphi(w)| = \ell(\overline{vw})$ for all edges $\overline{vw}$. We will investigate the topology of the space of planar realizations of linkages.

You may think of an abstract linkage as an ideal mechanical device consisting of a bunch of stiff rods (the edges) with length given by ℓ and sometimes attached at their ends by rotating joints. A planar realization is some way of placing this linkage in the plane.

If L is a finite graph, we let $\mathcal{V}(L)$ denote the set of vertices of L and let $\mathcal{E}(L)$ denote the set of edges of L.

We will often wish to fix some of the vertices of a linkage whenever we take a planar realization. So we say that a *linkage* $\mathcal{L}$ is a foursome (L,ℓ,V,μ) where (L,ℓ) is an abstract linkage, $V\subset\mathcal{V}(L)$ is a subset of its vertices, and $\mu\colon V\to\mathbb{C}$. So V is the set of fixed vertices and μ tells where to fix them. The configuration space of realizations is defined by:

$$\mathcal{C}(\mathcal{L}) = \left\{ \varphi\colon \mathcal{V}(L)\to\mathbb{C} \;\middle|\; \begin{array}{ll} \varphi(v)=\mu(v) & \text{if } v\in V, \\ |\varphi(v)-\varphi(w)| = \ell(\overline{vw}) & \text{if } \overline{vw}\in\mathcal{E}(L) \end{array} \right\}.$$

If w is a vertex of $\mathcal{L}$, we have a map $\rho_{\mathcal{L},w}\colon\mathcal{C}(\mathcal{L})\to\mathbb{C}$ given by $\rho_{\mathcal{L},w}(\varphi)=\varphi(w)$, the position of the vertex w. In this paper we will look at configuration spaces $\mathcal{L}$ and also the image of $\rho_{\mathcal{L},w}$.

The literature on linkages goes back over 300 years, see [CR] or [KM]. Moreover, humans have been using linkages for thousands of years in various mechanical devices.

I recall first hearing of planar linkages in a talk by Thurston at IAS, probably in the mid 1970s. I recall three results from this talk:

1. Any algebraic set is isomorphic to an algebraic set given by quadratic polynomials.
2. You can construct a linkage which will “sign your name”, i.e., there is a linkage $\mathcal{L}$ and vertices $v_1,\ldots,v_k$ so that $\cup_{i=1}^k \rho_{\mathcal{L},v_i}(\mathcal{C}(\mathcal{L}))$ is an arbitrarily close approximation of your signature.
3. Some result about realizing any compact smooth manifold as a configuration space of a linkage, which result I do not recall at all precisely.

As far as I can tell, Thurston never wrote these results up, so 3 must remain vague. Occasionally since then I have been contacted by an engineer interested in these results, but I could not recall anything about Thurston's proof so I could not help them. Then recently John Millson started asking me lots of questions on real algebraic sets. He and M. Kapovich were writing up proofs of the results 2 and 3 above. In the course of doing so, they discovered and solved some problems overlooked by previous literature. In this paper I will give a proof of 2 and 3 based upon the Kapovich-Millson proof in [KM]. In particular I will give proofs of the following:

Theorem 1.1. *(Thurston, Kapovich-Millson) If X is a compact real algebraic set, then there is a linkage $\mathcal{L}$ so that $\mathcal{C}(\mathcal{L})$ is the union of a number of copies of X, each analytically isomorphic to X. In fact there is a polynomial map from $\mathcal{C}(\mathcal{L})$ to X giving an analytically trivial covering of X.*

Theorem 1.2. *(Thurston, Kapovich-Millson) If $\alpha\colon [a,b] \to \mathbb{C}$ is any polynomial map, there is a linkage $\mathcal{L}$ and a vertex v of $\mathcal{L}$ so that $\rho_{\mathcal{L},v}$ traces out the curve α, i.e., $\alpha([a,b]) = \rho_{\mathcal{L},v}(\mathcal{C}(\mathcal{L}))$.*

Corollary 1.3. *(Thurston) There is a linkage which signs your name.*

The corollary follows from Theorem 1.2 because you can break up your signature into k segments, each of which is the image of a smooth curve, and then approximate each smooth curve by a polynomial curve. If you wish, you may even do this approximation in a way which preserves any cusps in your signature. By being careful, you can even show there is a linkage $\mathcal{L}$ which signs your name using just one output vertex v. So $\rho_{\mathcal{L},v}(\mathcal{C}(\mathcal{L}))$ is a close approximation of your signature, no matter how many strokes are needed, see [K1].

I should define some terms here. A *real algebraic set* is the set of solutions of a collection of real polynomial equations in $\mathbb{R}^n$. Note that if we view $\mathbb{C}$ as $\mathbb{R}^2$, any configuration space $\mathcal{C}(\mathcal{L})$ is an algebraic set in $\mathbb{R}^{2\mathcal{V}(\mathcal{L})}$, since it is the solutions of the polynomial equations $|y_i - y_j|^2 = \ell(ij)^2$ and $y_i = z_i$ for $i \in V$. Moreover these equations are quadratic at worst which is the reason for Thurston's result 1 above which says that quadratic algebraic sets are not at all special as algebraic sets go. [1]

We will use two notions of isomorphism. If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ then we say a homeomorphism $f\colon X \to Y$ is an *isomorphism* if f and f^{-1} are both restrictions of rational maps, for example polynomials. We say f is an *analytic isomorphism* if f and f^{-1} are both restrictions of analytic maps, i.e., maps locally given by power series. So any isomorphism is analytic, but the converse is not true.

[1] I will illustrate the proof of 1 with an example. If we make a new variable u and set $u = x^2$, then the cubic x^3 becomes the quadratic ux.

1.1. Cabled Linkages

Note that Theorem 1.1 is somewhat unsatisfying, since it does not give a complete characterization of all compact configuration spaces, but only a characterization up to finite analytically trivial covers.

So we introduce the following generalization of linkage which will allow a complete characterization of configuration spaces.

A *cabled linkage* is a quintuple (L, ℓ, V, μ, F) where (L, ℓ, V, μ) is a linkage and $F \subset \mathcal{E}(L)$. We will think of the edges in F as being flexible rather than rigid. A physical model for such a cabled linkage would have the edges in $\mathcal{E}(L) - F$ be rigid rods as before but the edges in F are just ropes or cables connecting two vertices. Thus in a planar realization, two vertices connected by an edge e in F would only be constrained to have distance $\leq \ell(e)$. If F is empty we get a classical linkage. The configuration space is given by:

$$\mathcal{C}(\mathcal{L}) = \left\{ \varphi \colon \mathcal{V}(L) \to \mathbb{C} \;\middle|\; \begin{array}{ll} \varphi(v) = \mu(v) & \text{if } v \in V, \\ |\varphi(v) - \varphi(w)| \leq \ell(\overline{vw}) & \text{if } \overline{vw} \in F \\ |\varphi(v) - \varphi(w)| = \ell(\overline{vw}) & \text{if } \overline{vw} \in \mathcal{E}(L) - F \end{array} \right\}.$$

From now on, the word linkage will refer to a cabled linkage. If we wish to refer to a linkage without any flexible edges, we will call it a classical linkage.

Because of the inequalities, the configuration space of a cabled linkage may no longer be a real algebraic set. However it is something which I will call a quasialgebraic set. I define a *quasialgebraic set* to be a subset of $\mathbb{R}^n$ of the form

$$\{ x \in \mathbb{R}^n \mid p_i(x) = 0,\, i = 1, \ldots, k \text{ and } q_j(x) \geq 0,\, j = 1, \ldots, m \}$$

for some polynomials p_i and q_j. I am not aware of any literature studying quasialgebraic sets. They lie somewhere strictly between real algebraic sets and semialgebraic sets. It's a bit of a strange category of spaces, since the union of two quasialgebraic sets need not be a quasialgebraic set.

Using cabled linkages, we get the following complete characterization of configuration spaces of cabled linkages.

Theorem 1.4. *(King) If X is a compact quasialgebraic set, then there is a cabled linkage $\mathcal{L}$ so that $\mathcal{C}(\mathcal{L})$ is analytically isomorphic to X. In fact there is a polynomial map from $\mathcal{C}(\mathcal{L})$ to X giving this isomorphism.*

Corollary 1.5. *(King) Up to analytic isomorphism, the set of configuration spaces of cabled linkages is exactly the set of spaces of the form $X \times \mathbb{R}^{2k}$ where X is a compact quasialgebraic set.*

So, for example using [AK] and [AT] any compact smooth manifold with boundary is diffeomorphic to some $\mathcal{C}(\mathcal{L})$ and any compact PL manifold with boundary is homeomorphic to some $\mathcal{C}(\mathcal{L})$.

1.2. Functoriality of $\mathcal{C}(\mathcal{L})$

Let $\mathcal{L}' \subset \mathcal{L}$ be a sublinkage. This means that $L' \subset L$, $\ell' = \ell|_{\mathcal{E}(L')}$, $V' \subset V$, $\mu' = \mu|_{V'}$, and $F' = F \cap \mathcal{E}(L')$. Then we have a natural map $\rho_{\mathcal{L},\mathcal{L}'} : \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}')$ obtained by restriction, i.e., $\rho_{\mathcal{L},\mathcal{L}'}(\varphi) = \varphi|_{\mathcal{V}(L')}$. Note that if L' is a single point v and V' is empty, then $\mathcal{C}(\mathcal{L}') = \mathbb{C}$ and with this identification we have $\rho_{\mathcal{L},\mathcal{L}'} = \rho_{\mathcal{L},v}$.

If we wished, we could generalize this by forming a category of linkages and linkage maps and noting that $\mathcal{C}$ is a contravariant functor. But we really only have a need to look at inclusion maps, or sometimes simple quotients, so we refrain from this generality.

Lemma 1.6. *If $\mathcal{L}' \subset \mathcal{L}$ and $\mathcal{L}'' \subset \mathcal{L}$ are two sublinkages then we have a natural identification of $\mathcal{C}(\mathcal{L}' \cup \mathcal{L}'')$ with the fiber product of the restriction maps $\rho_{\mathcal{L}',\mathcal{L}'\cap\mathcal{L}''} : \mathcal{C}(\mathcal{L}') \to \mathcal{C}(\mathcal{L}' \cap \mathcal{L}'')$ and $\rho_{\mathcal{L}'',\mathcal{L}'\cap\mathcal{L}''} : \mathcal{C}(\mathcal{L}'') \to \mathcal{C}(\mathcal{L}' \cap \mathcal{L}'')$.*

$$\begin{array}{ccc} \mathcal{C}(\mathcal{L}' \cup \mathcal{L}'') & \longrightarrow & \mathcal{C}(\mathcal{L}'') \\ \big\downarrow & & \big\downarrow \rho_{\mathcal{L}'',\mathcal{L}'\cap\mathcal{L}''} \\ \mathcal{C}(\mathcal{L}') & \xrightarrow{\rho_{\mathcal{L}',\mathcal{L}'\cap\mathcal{L}''}} & \mathcal{C}(\mathcal{L}' \cap \mathcal{L}'') \end{array}$$

Proof. This is because a planar realization of $\mathcal{L}' \cup \mathcal{L}''$ is just a planar realization of $\mathcal{L}'$ and a planar realization of $\mathcal{L}''$ which happen to agree on $\mathcal{L}' \cap \mathcal{L}''$. Thus

$$\mathcal{C}(\mathcal{L}' \cup \mathcal{L}'') = \{(\varphi', \varphi'') \in \mathcal{C}(\mathcal{L}') \times \mathcal{C}(\mathcal{L}'') \mid \rho_{\mathcal{L}',\mathcal{L}'\cap\mathcal{L}''}(\varphi') = \rho_{\mathcal{L}',\mathcal{L}'\cap\mathcal{L}''}(\varphi'')\}$$

is the fiber product. □

As a consequence of Lemma 1.6, the configuration space of the disjoint union of two linkages is the product of their configuration spaces.

