Tr. J. of Mathematics
23 (1999) , 257 – 264.

ON THE BOUNDEDNESS OF INTEGRAL OPERATORS

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Abstract

In this paper, we consider integal operators defined by kernel functions. It is well known the boundedness of such kind of operators as Shur Lemma type statements. But, the norm of operators was estimated by two integrals of kernel function. We obtain estimation of operators norm by one integral of kernel function.

1. Introduction

Let $L^p(R^n)$ be the space of all integrable functions with degree p and denote by $L^p_{loc}(R^n)$ the space of all local integrable functions with degree p in particular denote by $L^\infty_{loc}(R^n)$ the space of local bounded functions. Let $\mathcal{D}_1 \subset L^p(R^n)$ be a subspace of $L^p(R^n)$.

Consider a linear operator $T : \mathcal{D}_1 \to \mathcal{D}_2$, where $\mathcal{D}_2 \subset L^q(R^n)$.

Definition *Operator T is called an operator of type (p, q) if there exists a real number c such that for any $f \in \mathcal{D}_1$ it holds the following inequality $\| Tf \|_q \leq c \| f \|_p$, where $\| \cdot \|_p$ is a natural norm of the space $L_p(R^n)$ (see [3]).*

Note that if $\mathcal{D}_1$ is a dense set on $L^p(R^n)$ then the operator T has a bounded extension from $L^p(R^n)$ to $L^q(R^n)$. Let T be an integral operator with the kernel function $K(x, y)$ and operator T is defined from $C^\infty_0(R^n)$ to $L^2_{loc}(R^n)$. It is well known the statements about bonudedness of integral operators. In the statements of type Shur Lemma the L^2-norm of operators is estimated by both $\sup_y \int |K(x, y)|dx$ and $\sup_x \int |K(x, y)|dy$. i.e. by two integrals of kernel functions (see [1]).

We consider the integral operator of type

$$(Tf)(y) = \int_{R^n} \frac{K(x,y)f(x)dx}{|y|^n}.$$

*

Denote by σ_A the set of $R^n \times R^n$:

$$\sigma_A \equiv \{(x,y): \ x \in R^n, \ y \in R^n : |y| \le A|X|\},$$

where A is a fixed positive.

Our main result consists of the following Theorem.

Theorem 1.1. *If supp $K \subset \sigma_A$, and there exists a number $p > 2$ such that*

$$\int_{R^n} |K(\frac{y|x|}{|y|^2}, x)|^p dy = \rho(x) \in L^\infty(R^n),$$

then the integral operator (1.1) has a bounded extension on $L^p(R^n)$.

Now, consider also one variant of the statement for one-dimensional case. Let $K(\xi, x)$ be a measurable function. Assume that $suppK(.,x) \subset [a,b]$ and consider integral operator

$$(Tf)(x) = \int_R \frac{K(\frac{x}{y}, x)}{y} f(y)dy \tag{0.1}$$

where $f \in C_0^\infty(R \backslash \{0\})$.

It holds the following theorem.

Theorem 1.2. *Let there exists a number $p > 2$ such that*

$$\int_{R^n} |K(\xi, x)|^p d\xi = c(x) \in L^\infty(R),$$

then the integral operator (1.2) has a bounded extension on $L^2(R)$.

* 1991 mathematics Subject Classification: 44A15, 47B37

2. Shur Lemma on the Boundedness of Integral Operators

Let T_B be an integral operator with the kernel function $B(x,y)$ and opertor T_B is defined from $C_0^\infty(R^n)$ to $L^2_{loc}(R^n)$:

$$(T_B f)(x) = \int B(x,y)f(y)dy.$$

It is well known the Shur Lemma and its generalizations on the boundedness of the operators [1,3]. The following Lemma was proved in (see [2], p.47).

Lemma 2.1. *Let there exists non-negative numbers ε, c and a mesurable function $M(x,y)$ such that the following inequalities*

$$|B(x,y)| \le M(x,y),$$
$$\int_{R^n} M(x,y)|x|^{-\varepsilon}dx \le c|y|^{-\varepsilon},$$
$$\int_{R^n} M(x,y)|y|^{-\varepsilon}dy \le c|x|^{-\varepsilon}$$

hold. Then the integral operator T_B has a bounded extension on $L^2(R^n)$ and its norm is estimated by c.

Firstly we consider the application of Lemma 2.1 to the special integral operator.

Let $\rho \in L^\infty(R^n \times R^n)$ be a fixed function and $\mathcal{D} = C_0^\infty(R^n \backslash \{0\})$. Consider integral operator $T : \mathcal{D} \to L^2_{loc}(R^n)$ given by

$$Tf(y) = \int_{R^n} \frac{\rho(x,y)f(x)dx}{|x|^\alpha |y|^\beta}, \tag{0.2}$$

where α, β are fixed positive numbers and $\alpha + \beta = n$.
The following lemma is needed for the sequel.