Lemma 1.7. *Suppose $\mathcal{L}$ is a cabled linkage with no fixed vertices. Form a new $\mathcal{L}'$ from $\mathcal{L}$ by fixing exactly one vertex in each connected component of $\mathcal{L}$. If $\mathcal{L}$ has k connected components, then $\mathcal{C}(\mathcal{L})$ is isomorphic to $\mathcal{C}(\mathcal{L}') \times \mathbb{C}^k$.*

Proof. By the above remark on the configuration space of a disjoint union, it suffices to show this when $\mathcal{L}$ is connected, so $k = 1$. For the map from right to left, take any $\varphi' \in \mathcal{C}(\mathcal{L}')$ and any $z \in \mathbb{C}$. We then get a $\varphi \in \mathcal{C}(\mathcal{L})$ by letting φ be the composition of φ' and translation by z. □

So if we want to understand the topology configuration spaces, it suffices to consider only those cabled linkages with at least one fixed vertex in each connected component of L. Note that such configuration spaces must be compact since the distance between any two vertices in the same component of L is bounded by the sum of the lengths of the edges of a path connecting them.

Lemma 1.8. *Let $\mathcal{L}$ be a cabled linkage and let $v_1, \dots, v_m$ be vertices of $\mathcal{L}$ which are not fixed. Let $\mathcal{L}'$ be obtained from $\mathcal{L}$ by fixing the vertices $v_1, \dots, v_m$ to be at the* *points* $z_1, \ldots, z_m$. *Let* $p\colon \mathcal{C}(\mathcal{L}) \to \mathbb{C}^m$ *be the map* $(\rho_{\mathcal{L},v_1}, \ldots, \rho_{\mathcal{L},v_m})$. *Then* $\mathcal{C}(\mathcal{L}') = p^{-1}(z_1, \ldots, z_m)$.

Proof. Note that $\mathcal{C}(\mathcal{L}') \subset \mathcal{C}(\mathcal{L})$ and it must be exactly those φ with $\rho_{\mathcal{L},v_i}(\varphi) = \varphi(v_i) = z_i$. The lemma follows. □

Lemma 1.9. *Let* $\mathcal{L}$ *be a cabled linkage and let* $v_1, \ldots, v_m$ *be vertices of* $\mathcal{L}$ *which are not fixed. Let* $\mathcal{L}'$ *be obtained from* $\mathcal{L}$ *by adding new vertices* $u_1, \ldots, u_m$ *and new flexible edges* $u_i v_i$ *of length* b_i. *Fix the vertices* u_i *to points* $z_i \in \mathbb{C}$. *Let* $p\colon \mathcal{C}(\mathcal{L}) \to \mathbb{C}^m$ *be the map* $(\rho_{\mathcal{L},v_1}, \ldots, \rho_{\mathcal{L},v_m})$. *Then* $\mathcal{C}(\mathcal{L}')$ *is isomorphic to*

$$p^{-1}(\{w \in \mathbb{C}^m \mid b_i \geq |w_i - z_i|, i = 1, \ldots, m\}).$$

Proof. The inclusion $\mathcal{L} \subset \mathcal{L}'$ gives a map $\alpha\colon \mathcal{C}(\mathcal{L}') \to \mathcal{C}(\mathcal{L})$. Let $Y = p^{-1}(\{w \in \mathbb{C}^m \mid b_i \geq |w_i - z_i|, i = 1, \ldots, m\})$. We have a map $\beta\colon Y \to \mathcal{C}(\mathcal{L}')$ defined by $\beta(\varphi)(v) = \varphi(v)$ for v a vertex of $\mathcal{L}$ and $\beta(\varphi)(u_i) = z_i$. Note that $\alpha(\mathcal{C}(\mathcal{L}')) \subset Y$ and β is the inverse of $\alpha\colon \mathcal{C}(\mathcal{L}') \to Y$. □

The following are immediate from the definitions.

Lemma 1.10. *If* $\mathcal{L}' \subset \mathcal{L}$ *is a sublinkage then the map* $\rho_{\mathcal{L},\mathcal{L}'}\colon \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}')$ *is an (analytic) isomorphism if and only if it is onto and the position* $\varphi(v)$ *of each vertex* v *of* $\mathcal{L}$ *is a rational (resp. analytic) function of the positions* $\varphi(w_i)$ *of the vertices* w_i *in* $\mathcal{L}'$. *More generally, if* $Z \subset \mathcal{C}(\mathcal{L})$ *then the restriction* $\rho_{\mathcal{L},\mathcal{L}'}|_Z\colon Z \to \rho_{\mathcal{L},\mathcal{L}'}(Z)$ *is an (analytic) isomorphism if and only if for* $\varphi \in Z$, *the position* $\varphi(v)$ *of each vertex* v *of* $\mathcal{L}$ *is a rational (resp. analytic) function of the positions* $\varphi(w_i)$ *of the vertices* w_i *in* $\mathcal{L}'$.

Lemma 1.11. *Let* $\mathcal{L}$ *be a linkage and suppose* v *and* w *are two vertices of* $\mathcal{L}$. *Suppose that whenever there are edges* $\overline{vu}$ *and* $\overline{wu}$ *to the same vertex* u, *that* $\ell(\overline{vu}) = \ell(\overline{vw})$. *Suppose also that there is no edge* $\overline{vw}$. *Then we may form a linkage* $\mathcal{L}'$ *from* $\mathcal{L}$ *by identifying the vertices* v *and* w, *and identifying any edges* $\overline{vu}$ *and* $\overline{wu}$. *Moreover there is a natural identification of* $\mathcal{C}(\mathcal{L}')$ *with* $\{\varphi \in \mathcal{C}(\mathcal{L}) \mid \varphi(v) = \varphi(w)\}$.

1.3. How to put a rotating joint in the middle of an edge

In our linkages, rotating joints between two edges only occur at the ends of an edge. In real life, a mechanical linkage may have a joint in the middle. We may simulate such a joint as in Figure 1. In any realization φ, the point $\varphi(C)$ must lie on the line segment between $\varphi(A)$ and $\varphi(B)$. Thus when drawing linkages, it is allowable to draw a joint in the middle of an edge.

Note, this paragraph is optional. If you actually constructed the linkage in Figure 1, you would find it somewhat spongy, the position of the middle point is quite sensitive to small errors in length. This is reflected algebraically in the following. Suppose you look at the configuration space as a scheme, i.e., you focus on the polynomial equations defining $\mathcal{C}(\mathcal{L})$ rather than the point set of their solutions. Then in the above linkage, the configuration scheme is not reduced. (This is because of the nontransversality of the equations specifying the point $\varphi(C)$.) In [KM], where they take a scheme-theoretic point

Figure 1. How to put a joint in the middle of an edge

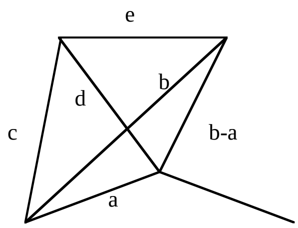

Figure 2. Stiffening a middle joint

of view, they get around this by essentially allowing joints in the middle of an edge and modifying the equations accordingly. One could also add a stiffening truss as shown in Figure 2 and the resulting configuration scheme is reduced. In Figure 2, $c^2 = a^2 + d^2$ and $e^2 = d^2 + (b - a)^2$.

1.4. The Square Linkage (actually a rhombus)

Let us now look at a very simple linkage and describe its configuration space. Consider the square linkage in Figure 3. As we will see soon, this is a sublinkage of a number of useful linkages. Let us try to find its configuration space if we fix the two bottom vertices A and B. It is tempting to believe this configuration space is a circle, that once the position of C is decided then the position of vertex D is determined. In fact, this is the standard mistake in classical papers on linkages. Kapovich and Millson noticed that there are other degenerate realizations, for example D could map to the same point as A, leaving C to rotate. Or C could map to the same point as B and D could rotate. As a result, the configuration space of Figure 3 is the union of three circles, each pair intersecting in a single point. A similar problem occurs for a rectangle which is not a square. Its configuration space is the union of two circles which intersect in two points.

In the linkages we wish to use, we will want to get rid of all these degenerate realizations. Kapovich and Millson did this by *rigidifying* the square. They added another edge between the midpoints of two opposite sides, see Figure 4. Note that for this new linkage, the degenerate realizations will not occur. In order not to clutter up our pictures, we will subsequently draw this extra edge as a dotted line, for example as in Figure 5.

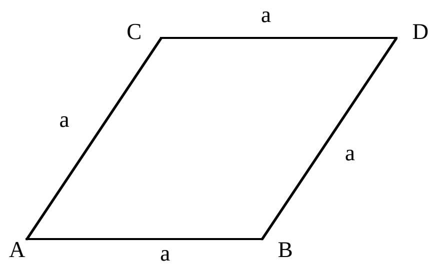

Figure 3. The Square Linkage, with an unexpected Configuration Space

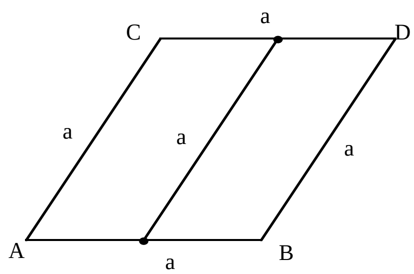

Figure 4. The Rigidified Square Linkage

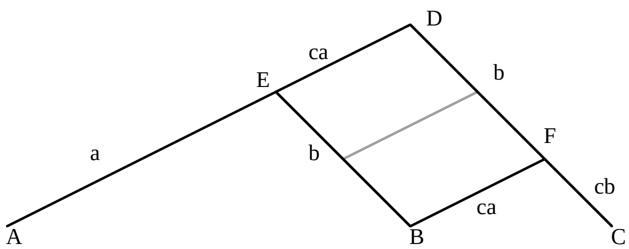

Figure 5. The Pantograph

2. Functional Linkages

Let us now look at a very useful linkage, the pantograph of Figure 5. The pantograph has been used as a mechanical device for centuries (without the rigidifying extra edge). It can do a number of useful things. For example, if A is fixed at 0 and B is at some point z, then C will be at the point $(1 + c)z$. This is an example of a functional linkage, since it can be used to evaluate the function $z \mapsto \lambda z$.