Lemma 2.2. *Let A be a fixed positive number. If $supp\rho \subset \sigma_A$ and $\alpha < \beta$ then the operator T defined by the formula (2.1) has a type (2.2).*

Proof. Without loss of generality we can assume that $A = 1$. Let $\alpha < \varepsilon < \beta$ be a fixed positive number. Then it is easy to show that the following inequalities:

$$\int_{R^n} \frac{|\rho(x,y)dx}{|x|^{\varepsilon}|x|^{\alpha}|x|^{\beta}} \leq \frac{H}{|y|^{\varepsilon}},$$
$$\int_{R^n} \frac{|\rho(x,y)dy}{|x|^{\alpha}|y|^{\beta}|y|^{\varepsilon}} \leq \frac{H}{|x|^{\varepsilon}}$$

hold, where H is a some constant.

Consequently, by the Lemma 2.1 the integral operator T has a bounded extension on $L^2(R^n)$, therefore, it has a type (2,2) and the proof is completed □

3. Proof of Theorems

In this section we will prove the Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 We show that the integral operator (1.1) has a type (2.2). Without loss of generality we can assume that $A = 1$. Let $f \in C_0^{\infty}(R^n \backslash \{0\})$ be a fixed function. Then we have

$$|Tf(y)|^2 = \int_{R^n}\int_{R^n} \frac{K(y,x_1)\overline{K(y,x_2)}}{|y|^{2n}} f(x_1)\overline{f(x_2)}dx_1dx_2.$$

We will represent the integral $|Tf(y)|^2$ as a sum of two integrals.

$$\begin{aligned}|Tf(y)|^2 &= \int\int_{|x_1|\geq|x_2|} \frac{K(y,x_1)\overline{K(y,x_2)}}{|y|^{2n}} f(x_1)\overline{f(x_2)}dx_1dx_2 \\ &+ \int\int_{|x_1|\leq|x_2|} \frac{K(y,x_1)\overline{K(y,x_2)}}{|y|^{2n}} f(x_1)\overline{f(x_2)}dx_1dx_2\end{aligned}$$

We will prove that the following inequality

$$\int_{R^n}\int\int_{|x_1|\geq|x_2|} \frac{\left|K(y,x_1)\overline{K(y,x_2)}\right|}{|y|^{2n}} \left|f(x_1)\overline{f(x_2)}\right| dx_1dx_2dy \leq L \parallel f \parallel^2 \qquad (0.3)$$

holds, where L is a positive number. Let q be an adjoint number to p, i.e., $\frac{1}{p}+\frac{1}{q}=1$. Then by using the Hölder's inequality we obtain

$$\begin{aligned}
&\int_{R^n} dy \int\int_{|x_1|\geq|x_2|} \frac{\left|K(y,x_1)\overline{K(y,x_2)}\right|}{|y|^{2n}} |f(x_1)f(x_2)|\, dx_1dx_2 \\
=& \int\int_{|x_1|\geq|x_2|} |f(x_1)f(x_2)|\, dx_1dx_2 \int_{R^n} \frac{\left|K(y,x_1)\overline{K(y,x_2)}\right|}{|y|^{2n}} dy \leq \\
\leq& \int\int_{|x_1|\geq|x_2|} |f(x_1)f(x_2)|\, dx_1dx_2 \\
\times& \left(\int_{R^n} \frac{|K(y,x_2)|^p\, dy}{|y|^{2n}}\right)^{\frac{1}{p}} \left(\int_{R^n} \frac{|K(y,x_1)|^q\, dy}{|y|^{2n}}\right)^{\frac{1}{q}}
\end{aligned}$$

Changing of variables by $y = \frac{z|x_2|}{|z|^2}$ we get for the jacobian the relation

$$J(x_2,z) = \frac{|x_2|^n}{|z|^{2n}} a(x_2,z),$$

where $a(x_2,z)$ is some uniformly bounded function on $R^n \times R^n$.
It is clear that if $(y,x_2) \in \sigma$, then $|z| = \frac{|x_2|}{|y|} \leq 1$. Consequently, we obtain

$$\begin{aligned}
\int_{R^n} \frac{|K(y,x_2)|^p}{|y|^{2n}} dy &= \int_{|z|\leq 1} \frac{\left|K(\frac{z|x_2|}{|z|^2},x_2)\right|^p}{|x_2|^n} |a(x_2,z)dz \\
&\leq \frac{\|\, a\,\|_\infty}{|x_2|^n} \int_{|z|\leq 1} \left|K(\frac{z|x_2|}{|z|^2},x_2)\right|^p dz.
\end{aligned}$$