Notice however that B cannot move to every point of the plane. Since A is fixed at 0, the point B is constrained to lie in the annulus between the circles of radius $a - b$ and $a + b$. Moreover, for any position of the point B in the interior of this annulus, there will be exactly two configurations depending on which side of the line $\overline{AB}$ the point D lies. This generalizes to the following:

A linkage $\mathcal{L}$ is *quasifunctional* for a map $f\colon \mathbb{C}^n \to \mathbb{C}^m$ if there are vertices $w_1, \ldots, w_n$ and $v_1, \ldots, v_m$ of $\mathcal{L}$ so that if $p\colon \mathcal{C}(\mathcal{L}) \to \mathbb{C}^m$ is $(\rho_{\mathcal{L},v_1}, \ldots, \rho_{\mathcal{L},v_m})$ and $q\colon \mathcal{C}(\mathcal{L}) \to \mathbb{C}^n$ is $(\rho_{\mathcal{L},w_1}, \ldots, \rho_{\mathcal{L},w_n})$ then $p = f \circ q$. The set $q(\mathcal{C}(\mathcal{L}))$ is called the *domain* of the quasifunctional linkage. We call p the input map and call q the output map.

If in addition, there is a $U \subset q(\mathcal{C}(\mathcal{L}))$ so that the restriction $q|\colon q^{-1}(U) \to U$ is an analytically trivial covering map, we say that $\mathcal{L}$ is *functional* for f with *restricted domain* U.

Moreover, if $q\colon \mathcal{C}(\mathcal{L}) \to q(\mathcal{C}(\mathcal{L}))$ is an analytic isomorphism we say that $\mathcal{L}$ is *strongly functional* for f. As a linguistic convenience, if $\mathcal{L}$ is strongly functional for f and $U \subset q(\mathcal{C}(\mathcal{L}))$ is any subset of the domain, we say that $\mathcal{L}$ is strongly functional for f with restricted domain U.

We call $w_1, \ldots, w_n$ the input vertices and call $v_1, \ldots, v_m$ the output vertices. Repetitions of vertices are allowed, although they are not necessary for the results in this paper.

So if $\mathcal{L}$ is functional for f, then over U the configuration space is a bunch of copies of the graph of f. If the configuration space is just one copy of the graph of f it is strongly functional.

It is very hard for a classical linkage to be strongly functional, because its configuration space is a real algebraic set and polynomial maps on real algebraic sets have a mod 2 degree theory [AK, Prop. 2.3.2]. If $\mathcal{L}$ is a classical linkage and $\mathcal{C}(\mathcal{L})$ is compact, the mod 2 degree of the map q must be even (since for any point z outside the image of q, the fiber $q^{-1}(z)$ has an even number of points). Thus if U is open, the degree of the covering $q|\colon q^{-1}(U) \to U$ must be even. So for a classical linkage to be strongly functional with open restricted domain, we must have $\mathcal{C}(\mathcal{L})$ noncompact, i.e., some component of L has no fixed vertices. A moment's thought will then confirm that only coordinate projections, constant maps, or products of the two could have strongly functional classical linkages with open restricted domain.

But cabled linkages are a different matter. Since their configuration spaces are only quasialgebraic sets, there is no degree theory to get in the way of having strongly functional linkages. Indeed we will show the following:

Theorem 2.1. *(Kapovich and Millson) For any real polynomial map $g(w, \overline{w})\colon \mathbb{C}^n \to \mathbb{C}^m$ and any compact $U \subset \mathbb{C}^n$ there is a functional classical linkage $\mathcal{L}$ for g with restricted domain U.*

Theorem 2.2. *(King) For any real polynomial map $g(w, \overline{w})\colon \mathbb{C}^n \to \mathbb{C}^m$ and any compact $U \subset \mathbb{C}^n$ there is a strongly functional cabled linkage $\mathcal{L}$ for g with restricted domain U.*

In [K2], among other things, we investigate which functions besides polynomials have functional linkages.

3. Proofs of Theorems 1.1, 1.2, and 1.4

Given Theorems 2.1 and 2.2, we may easily prove Theorems 1.1, 1.2, and 1.4. To prove Theorem 1.1, take any compact real algebraic set $X \subset \mathbb{C}^n$, find a polynomial g so $X = g^{-1}(0)$, then use Theorem 2.1 to get a classical linkage $\mathcal{L}'$ which is functional for g with restricted domain X. Now construct the linkage $\mathcal{L}$ by taking $\mathcal{L}'$ and fixing all the output vertices to 0. Then by Lemma 1.8,

$$\begin{aligned}\mathcal{C}(\mathcal{L}) &= \{\varphi \in \mathcal{C}(\mathcal{L}') \mid \varphi(v_i) = 0, i = 1, \ldots, m\}\\ &= p^{-1}(0) = (g \circ q)^{-1}(0) = q^{-1}(X).\end{aligned}$$

But $q^{-1}(X)$ is an analytically trivial covering of X.

To prove Theorem 1.2, take a linkage $\mathcal{L}'$ which is functional for $z \mapsto \alpha(z + \overline{z})$ and with restricted domain the circle C with radius $(b - a)/4$ and center $(b + a)/4$. Form $\mathcal{L}$ from $\mathcal{L}'$ by attaching to its input an edge of length $(b - a)/4$ with a vertex fixed at $(b + a)/4$. The end of this edge will trace out the circle C, but the function $z \mapsto z + \overline{z}$ maps C onto the interval $[a, b]$, hence the output vertex of $\mathcal{L}$ will trace out $\alpha([a, b])$. Alternatively, we show below that there is a linkage $\mathcal{L}''$ with a vertex w which traces any interval, that is, $\rho_{\mathcal{L}'',w}(\mathcal{C}(\mathcal{L}'')) = [a, b]$. We could hook this linkage up to a functional linkage for α.

Theorem 1.4 is proven similarly to Theorem 1.1, using cabled linkages and Theorem 2.2 instead of Theorem 2.1. In particular, take any compact quasialgebraic set $X \subset \mathbb{C}^n$, find a polynomial $g(z, \overline{z})\colon \mathbb{C}^n \to \mathbb{C}^m$ so the image of g is contained in $\mathbb{R}^m$ and

$$X = \{x \in \mathbb{C}^n \mid g_i(x) = 0, i \le k \text{ and } g_i(x) \ge 0, i > k\}.$$

Use Theorem 2.2 to get a cabled linkage $\mathcal{L}'$ which is strongly functional for g with restricted domain X. By compactness of X we may choose a real number b so that $g_i(x) \le 2b$ for all $i > k$ and $x \in X$.

Now construct the linkage $\mathcal{L}$ by taking $\mathcal{L}'$, fixing all the output vertices v_i for $i \le k$ to 0, and adding a new vertex u_i and a new flexible edge from v_i to u_i for each $k < i \le m$. We fix the new vertices u_i to the point b and make the length of each new flexible edge b. (We call this operation tethering v_i to u_i.) Since each g_i is real valued, the effect of each new flexible edge is to restrict $0 \le g_i(x) \le 2b$. Then by Lemma 1.8 and Lemma 1.9,

$$\mathcal{C}(\mathcal{L}) = \{\varphi \in \mathcal{C}(\mathcal{L}') \mid \varphi(v_i) = 0, i \le k \text{ and } 0 \le \varphi(v_i) \le 2b, k < i \le m\} = q^{-1}(X).$$

But $q| : q^{-1}(X) \to X$ is an analytic isomorphism, so $\mathcal{C}(\mathcal{L})$ is analytically isomorphic to X.

To see Corollary 1.5, note that if X is a compact quasialgebraic set, Theorem 1.4 gives a cabled linkage $\mathcal{L}'$ so that $\mathcal{C}(\mathcal{L}')$ is isomorphic to X. Now let $\mathcal{L}$ be the union of $\mathcal{L}'$ and k disjoint vertices. Then $\mathcal{C}(\mathcal{L}) = \mathcal{C}(\mathcal{L}') \times \mathbb{R}^{2k}$. On the other hand, if $\mathcal{L}$ is any cabled linkage, let $\mathcal{L}'$ be a linkage obtained by fixing one vertex in every component of $\mathcal{L}$ which does not already have a fixed vertex. Let k be the number of such vertices we fixed. Then $\mathcal{C}(\mathcal{L})$ is isomorphic to $\mathcal{C}(\mathcal{L}') \times \mathbb{R}^{2k}$, and $\mathcal{C}(\mathcal{L}')$ is a compact quasialgebraic set, see Lemma 1.7.

4. Constructing Polynomial Functional Linkages

So it remains to prove Theorems 2.1 and 2.2. Notice first that any polynomial map $f(z, \overline{z}) \colon \mathbb{C}^n \to \mathbb{C}^m$ can be written as a composition of the following two types of polynomial maps:

- $g \colon \mathbb{C}^k \to \mathbb{C}^k \times \mathbb{C}$ given by $g(z) = (z, h(z))$ where $h(z)$ is $z_i + z_j$, $z_i z_j$, $\overline{z}_i$, or a constant.
- a projection $\mathbb{C}^k \to \mathbb{C}^m$ onto some of the coordinates, $m < k$.

This corresponds to how we calculate polynomials, as a series of elementary arithmetic operations on partial results, and throwing away those results no longer needed.

Consequently, because of the following lemmas, it suffices to find functional linkages for the polynomials $z + w$, zw, $\overline{z}$, a constant, and coordinate projection, all with arbitrarily large compact restricted domain. The last two are trivial. For the constant, just take a fixed output vertex. For the projection, use a linkage with k vertices and no edges, make all k vertices inputs, and select m output vertices.

Lemma 4.1. *Let $\mathcal{L}$ and $\mathcal{L}'$ be (strong) functional linkages for functions $f \colon \mathbb{C}^n \to \mathbb{C}^m$ and $g \colon \mathbb{C}^m \to \mathbb{C}^k$ with restricted domains U and U'. Suppose that $U \cap f^{-1}(U')$ is nonempty. Form a linkage $\mathcal{L}''$ by taking the disjoint union of $\mathcal{L}$ and $\mathcal{L}'$ and then identifying each output vertex of $\mathcal{L}$ with the corresponding input vertex of $\mathcal{L}'$. Then $\mathcal{L}''$ is a (strong) functional linkage for $g \circ f$ with restricted domain $U \cap f^{-1}(U')$.*

Proof. First assume that there are no duplications of the input vertices of $\mathcal{L}'$ and also no duplications of the output vertices of $\mathcal{L}$. Note that $\mathcal{L}''$ is the union of $\mathcal{L}$ and $\mathcal{L}'$, and their intersection is a linkage with m vertices and no edges. Let ρ_1 and ρ_2 be the input and output maps of $\mathcal{L}$ and let ρ_3 and ρ_4 be the input and output maps of $\mathcal{L}'$. By Lemma 1.6, we know that $\mathcal{C}(\mathcal{L}'')$ is the fiber product of ρ_2 and ρ_3,

$$\mathcal{C}(\mathcal{L}'') = \{(\varphi, \varphi') \in \mathcal{C}(\mathcal{L}) \times \mathcal{C}(\mathcal{L}') \mid \rho_2(\varphi) = \rho_3(\varphi')\}$$

so that $\rho_{\mathcal{L}'',\mathcal{L}}$ and $\rho_{\mathcal{L}'',\mathcal{L}'}$ are induced by projection. Note that

$$g \circ f \circ \rho_1 \circ \rho_{\mathcal{L}'',\mathcal{L}} = g \circ \rho_2 \circ \rho_{\mathcal{L}'',\mathcal{L}} = g \circ \rho_3 \circ \rho_{\mathcal{L}'',\mathcal{L}'} = \rho_4 \circ \rho_{\mathcal{L}'',\mathcal{L}'}$$

so $\mathcal{L}''$ is quasifunctional for $g \circ f$.

Now let us see that we can take the restricted domain to be $U \cap f^{-1}(U')$. The restriction of ρ_1 to $U \cap f^{-1}(U')$ is an analytically trivial cover since the restriction to U is, so we only need show that $\rho_{\mathcal{L}'',\mathcal{L}}$ restricts to an analytically trivial cover of $\rho_1^{-1}(U \cap f^{-1}(U')) = \rho_1^{-1}(U) \cap \rho_2^{-1}(U')$. We know that there is a finite set F and an analytic isomorphism $\alpha \colon U' \times F \to \rho_3^{-1}(U')$ so that $\rho_3 \circ \alpha$ is projection to U'. Now

$$\begin{aligned} \rho_{\mathcal{L}'',\mathcal{L}}^{-1}(\rho_2^{-1}(U')) &= \{(\varphi, \varphi') \mid \rho_2(\varphi) = \rho_3(\varphi') \in U'\} \\ &= \{(\varphi, \alpha(\rho_2(\varphi), c)) \mid \rho_2(\varphi) \in U' \text{ and } c \in F\}. \end{aligned}$$

So we have an analytic trivialization

$$\alpha' \colon \rho_2^{-1}(U') \times F \to \rho_{\mathcal{L}'',\mathcal{L}}^{-1}(\rho_2^{-1}(U'))$$

given by $\alpha'(\varphi, c) = (\varphi, \alpha(\rho_2\varphi, c))$.

If $\mathcal{L}$ and $\mathcal{L}'$ are strong functional linkages, we may as well suppose U and U' are as large as possible, so $U = \rho_1(\mathcal{C}(\mathcal{L}))$, $U' = \rho_3(\mathcal{C}(\mathcal{L}'))$. Note that all the covers are one-fold and hence are analytic isomorphisms to $U \cap f^{-1}(U')$. But the input map of $\mathcal{L}''$ is $\rho_1 \circ \rho_{\mathcal{L}'',\mathcal{L}}$ and so its domain is $\rho_1(\rho_{\mathcal{L}'',\mathcal{L}}(\mathcal{C}(\mathcal{L}'')) = \rho_1(\{\varphi \in \mathcal{C}(\mathcal{L}) \mid \rho_2(\varphi) \in U'\} = U \cap f^{-1}(U')$. So the input map of $\mathcal{L}''$ is an analytic isomorphism.

If there are duplications in the input and output vertices things get more complicated. For the purposes of this paper we never need to use duplicated vertices, but for the sake of generality we provide the proof. Let $\Delta_{ij} = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid z_i = z_j\}$. Let $v_1, \ldots, v_m$ be the input vertices of $\mathcal{L}'$ and let $w_1, \ldots, w_m$ be the output vertices of $\mathcal{L}$.

Suppose $v_i = v_j$. Then we must have $U' \subset \Delta_{ij}$. Also, in $\mathcal{L}''$ we end up identifying w_i with w_j. Let us first see whether we can do so. Suppose w is another vertex so that $\overline{ww_i}$ and $\overline{ww_j}$ are both edges of $\mathcal{L}$. Since $U \cap f^{-1}(U')$ is nonempty, there is a $\varphi \in \mathcal{C}(\mathcal{L})$ so that $\rho_1(\varphi) \in U \cap f^{-1}(U')$. Hence $\rho_2(\varphi) = f(\rho_1(\varphi)) \in U' \subset \Delta_{ij}$, and so $\varphi(w_i) = \varphi(w_j)$. So $\ell(\overline{ww_i}) = |\varphi(w) - \varphi(w_i)| = |\varphi(w) - \varphi(w_j)| = \ell(\overline{ww_j})$. There could not be an edge $\overline{w_iw_j}$ since $0 \neq \ell(\overline{w_iw_j}) = |\varphi(w_i) - \varphi(w_j)| = 0$. So by Lemma 1.11 we are allowed to take the quotient linkage $\mathcal{L}_1$ of $\mathcal{L}$, identifying w_i and w_j. By Lemma 1.11 we also see that $\mathcal{L}_1$ is still functional for f but the domain has been restricted to $U \cap f^{-1}(\Delta_{ij})$. Do this identification for each pair i, j with $v_i = v_j$ and we eventually get a functional linkage $\mathcal{L}_2$ for f with domain $U_2 = U \cap f^{-1}(\Delta)$ for some $\Delta \supset U'$.

Now suppose $w_i = w_j$. Then we must have $f(U) \subset \Delta_{ij}$. Also, in $\mathcal{L}''$ we end up identifying v_i with v_j. Let us see whether we can do so. Suppose v is another vertex so that $\overline{vv_i}$ and $\overline{vv_j}$ are both edges of $\mathcal{L}'$. Since $U \cap f^{-1}(U')$ is nonempty, we know $\Delta_{ij} \cap U'$ is nonempty, so there is a $\varphi \in \mathcal{C}(\mathcal{L}')$ so that $\rho_3(\varphi) \in \Delta_{ij} \cap U'$, and hence $\varphi(v_i) = \varphi(v_j)$. So as above, Lemma 1.11 will allow us to take the quotient linkage $\mathcal{L}_1'$ identifying v_i and v_j. By Lemma 1.11 we also see that $\mathcal{L}_1'$ is still functional for g but the domain has been restricted to $U' \cap \Delta_{ij}$. Do this identification for each pair i, j with $w_i = w_j$ and we eventually get a functional linkage $\mathcal{L}_2'$ for g with domain $U_2' = U' \cap \Delta'$ for some $\Delta' \supset f(U)$.

After doing all this, we have $\mathcal{L}''$ is the union of $\mathcal{L}_2$ and $\mathcal{L}_2'$, and we may finish the proof as above. The only thing to check is that $U_2 \cap f^{-1}(U_2') = U \cap f^{-1}(U')$. But $U_2 \cap f^{-1}(U_2') = U \cap f^{-1}(\Delta) \cap f^{-1}(U') \cap f^{-1}(\Delta') = U \cap f^{-1}(U')$ since $U' \subset \Delta$ and $U \subset f^{-1}(\Delta')$. □

Lemma 4.2. *Let $\mathcal{L}$ be a (strong) functional linkage for a function $f \colon \mathbb{C}^n \to \mathbb{C}^m$ with restricted domain U. Form a functional linkage $\mathcal{L}'$ by taking $\mathcal{L}$ but taking the output vertices of $\mathcal{L}'$ to be the concatenation of the input and output vertices of $\mathcal{L}$. Then $\mathcal{L}'$ is a (strong) functional linkage for $(f, g) \colon \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^n$ with restricted domain U, where g is the identity.*

This lemma is trivial. If we wished we could generalize it to a way of combining functional linkages for any f and g to a linkage for (f, g), just identify the inputs. But we have no need to do so.

4.1. Elementary Polynomial Functional Linkages

So it suffices to find (strong) functional linkages for the polynomials $z + w$, zw, and $\overline{z}$, with arbitrarily large compact restricted domain. In fact we will not directly construct functional linkages for $z + w$, zw, and $\overline{z}$, but will do so for other functions which can be composed to give them. In particular we will construct (strong) functional linkages for:

1. Translation: $z \mapsto z + z_0$, with restricted domain any compactum in $\mathbb{C}$.
2. Real scalar multiplication: $z \mapsto \lambda z$, with restricted domain a disc $\{|z - z_0| \leq r\}$ for some z_0 and r as large as we wish.
3. Average: $(z, w) \mapsto (z+w)/2$, with restricted domain $\{(z, w) \mid |z-z_0| \leq r, |w+z_0| \leq r\}$ for some z_0 and r as large as we wish.
4. Inversion through a circle: $z \mapsto t^2/\overline{z}$, with restricted domain $\{|z - z_0| \leq r\}$ for any specified z_0 with $|z_0| = t$ and any $r \leq t/2$.
5. Complex conjugation: $z \mapsto \overline{z}$, with restricted domain $\{|z-z_0| \leq r\}$ for some z_0 and r as large as we wish.

From these functional linkages, we may compose to get functional linkages for $z + w$, zw, and $\overline{z}$ with any compact restricted domain K. Note that we may always restrict the domain of a functional linkage further, so it suffices to find functional linkages with arbitrarily large compact restricted domains, for example (products of) balls of radius r.

2.1. To get $z \mapsto \lambda z$ with restricted domain $|z| \leq r$ for λ real, first use 2 above to get a functional linkage for $z \mapsto \lambda z$, with restricted domain $\{|z - z_0| \leq r\}$. Then use 1 to get a functional linkage for $z \mapsto z + z_0$ with restricted domain $|z| \leq r$. Using Lemma 4.1, compose these two to get a functional linkage for $z \mapsto \lambda z + \lambda z_0$ with restricted domain $|z| \leq r$. Now using 1 and Lemma 4.1, compose with a translation by $-\lambda z_0$ to get our desired linkage for $z \mapsto \lambda z$ with restricted domain $|z| \leq r$.

3.1. To get $(z, w) \mapsto z + w$ with restricted domain $|z| \leq r$, $|w| \leq r$, find a functional linkage for the average 3 above. Then using 1, find functional linkages for $z \mapsto z+z_0$ and $z \mapsto z - z_0$, both with restricted domain $|z| \leq r$. Their disjoint union is functional for $(z, w) \mapsto (z + z_0, w - z_0)$ with restricted domain $|z| \leq r$, $|w| \leq r$. Using Lemma 4.1 and composing with the first linkage, we get a functional linkage for $(z, w) \mapsto (z + w)/2$ with restricted domain $|z| \leq r$, $|w| \leq r$. Now using 2.1 with $\lambda = 2$ and Lemma 4.1, we get a functional linkage for $(z, w) \mapsto z+w$ with restricted domain $|z| \leq r$, $|w| \leq r$.

5.1. To get $z \mapsto \overline{z}$, with restricted domain $|z| \leq r$, first use 5 to get a functional linkage for $z \mapsto \overline{z}$ with restricted domain $|z - z_0| \leq r$. Then compose with translations by z_0 and $-\overline{z}_0$.

Figure 6. The Tensor Lamp Linkage

6. We will do some algebraic manipulation to get multiplication $(z, w) \mapsto zw$. First, note that $zw = ((z+w)^2 - (z-w)^2)/4$. So it suffices to find a functional linkage for $z \mapsto z^2$ with restricted domain $|z| \leq r$. Next, note that if $h(z) = t^2/\overline{z}$ then
$$t^2 - th((h(t+z) + h(t-z))/2) = z^2$$
Consequently, we may get a functional linkage for z^2 by using Lemma 4.1 and functional linkages we have found above. In particular, the reader who wishes to work out the details will find it is good to use (three copies of) a functional linkage for h by choosing some $t > 2r$, and specifying that its restricted domain be $\{|z-t| \leq r\}$.

So we have now reduced to finding functional linkages 1-5 above. In doing so, the following Lemma will be useful. Its proof may be safely left to the reader. It is, for example, a special case of the theorem that a proper submersion is a locally trivial fibration.

Lemma 4.3. *Let $f\colon M \to \mathbb{R}^m$ be a smooth map from a compact m dimensional manifold with boundary. Let $S \subset M$ be the set of critical points of f, the points where df has rank $< m$. Let U be contained in a connected component of $\mathbb{R}^m - f(S \cup \partial M)$. Then $f|\colon f^{-1}(U) \to U$ is a covering projection.*

In our usage, $m = 2$, f is analytic, and U is often contractible, so f restricts to an analytically trivial covering of U, thus $f^{-1}(U)$ is analytically isomorphic to $U\times$ a finite set. As another application, we will use the consequence that $f(M)$ is the union of $f(S \cup \partial M)$ and some connected components of $\mathbb{R}^m - f(S \cup \partial M)$.

4.2. A simple Linkage, a key to understanding more complicated Linkages

It will be useful to look first at a very simple linkage $\mathcal{L}$, as shown in the left half of Figure 6. The vertex A is fixed at some point z_1, but B and C are not fixed. The two edges $\overline{AB}$ and $\overline{BC}$ are rigid. We assume that $b \leq a$.

Clearly $\mathcal{C}(\mathcal{L})$ is a torus $S^1 \times S^1$, we may identify $(u, v) \in S^1 \times S^1$ with φ_{uv} where $\varphi_{uv}(A) = z_1$, $\varphi_{uv}(B) = z_1 + au$, and $\varphi_{uv}(C) = z_1 + au + bv$. Note that $\rho_{\mathcal{L},C}\colon \mathcal{C}(\mathcal{L}) \to \mathbb{C}$ is then the map $\rho_{\mathcal{L},C}(u, v) = z_1 + au + bv$ which has critical set where $u = \pm v$. The image of the critical set is the two circles $\{|z - z_1| = a - b\} \cup \{|z - z_1| = a + b\}$, so by Lemma 4.3 we see that the image of $\rho_{\mathcal{L},C}$ is the annulus $\{a - b \leq |z - z_1| \leq a + b\}$ and moreover

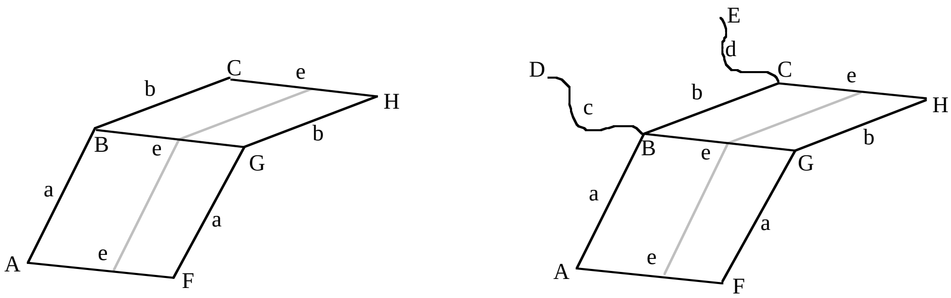

Figure 7. Translation Linkage

$\rho_{\mathcal{L},C}$ restricts to a double cover of the open annulus $\{a-b < |z-z_1| < a+b\}$. In fact this double cover is analytically trivial. (The masochistic reader may verify that the inverse of $\rho_{\mathcal{L},C}$ takes B to the points $z_1 + (z-z_1)(\alpha + a^2 - b^2 \pm \beta)/(2\alpha)$ where $\alpha = |z-z_1|^2$ and $\beta = \sqrt{(\alpha + a^2 - b^2)^2 - 4a^2\alpha}$.) In applications below, we will usually only focus on some disc inside the annulus $\{|z-z_0| \le r\}$ where, say $z_0 = z_1 + a$ and $0 < r < b$. Then $\rho_{\mathcal{L},C}$ restricts to an analytically trivial double cover of this disc.

When working with cabled linkages, it will be convenient to modify this linkage so that $\rho_{\mathcal{L},C}$ is an analytic isomorphism to some disc $\{|z-z_0| \le d\}$. We do this by tethering the vertices B and C to fixed vertices D and E so that their movement is restricted, see the cabled linkage on the right half of Figure 6. Consider first the sublinkage $\mathcal{L}'$ formed by A, B, C, and D, with rigid edges $\overline{AB}$ and $\overline{BC}$, and a flexible edge $\overline{BD}$ of length c, where D is fixed at some point z_2 (and A is fixed at z_1 as before). We have

$$\begin{aligned}\mathcal{C}(\mathcal{L}') &= \rho^{-1}_{\mathcal{L},B}(\{\,|z-z_2| \le c\,\}) \\ &= \{\,(u,v) \in S^1 \times S^1 \mid c \ge |z_1 + au - z_2|\,\} = T \times S^1\end{aligned}$$

for some arc T of S^1 as long as we choose z_2 and c so that $c - a < |z_1 - z_2| < c + a$. For convenience, we choose $c = \sqrt{2}a$ and $z_2 = z_1 + aw_0$ for some $w_0 \in S^1$. Then T will be the semicircle between $\pm\sqrt{-1}w_0$ which contains w_0. By Lemma 4.3, we know that $\rho_{\mathcal{L},C}$ restricts to an analytically trivial covering of $\{|z - z_1 - \sqrt{-1}aw_0| < b\}$. But by checking the inverse image of a point, for example $z_1 + \sqrt{-1}aw_0$, we see that it is a one-fold cover, hence an analytic isomorphism. So now in $\mathcal{L}$, if we fix E at $z_1 + \sqrt{-1}aw_0$ and pick $d < b$, we see that $\rho_{\mathcal{L},C}\colon \mathcal{C}(\mathcal{L}) \to \mathbb{C}$ is an analytic isomorphism to its image $|z - z_1 - \sqrt{-1}aw_0| \le d$.

4.3. A Functional Linkage for Translation

Now let us find a functional linkage for translation 1 above with restricted domain $|z| \le r$. Consider the linkages $\mathcal{L}$ in Figure 7, which we will show to be functional for $z \mapsto z + z_0$ with restricted domain $|z| \le r$. The right hand cabled linkage will be strongly functional.

Choose $a > 2r$, and let $e = |z_0|$. The vertices A and F are fixed at z_1 and $z_1 + z_0$ respectively, where z_1 will be determined later. We let C be the input vertex and H be the output vertex. The parallelograms $ABGF$ and $BCHG$ are rigidified. Thus $\overline{AF}$, $\overline{BG}$, and $\overline{CH}$ are parallel, and so for any $\varphi \in \mathcal{C}(\mathcal{L})$ we must have $\varphi(H) = \varphi(C) + z_0$. So $\mathcal{L}$ is quasifunctional and we must only check that the restricted domain can be $|z| \leq r$. Note that the linkage $\mathcal{L}'$ of Figure 6 is a sublinkage of $\mathcal{L}$. We claim by Lemma 1.10 that $\rho_{\mathcal{L},\mathcal{L}'} \colon \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}')$ is an isomorphism. This is because the positions of F, G, H, and the other three unnamed vertices used to rigidify the quadrilaterals are all polynomial functions of the positions of A, B, and C. Now the fact that $\rho_{\mathcal{L},C}$ doubly covers $|z| \leq r$ (for the left hand classical linkage) or singly covers $|z| \leq r$ (for the right hand cabled linkage) follows from the above discussion of $\mathcal{C}(\mathcal{L}')$ in Figure 6, as long as we make appropriate choices of z_1, a, b, and w_0. For example, we may choose b so $r < b < a$, choose $w_0 = 1$ and $z_1 = -a\sqrt{-1}$.

4.4. A Functional Linkage for real scalar Multiplication

Now let us find a functional linkage $\mathcal{L}$ for real scalar multiplication 2 above. We have already seen this linkage, the pantograph in Figure 5. We consider three cases, $\lambda > 1$, $0 < \lambda < 1$, and $\lambda < 0$. The remaining cases $\lambda = 0$ or $\lambda = 1$ are trivial functions which have trivial functional linkages.

If $\lambda > 1$ we take $c = \lambda - 1$, let B be the input vertex and let C be the output vertex, and fix A at 0. Note that for any $\varphi \in \mathcal{C}(\mathcal{L})$ we have $\varphi(C) = \lambda\varphi(B)$. So $\mathcal{L}$ is quasifunctional. Consider the sublinkage $\mathcal{L}'$, with vertices A, E, and B and edges $\overline{AE}$ and $\overline{EB}$. Note by Lemma 1.10 that $\rho_{\mathcal{L},\mathcal{L}'} \colon \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}')$ is an isomorphism since the positions of D, F, and C are polynomial functions of A, E, and B. By the discussion of the linkage in Figure 6, we know that if a and b are chosen appropriately, then $\rho_{\mathcal{L},B}$ double covers some disc $|z - z_0| \leq r$. To get a strong functional linkage, we add two vertices and tether E and B to them with appropriate length cables. By the discussion of the right hand linkage of Figure 6, we know that $\rho_{\mathcal{L},B}$ singly covers some disc $|z - z_0| \leq r$.

If $0 < \lambda < 1$, we take $c = 1/\lambda - 1$, let C be the input vertex, let B be the output vertex, and fix A at 0. By considering the sublinkage with vertices A, D, and C, we see as above that with appropriate choices of a and b, the linkage will be functional for $z \mapsto \lambda z$ with restricted domain $|z - z_0| \leq r$. To get a strongly functional linkage, tether D and C appropriately.

If $\lambda < 0$, we take $c = -\lambda$, let A be the input vertex, C be the output vertex, and fix B at 0. Letting $\mathcal{L}'$ be the sublinkage with vertices A, E, and B, we see as above that $\mathcal{L}$ is functional for $z \mapsto \lambda z$ with restricted domain $|z - z_0| \leq r$. To get a strongly functional linkage, tether E and A appropriately.

4.5. A Functional Linkage for the Average

Now let us find a functional linkage $\mathcal{L}$ for the average 3. Again $\mathcal{L}$ will be the pantograph of Figure 5. The input vertices will be A and C. The output vertex will be B. We let $c = 1$. Note $\mathcal{L}$ is quasifunctional for $(z, w) \mapsto (z + w)/2$.

Let $\mathcal{L}'$ be the linkage obtained from $\mathcal{L}$ by fixing A to the point 0. We know by Lemma 1.7 that there is an isomorphism $\alpha\colon \mathcal{C}(\mathcal{L}') \times \mathbb{C} \to \mathcal{C}(\mathcal{L})$ where $\alpha(\varphi, w)(v) = \varphi(v) + w$ for any vertex v of $\mathcal{L}$. Thus $\rho_{\mathcal{L},C} \circ \alpha(\varphi, w) = w + \rho_{\mathcal{L}',C}(\varphi)$ and $\rho_{\mathcal{L},A} \circ \alpha(\varphi, w) = w$. Note that $\mathcal{L}'$ is the functional linkage for $z \mapsto z/2$ which we have already studied. So we know for appropriate choices of a and b, that $\rho_{\mathcal{L}',C}\colon \mathcal{C}(\mathcal{L}') \to \mathbb{C}$ double covers some disc $|z - 2z_0| \leq 2r$. Consequently, $(\rho_{\mathcal{L},C}, \rho_{\mathcal{L},A})\colon \mathcal{C}(\mathcal{L}) \to \mathbb{C}^2$ double covers the set $|z - 2z_0 - w| \leq 2r$, since $(\rho_{\mathcal{L},C}, \rho_{\mathcal{L},A}) \circ \alpha$ does. In particular, $(\rho_{\mathcal{L},C}, \rho_{\mathcal{L},A})$ is an analytically trivial double cover of the subset $|z - z_0| \leq r$, $|w + z_0| \leq r$.

Now let us construct a strong functional cabled linkage for the average. As usual, we will do so by tethering some vertices of $\mathcal{L}$, in particular, A, C, and D. In fact $|z_0| > r$ already, but in any case we may shrink r so this is true. We will first tether A to a fixed vertex at $-z_0$ with a cable of length r. Then tether C to a fixed vertex at z_0 with a cable of length r. From the above analysis and Lemma 1.9, the resulting linkage $\mathcal{L}'$ will have $\mathcal{C}(\mathcal{L}')$ be an analytically trivial double cover of the product of discs

$$K = \{(z, w) \mid |z - z_0| \leq r, |w + z_0| \leq r\}.$$

We can compute the inverse of this double cover explicitly, for convenience we choose our linkage so that $a = b$, and of course $c = 1$. If $(\rho_{\mathcal{L},C}, \rho_{\mathcal{L},A})(\varphi) = (z, w)$ for $(z, w) \in K$, then

$$\varphi(D) = \gamma(z, w, f) = (z + w)/2 + f\sqrt{-1}\beta(z, w)(w - z)/|w - z|$$

where $f = \pm 1$ and where $\beta\colon K \to (0, \infty)$ is

$$\beta(z, w) = \sqrt{4a^2 - |w - z|^2/4}.$$

Note $\gamma(z_0, -z_0, 1) \neq \gamma(z_0, -z_0, -1)$. So by shrinking r if necessary we may find a d so that for all $(z, w) \in K$ we have

$$\begin{aligned} |\gamma(z, w, 1) - \gamma(z_0, -z_0, 1)| &< d \\ |\gamma(z, w, -1) - \gamma(z_0, -z_0, 1)| &\geq d. \end{aligned}$$

So if we tether D to a fixed vertex at $\gamma(z_0, -z_0, 1)$ with a cable of length d, the resulting cabled linkage will be strongly functional for $(z, w) \mapsto (z + w)/2$ with restricted domain K.

The only problem is that since we shrunk r, the restricted domain is no longer as large as we want. So we do a final step, where we take our cabled linkage and rescale it by some large factor N. In other words, all edge lengths are multiplied by N, and if a vertex v is fixed at some z, we instead fix it at Nz. The resulting linkage is still strongly functional for $z \mapsto (z + w)/2$, since $N(z/N + w/N)/2 = (z + w)/2$, but its restricted domain is $\{(z, w) \mid |z - Nz_0| \leq Nr, |w + Nz_0| \leq Nr\}$ and hence as big as we wish.

4.6. A Functional Linkage for Inversion through a Circle

We must now find a functional linkage $\mathcal{L}$ for inversion through the circle. For this we will use the Peaucellier inversor of Figure 8. The linkage at the left is the full linkage, the one at the right just has the basics. The extra vertices and edges are only needed to

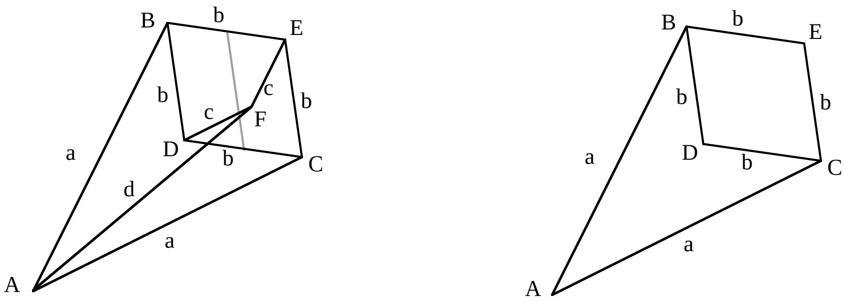

Figure 8. The Peaucellier Inversor

eliminate some degenerate configurations when B and C or D and E coincide. This is slightly modified from the linkage given in [KM], to make $\mathcal{C}(\mathcal{L})$ easier to compute.

We fix A at 0, set $t^2 = a^2 - b^2$, $c < b < a$, set $d = \sqrt{t^2 + c^2}$, let the input vertex be D and the output be E. Let us see why $\mathcal{L}$ is quasifunctional for $z \mapsto t^2/\overline{z} = t^2 z/|z|^2$. Pick any $\varphi \in \mathcal{C}(\mathcal{L})$. Note first that because $c < b$ we are guaranteed that $\varphi(B) \neq \varphi(C)$. Note next that the lines from $\varphi(A)$, $\varphi(D)$, and $\varphi(E)$ to the midpoint of the line segment $\varphi(B)\varphi(C)$ are all perpendicular to $\varphi(B)\varphi(C)$, hence they are collinear. So $\varphi(D)$ is a multiple of $\varphi(E)$. Solving triangles shows that if $s = |\varphi(B) - \varphi(C)|/2$, then

$$
\begin{aligned}
|\varphi(D)| &= \sqrt{a^2 - s^2} \pm \sqrt{b^2 - s^2} \\
|\varphi(E)| &= \sqrt{a^2 - s^2} \mp \sqrt{b^2 - s^2}
\end{aligned}
$$

from which we see that $|\varphi(D)|\,|\varphi(E)| = (a^2 - b^2)$. So $\mathcal{L}$ is quasifunctional for $z \mapsto t^2/\overline{z}$.

Now let us check that for any $0 < r \leq t/2$ then for appropriate choices of a, b, and c, we may take the restricted domain to be $|z - z_0| \leq r$, for any z_0 with $|z_0| = t$. By being more careful below, we would really only need $r < t$, but it is not worth the extra bother. As r gets closer to t, the lengths a, b, and c need to get much bigger than t.

Now $\mathcal{L}$ contains a sublinkage $\mathcal{L}'$ with vertices A, F, and D, and edges $\overline{AF}$ and $\overline{FD}$. This is just our old friend from Figure 6. However, the map $\rho_{\mathcal{L},\mathcal{L}'} : \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}')$ is no longer an isomorphism as it has been in previous linkages. We will show that it is an analytically trivial double cover. (The nontrivial covering translation is obtained from reflection about the line $\overline{AE}$.) To see that it is a double cover, we will solve for $\varphi(B)$, $\varphi(C)$, and $\varphi(E)$ in terms of $\varphi(D)$. In particular, if $f = \pm 1$ and

$$
\begin{aligned}
\beta(u, v) &= \sqrt{b^2 - |u - v|^2/4} \\
\gamma(u, v, f) &= (u + v)/2 + f\sqrt{-1}\beta(u, v)u/|u|
\end{aligned}
$$

then

$$\begin{aligned}\varphi(E) &= t^2/\overline{\varphi(D)}\\ \varphi(B) &= \gamma(\varphi(D),\varphi(E),f)\\ \varphi(C) &= \varphi(D)+\varphi(E)-\varphi(B)\end{aligned}$$

Note that $\beta(\varphi(D),\varphi(E))$ is half the distance from $\varphi(B)$ to $\varphi(C)$, so it is never 0. So $\varphi(B)$ is an analytic function of $\varphi(D)$ and f. Note also that

$$\begin{aligned}|\varphi(E)-\varphi(F)|^2 &= |\varphi(E)|^2-\varphi(E)\overline{\varphi(F)}-\varphi(F)\overline{\varphi(E)}+|\varphi(F)|^2\\ &= t^2/|\varphi(D)|^2(t^2-\varphi(D)\overline{\varphi(F)}-\varphi(F)\overline{\varphi(D)})+d^2\\ &= t^2/|\varphi(D)|^2(t^2+|\varphi(D)-\varphi(F)|^2-|\varphi(D)|^2-|\varphi(F)|^2)+d^2\\ &= t^2/|\varphi(D)|^2(t^2+c^2-|\varphi(D)|^2-d^2)+d^2\\ &= -t^2+d^2=c^2\end{aligned}$$

so our solution does lie in $\mathcal{C}(\mathcal{L})$.

Hence the above maps give an analytic isomorphism

$$\alpha\colon \mathcal{C}(\mathcal{L}')\times\{1,-1\}\to\mathcal{C}(\mathcal{L})$$

Putting this all together with the analysis of Figure 6, we see that $\rho_{\mathcal{L},D}$ restricts to an analytic fourfold cover of the annulus $d-c<|z|<d+c$. So pick $c=t$, then $d=\sqrt{2}t$. If $|z_0|=t$ and $|z-z_0|\le r\le t/2$, then $d-c<t/2\le|z|\le 3t/2<c+d$. So $\rho_{\mathcal{L},D}$ restricts to an analytic fourfold cover of the disc $|z-z_0|\le r$, and thus $\mathcal{L}$ is functional for inversion with restricted domain $|z-z_0|\le r$.

We could get a strong functional cabled linkage by taking $\mathcal{L}$ above, and tethering F, D, and B to fixed vertices, using appropriate length cables. (Those for F and D could be chosen as in Figure 6, and for example if $b=6t$, $a=\sqrt{37}t$, and $e=5t$ we could tether B to $z_0(1+b\sqrt{-1})$ with a cable of length e.)

However, it is easier to modify the linkage $\mathcal{L}$ by eliminating the vertex F and edges $\overline{AF}$, $\overline{DF}$ and $\overline{EF}$, and replacing them with a flexible edge $\overline{DE}$ of length $2c$. This has the same effect (preventing B and C from coinciding) but the configuration space is simpler. So let $\mathcal{L}'$ be the cabled linkage obtained by the above modification, and also by tethering C and D to fixed vertices. By the analysis of Figure 6, we may do this tethering of C and D in such a way that if $\mathcal{L}''$ is the sublinkage formed by $\overline{AC}$ and $\overline{CD}$ then $\rho_{\mathcal{L}'',D}\colon\mathcal{C}(\mathcal{L}'')\to\mathbb{C}$ is an isomorphism to its image $|z-z_0'|\le r'$ where $z_0'=\sqrt{-1}aw_0$ and $w_0\in S^1$ can be chosen arbitrarily, and r' is any positive number less than b. Choose $w_0=-\sqrt{-1}z_0/t$, and choose for example $a=5t/3$, $b=4t/3$, $c=t$, $r'=7t/6$. Then if $|z-z_0|\le r$ we have

$$|z-z_0'|\le r+|z_0-z_0'|=r+a-t<7t/6=r'$$

so $\rho_{\mathcal{L}'',D}$ restricts to an analytic isomorphism to $\{|z-z_0|\le r\}$. In fact we may as well tether D to a vertex fixed at z_0 by a cable of length r and assume $\rho_{\mathcal{L}'',D}$ is an analytic

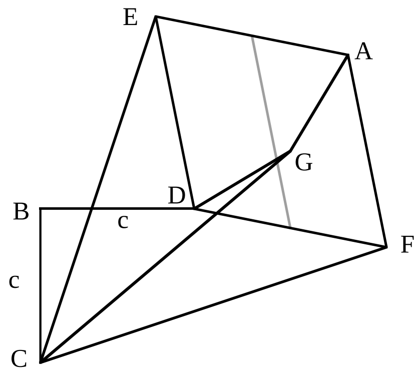

Figure 9. With B and C fixed, A will trace out a straight line

isomorphism to $\{|z - z_0| \leq r\}$. Note that if $|z - z_0| \leq r$, then

$$|z - t^2/\overline{z}| = |\,|z| - t^2/|z|\,| \leq t^2/(t-r) - (t-r) \leq 3t/2 \leq 2c$$

Consequently, by Lemma 1.10 the map $\rho_{\mathcal{L}',\mathcal{L}''}$ is an analytic isomorphism, since it is onto, and if $\varphi \in \mathcal{C}(\mathcal{L}')$ we know $\varphi(E) = t^2/\varphi(D)$ and $\varphi(B) = \varphi(D) + \varphi(E) - \varphi(C)$. So $\mathcal{L}'$ is strongly functional for inversion with restricted domain $\{|z - z_0| \leq r\}$.

4.7. How to draw a straight line

The only remaining function, complex conjugation, will require that we find a linkage so that some vertex is constrained to lie in some line segment. Watt tried to find such a linkage for his steam engine, but ended up getting only an approximate straight line. But such a linkage was found in the 1860's by Peaucellier and was actually used for a while soon after in a ventilating scheme for the British House of Parliament.

So in this section we will construct a linkage $\mathcal{L}$ with a vertex A so that, $\rho_{\mathcal{L},A}(\mathcal{C}(\mathcal{L})) = [a, b]$, and $\rho_{\mathcal{L},A}$ restricted to (a, b) is an analytically trivial cover. In the cabled linkage case we ask that $\rho_{\mathcal{L},A}$ be an analytic isomorphism from $\mathcal{C}(\mathcal{L})$ to $[a, b]$. This linkage $\mathcal{L}$ is obtained by taking the input of a functional linkage for inversion through a circle and forcing this input to lie in a circle going through the origin. But when we invert a circle through the origin, we get a straight line. Now it is just a matter of translating and rotating it and rescaling, to make this line be any interval on the real axis. This linkage $\mathcal{L}'$ is shown in Figure 9.

This linkage is a union of two sublinkages $\mathcal{L}''$ and $\mathcal{L}'$. The linkage $\mathcal{L}''$ is a functional linkage for inversion through a circle of radius $2c$ as in Figure 8, only we translate C to some point $z_0 = x_0 - 2c\sqrt{-1}$ where the real number x_0 is to be determined later. (The only effect of x_0 is to translate the output A along the real axis). The restricted domain of this functional linkage is $|z - x_0| \leq c$. After taking account of the translation of C, $\mathcal{L}''$ is functional for $f(z) = z_0 + 4c^2/(\overline{z - z_0})$.

The linkage $\mathcal{L}'$ has vertices B, C, and D, and edges $\overline{BC}$ and $\overline{BD}$ of length c. We fix C at the point z_0 and fix B at the point $z_0 + c\sqrt{-1}$

Let T be the circle of radius c around $z_0+c\sqrt{-1}$. Now $\mathcal{C}(\mathcal{L}') = T$ and we know $\mathcal{C}(\mathcal{L})$ is the fiber product of $\rho_{\mathcal{L}',D}\colon \mathcal{C}(\mathcal{L}') \to \mathbb{C}$ (which is just inclusion of T) and $\rho_{\mathcal{L}'',D}\colon \mathcal{C}(\mathcal{L}'') \to \mathbb{C}$ (which is the input map of $\mathcal{L}''$). Hence $\mathcal{C}(\mathcal{L}) = \rho^{-1}_{\mathcal{L}'',D}(T) \subset \mathcal{C}(\mathcal{L}'')$.

So in the classical linkage case we must do a bit more work to see what this configuration space is. Let d be the length of $\overline{DG}$ and $\overline{AG}$. Recall from the analysis of the linkage of Figure 8 that $\mathcal{C}(\mathcal{L}'')$ is an analytically trivial double cover of the linkage $\mathcal{L}'''$ with vertices C, D, and G, and with edges DG of length d and $\overline{CG}$ of length $e = \sqrt{4c^2+d^2}$. From the analysis of Figure 6, we then see that:

$$\rho_{\mathcal{L}'',D}(\mathcal{C}(\mathcal{L}'')) = \rho_{\mathcal{L}''',D}(\mathcal{C}(\mathcal{L}''')) = \{e-d \leq |z-z_0| \leq e+d\}$$

Moreover, $\rho_{\mathcal{L}'',D}\colon \mathcal{C}(\mathcal{L}'') \to \mathbb{C}$ restricts to an analytically trivial fourfold cover of the open annulus $\{e-d < |z-z_0| < e+d\}$. So $\rho_{\mathcal{L},D}(\mathcal{C}(\mathcal{L})) = T \cap \{e-d \leq |z-z_0| \leq e+d\}$ and $\rho_{\mathcal{L},D}$ restricts to an analytically trivial fourfold cover of $T \cap \{e-d < |z-z_0| < e+d\}$.

As a consequence, $\rho_{\mathcal{L},A}(\mathcal{C}(\mathcal{L})) = f(T \cap \{e-d \leq |z-z_0| \leq e+d\})$. Also, since $f|\colon \mathbb{C}-z_0 \to \mathbb{C}-z_0$ is a diffeomorphism and $\rho_{\mathcal{L},A} = f\circ\rho_{\mathcal{L},D}$, we know that $\rho_{\mathcal{L},A}$ restricts to a fourfold cover over $f(T\cap\{e-d<|z-z_0|<e+d\})$. But $T\cap\{e-d\leq|z-z_0|\leq e+d\}$ is an arc of the circle T, and its image under f is a line segment on the real axis. By adjusting the constants x_0, d and c we can get any line segment of the real axis we desire. (The line segment we get is the one between the two points $x_0 \pm \sqrt{2d^2+2de}$).

The analysis in the cabled linkage case is a bit easier. Choose $\mathcal{L}''$ strongly functional. By tethering D to a vertex fixed at x_0 with a cable of length c, we may then assume that $\rho_{\mathcal{L}'',D}\colon \mathcal{C}(\mathcal{L}'') \to \mathbb{C}$ is an analytic isomorphism to its image $\{|z-x_0| \leq c\}$. Then $\mathcal{C}(\mathcal{L}) = \rho^{-1}_{\mathcal{L}'',D}(T)$ is analytically isomorphic to $T\cap\{|z-x_0|\leq c\}$ which is again an arc of T, which f takes to a line segment. (In fact to the segment $[x_0-2c/\sqrt{3}, x_0+2c/\sqrt{3}]$.) So $\rho_{\mathcal{L},A}$ is an analytic isomorphism from $\mathcal{C}(\mathcal{L})$ to this segment.

4.8. A Functional Linkage for Complex Conjugation

Finally we need a functional linkage $\mathcal{L}$ for complex conjugation. Our first step is to take two linkages which draw straight lines, as in the previous section. In particular, choose $0<a<b$ and take $\mathcal{L}'$ with $\rho_{\mathcal{L}',A}(\mathcal{C}(\mathcal{L}')) = [a,b]$, and $\mathcal{L}''$ with $\rho_{\mathcal{L}'',B}(\mathcal{C}(\mathcal{L}'')) = [-b,-a]$. Make sure $\rho_{\mathcal{L}',A}$ and $\rho_{\mathcal{L}'',B}$ restricted to (a,b) and $(-b,-a)$ are analytically trivial covers. In the cabled linkage case we ask that $\rho_{\mathcal{L}',A}$ and $\rho_{\mathcal{L}'',B}$ be analytic isomorphisms to $[a,b]$ and $[-b,-a]$.

Pick $c>b$ and form a new linkage $\mathcal{L}$ from the disjoint union of $\mathcal{L}'$ and $\mathcal{L}''$, by putting a rigidified square with side length c between A and B.

Note then that if C is the input vertex and D is the output vertex, then $\mathcal{L}$ is quasi-functional for $z \mapsto \overline{z}$. We have an analytic isomorphism

$$\alpha\colon \mathcal{C}(\mathcal{L}') \times \mathcal{C}(\mathcal{L}'') \times \{1,-1\} \to \mathcal{C}(\mathcal{L})$$

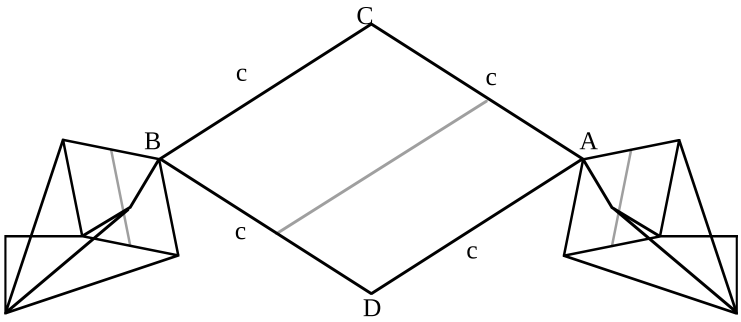

Figure 10. Complex Conjugation

determined by $\rho_{\mathcal{L},\mathcal{L}'}(\alpha(\varphi,\psi,f)) = \varphi$, $\rho_{\mathcal{L},\mathcal{L}''}(\alpha(\varphi,\psi,f)) = \psi$, and

$$\begin{aligned}\alpha(\varphi,\psi,f)(C) &= (\varphi(A)+\psi(B))/2 + f\sqrt{-1}\sqrt{c^2-(\varphi(A)-\psi(B))^2/4}\\ \alpha(\varphi,\psi,f)(D) &= (\varphi(A)+\psi(B))/2 - f\sqrt{-1}\sqrt{c^2-(\varphi(A)-\psi(B))^2/4}\end{aligned}$$

Using Lemma 4.3, we see that

$$\rho_{\mathcal{L},C}(\mathcal{C}(\mathcal{L})) = \{z \in \mathbb{C} \mid c \geq |z-a|, c \leq |z-b|, c \geq |z+a|, c \leq |z+b|\}$$

and moreover $\rho_{\mathcal{L},C}$ restricts to an analytically trivial cover of $U = \{z \in \mathbb{C} \mid c > |z-a|, c < |z-b|, c > |z+a|, c < |z+b|\}$. But note that if we choose a, b, and c large enough then U contains $|z-z_0| \leq r$ for some z_0. For example, we may take $a = 4r$, $b = 8r$, $c = 10r$, and $z_0 = 8r\sqrt{-1}$. So $\mathcal{L}$ is functional for $\overline{z}$ with restricted domain $|z-z_0| \leq r$.

In the cabled linkage case, we see that $\mathcal{L}$ is in fact strongly functional, since with the choices above, there is unique (φ,ψ,f) with $\alpha(\varphi,\psi,f)(C) = z_0$, namely $f = 1$, and $\varphi = \rho^{-1}_{\mathcal{L}',A}(6r)$, and $\psi = \rho^{-1}_{\mathcal{L}'',B}(-6r)$. So $\rho_{\mathcal{L},C}$ restricts to a one fold cover of $|z-z_0| \leq r$.

5. Miscellaneous Comments

One could also look at something I call a semiconfiguration space of a linkage $\mathcal{L}$. In a semiconfiguration space, you only record the position of some of the vertices, but ignore the positions of the rest. In other words it is a projection of the configuration space to some coordinate k-plane, if you ignore all but k vertices. Semiconfiguration spaces are semialgebraic sets (i.e., finite unions of differences of quasialgebraic sets). In another paper [K1], I give a complete characterization of semiconfiguration spaces. In particular, any compact semialgebraic set $K \subset \mathbb{C}^m$ is the semiconfiguration space of a linkage.

So for example, take any compact polyhedron K in $\mathbb{C}^m$. Then K is a semialgebraic set (since a simplex is a quasialgebraic set). Consequently, K is the semiconfiguration space of a linkage. Put another way, there is a linkage $\mathcal{L}$ and m vertices of $\mathcal{L}$ so that if we look at the image of these m vertices in $\mathbb{C}^m$, they exactly trace out K.

Looking at Lemma 1.7 brings up the question of how the number of fixed vertices affects the topology of $\mathcal{C}(\mathcal{L})$. In fact, for any linkage $\mathcal{L}$, there is a linkage $\mathcal{L}'$ with only three fixed vertices so that $\mathcal{C}(\mathcal{L})$ is isomorphic to $\mathcal{C}(\mathcal{L}')$. To see this, let $\mathcal{L}''$ be the linkage obtained from $\mathcal{L}$ by adding three vertices v_0, v_1, v_2, fixing them at $0, 1$, and $\sqrt{-1}$ respectively, and then for each fixed vertex v of $\mathcal{L}$, we add three edges, $\overline{vv_0}$ of length $|z|$, $\overline{vv_1}$ of length $|z-1|$ and $\overline{vv_2}$ of length $|z-\sqrt{-1}|$, where $z \in \mathbb{C}$ is the point where v is fixed. (This assumes z is not 0, 1 or $\sqrt{-1}$. If z is one of these values, we identify v with the corresponding v_i.) Let $\mathcal{L}'$ be obtained from $\mathcal{L}''$ by unfixing all vertices except v_0, v_1, v_2. Then $\mathcal{L}' \subset \mathcal{L}''$ and $\mathcal{L} \subset \mathcal{L}''$ and by Lemma 1.10, both the maps $\rho_{\mathcal{L}'',\mathcal{L}}$ and $\rho_{\mathcal{L}'',\mathcal{L}'}$ are isomorphisms.

Now what about linkages with exactly two fixed vertices? This was the main focus of [KM]. Assume first that the images of the two fixed vertices are different, otherwise we could identify them and obtain a linkage with just one fixed vertex and isomorphic $\mathcal{C}(\mathcal{L})$. If $\mathcal{L}$ has exactly two fixed vertices, fixed at different points of $\mathbb{C}$, there is an involution $\tau\colon \mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L})$ given by reflection about the line T through the two fixed points. Usually this involution is nontrivial, it is only trivial when all vertices are forced to lie on the line T. By Lemma 5.1 below, this can only occur if $\mathcal{C}(\mathcal{L})$ is a finite number of points. So we have a restriction on the topology of $\mathcal{C}(\mathcal{L})$, it is either a point or it supports a nontrivial involution.

If the linkage $\mathcal{L}$ has exactly one fixed vertex (or if all fixed vertices are fixed to the same point) and there is an edge containing this vertex, then $\mathcal{C}(\mathcal{L})$ is isomorphic to $\mathcal{C}(\mathcal{L}') \times S^1$ for some linkage $\mathcal{L}'$ with two fixed vertices. The linkage $\mathcal{L}'$ is obtained from $\mathcal{L}$ by fixing the other vertex on an edge containing the fixed vertex. The S^1 factor comes from rotating a planar realization of $\mathcal{L}'$ around the image of the fixed vertex of $\mathcal{L}$.

Finally Lemma 1.7 gives restrictions on the topology of $\mathcal{C}(\mathcal{L})$ if there are no fixed vertices.

I presume that the conclusion to Lemma 5.1 below could be sharpened to say that $\mathcal{C}(\mathcal{L})$ is a single point. I presume also that Lemma 5.1 remains true for cabled linkages. But in any case we have:

Lemma 5.1. *Suppose $\mathcal{L}$ is a classical linkage so that for some line $T \subset \mathbb{C}$, we have $\varphi(v) \in T$ for all vertices v of $\mathcal{L}$ and all $\varphi \in \mathcal{C}(\mathcal{L})$. Then $\mathcal{C}(\mathcal{L})$ is discrete.*

Proof. Suppose not. Let $\mathcal{L}$ be such a linkage with the least number of vertices so that $\mathcal{C}(\mathcal{L})$ is not discrete. Then for some vertex v_0 of $\mathcal{L}$, there is a one parameter family φ_t in $\mathcal{C}(\mathcal{L})$, $t \in (-b, b)$ so that $\varphi_t(v_0) \neq \varphi_0(v_0)$ for all $t \in (0, b)$. So $\varphi_t(v_0) = \varphi_0(v_0) + \alpha(t)z_0$ for some continuous α and constant z_0 parallel to T.

Let $v_1, v_2, \ldots, v_k$ be the vertices of $\mathcal{L}$ so that there is an edge in $\mathcal{L}$ between v_i and v_0. Order them so that $\varphi_0(v_i) \leq \varphi_0(v_j)$ if $i < j$. (For convenience we identify T isometricly with $\mathbb{R}$.) Let n be such that $\varphi_0(v_0) \geq \varphi_0(v_i)$ for all $i \leq n$ and $\varphi_0(v_0) < \varphi_0(v_i)$ for all $i > n$. Since there are no zero length edges, we must have $\varphi_0(v_0) \neq \varphi_0(v_i)$ for all $1 \leq i \leq k$. We must have $\ell(\overline{v_0v_i}) = |\varphi_t(v_0) - \varphi_t(v_i)|$. So by continuity, we must have $\varphi_t(v_i) = \varphi_0(v_i) + \alpha(t)z_0$ for all $i \leq k$ and $t \in (-b, b)$. In particular, if $\varphi_0(v_i) = \varphi_0(v_j)$, then $\varphi_t(v_i) = \varphi_t(v_j)$ for all $t \in (-b, b)$.

Note we must have $k \geq 2$ since if $k = 0$, then there are no constraints on the position of $\varphi(v_0)$ for $\varphi \in \mathcal{C}(\mathcal{L})$, and if $k = 1$, the only constraint is that $\varphi(v_0)$ lie in a certain circle about $\varphi(v_1)$. In neither case is $\varphi(v_0)$ forced to be in T.

Consider the following linkage $\mathcal{L}'$. The vertices of $\mathcal{L}'$ are all the vertices of $\mathcal{L}$ except for v_0. However, we identify two vertices v_i and v_j connected to v_0 if $\varphi_0(v_i) = \varphi_0(v_j)$. There is an edge in $\mathcal{L}'$ between vertices v and w if either:

1. There is an edge between v and w in $\mathcal{L}$. In this case, $\ell'(\overline{vw}) = \ell(\overline{vw})$, they have the same length.
2. They are both connected to v_0, i.e., $v = v_i$ and $w = v_j$ for some $1 \leq i \neq j \leq k$. In this case $\ell'(\overline{v_iv_j})$=$|\varphi_0(v_i) - \varphi_0(v_j)|$.

Note that by restricting φ_t to $\mathcal{L}'$ we get a parameterized family $\varphi'_t \in \mathcal{C}(\mathcal{L}')$ so that $\varphi'_t(v_i) \neq \varphi'_0(v_i)$ for $t \in (0, b)$.

Take any $\varphi \in \mathcal{C}(\mathcal{L}')$. We will show that $\varphi(v) \in T$ for all vertices v of $\mathcal{L}'$. But considering φ'_t, we see that $\mathcal{C}(\mathcal{L}')$ is not discrete, which contradicts minimality of $\mathcal{L}$.

Suppose that $n \neq 0, k$, and thus $\varphi_0(v_1) < \varphi_0(v_0) < \varphi_0(v_k)$. For any $1 < i \leq n$ then:

$$\begin{aligned}\ell'(\overline{v_1v_i}) + \ell'(\overline{v_iv_k}) &= (\ell(\overline{v_0v_1}) - \ell(\overline{v_0v_i})) + (\ell(\overline{v_0v_k}) + \ell(\overline{v_0v_i})) \\ &= \ell(\overline{v_0v_1}) + \ell(\overline{v_0v_k}) = \ell'(\overline{v_1v_k}).\end{aligned}$$

For any $n < i < k$ then:

$$\begin{aligned}\ell'(\overline{v_1v_i}) + \ell'(\overline{v_iv_k}) &= (\ell(\overline{v_0v_1}) + \ell(\overline{v_0v_i})) + (\ell(\overline{v_0v_k}) - \ell(\overline{v_0v_i})) \\ &= \ell(\overline{v_0v_1}) + \ell(\overline{v_0v_k}) = \ell'(\overline{v_1v_k})\end{aligned}$$

so $\varphi(v_i)$ lies on the line segment between $\varphi(v_1)$ and $\varphi(v_k)$. Moreover, if we identify this line segment isometricly with $[\varphi_0(v_1), \varphi_0(v_k)]$ we must have $\varphi(v_i) = \varphi_0(v_i)$. To be precise, there is a Euclidean motion β of $\mathbb{C}$ so that $\varphi(v_i) = \beta\varphi_0(v_i)$ for all $1 \leq i \leq k$. So from φ we get a $\varphi' \in \mathcal{C}(\mathcal{L})$ given by $\varphi'(v) = \varphi(v)$ for $v \neq v_0$ and $\varphi'(v_0) = \beta\varphi_0(v_0)$. By our supposition, all $\varphi'(v)$ must lie on the line T. Hence $\varphi(v) \in T$ for all vertices v of $\mathcal{L}'$.

Now suppose that $n = 0$. Then:

$$\begin{aligned}\ell'(\overline{v_1v_i}) + \ell'(\overline{v_iv_k}) &= (\ell(\overline{v_0v_i}) - \ell(\overline{v_0v_1})) + (\ell(\overline{v_0v_k}) - \ell(\overline{v_0v_i})) \\ &= \ell(\overline{v_0v_k}) - \ell(\overline{v_0v_1}) = \ell'(\overline{v_1v_k}).\end{aligned}$$

On the other hand, if $n = k$, then:

$$\begin{aligned}\ell'(\overline{v_1v_i}) + \ell'(\overline{v_iv_k}) &= (\ell(\overline{v_0v_1}) - \ell(\overline{v_0v_i})) + (\ell(\overline{v_0v_i}) - \ell(\overline{v_0v_k})) \\ &= \ell(\overline{v_0v_1}) - \ell(\overline{v_0v_k}) = \ell'(\overline{v_1v_k})\end{aligned}$$

So similarly, we see that the vertices of $\mathcal{L}'$ are forced to lie in T. □

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Department of Mathematics, University of Maryland, College Park, Maryland, 20742
E-mail address: hking@math.umd.edu