Hence

$$\int_{R^n} \frac{|K(y,x_2)|^p}{|y|^{2n}} dy \leq \frac{\rho_1(x_2)}{|x_2|^n}$$

where $\rho_1 \in L^\infty(R^n)$.
At the same time by using Hölder's inequality we get

$$\begin{aligned}
\int_{R^n} \frac{|K(y,x_1)|^q}{|y|^{2n}} dy &= \int_{|z|\leq 1} \frac{\left|K(\frac{z|x_1|}{|z|^2},x_1)\right|^q}{|x_1|^n} |a(x_1,z)dz \\
&\leq \frac{b_n \|\, a\,\|_\infty}{|x_1|^n} \left(\int_{|z|\leq 1} \left|K(\frac{z|x_1|}{|z|^2},x_1)\right|^p dz\right)^{\frac{q}{p}},
\end{aligned}$$

where b_n is some constant. Thus we have

$$\int_{R^n} \frac{|K(y,x_1)|^q}{|y|^{2n}} dy \le \frac{\rho_2(x_1)}{|x_1|^n}$$

where $\rho_2 \in L^\infty(R^n)$.

Let $\chi(x_1,x_2)$ be an indicator function of the set

$$\sigma_1 = \{(x_1,x_2) \in R^n \times R^n : |x_1| \ge |x_2|\}.$$

Consider an integral operator

$$T_1 f(x_1) = \int_{R^n} \frac{\chi(x_1,x_2)\rho_1(x_2)\rho_2(x_1)}{|x_1|^{\frac{n}{q}}|x_2|^{\frac{n}{p}}} f(x_2)dx_2.$$

By the Lemma 2.2 the integral operator T_1 has a type (2,2) and consequently has a bounded extension on $L^2(R^n)$. This implies that for any $f \in L^2(R^n)$ it holds the following inequality

$$\| T_1 f \| \le M \| f \|, \tag{0.4}$$

where M is a constant and norm $\| \cdot \|$ is $L^2(R^n)$.

It is not hard to prove that the following inequality

$$\int_{R^n} dy \int_{|x_1|\ge|x_2|} \frac{\left|K(y,x_1)\overline{K(y,x_2)}\right|}{|y|^{2n}} |f(x_1)f(x_2)|dx_1dx_2 \le \int_{R^n} |f(x_1)|\,|T_1 f(x_1)|dx_1$$

holds.

Therefore we get (3.1) by using (3.2) and Schwartz inequality. The case $|x_2| \ge |x_1|$ may be considered by the analogy. This completes the proof of Theorem 1.2.

Corollary 3.1 *Let $K(y,x)$ be a homogeneous function of degree zero with respect to y and $suppK(y,x) \subset \sigma_a$. If the following relation*

$$\int_{|y|\le 1} |K(y,x)|^p ds(y) \in L^\infty(R^n)$$

holds for some $p > 2$ *, where ds(y) is the Lebesque measure on* S^{n-1} *, then the integral operator (1.1) has a bounded extension on* $L^2(R^n)$ *.*

The Corollary 3.1 is proved by the analogy of Theorem 1.1.

Proof of Theorem 1.2 The noting by $T^\star$ adwoint operator of T we see

$$(T^\star f)(y) = \int_R \frac{K(\frac{x}{y}, x)}{y} f(x)dx.$$

We show that $T^\star$ has a type (2,2). Let $f \in C_0^\infty(R \backslash \{0\})$ be a fixed function.

Then we have

$$\begin{aligned} |T^\star f(y)|^2 \quad &= \quad \left| \int\int_{|x_1| \geq |x_2|} \frac{K(\frac{x_1}{y}, x_1), \overline{K(\frac{x_2}{y}, x_2)} f(x_1) \overline{f(x_2)}}{y^2} dx_1 dx_2 + \right. \\ &+ \quad \left. \int\int_{|x_1| \leq |x_2|} \frac{K(\frac{x_1}{y}, x_1), \overline{K(\frac{x_2}{y}, x_2)} f(x_1) \overline{f(x_2)}}{y^2} dx_1 dx_2 \right| . \end{aligned}$$

Now we prove the analogy of inequality (3.1). By using the Hölder's inequality we have

$$\int dy \left| \int\int_{|x_1| \geq |x_2|} \frac{K(\frac{x_1}{y}, x_1), \overline{K(\frac{x_2}{y}, x_2)} f(x_1) \overline{f(x_2)}}{y^2} dx_1 dx_2 \right| \leq$$

$$\leq \int\int_{|x_1| \geq |x_2|} |f(x_1) f(x_2)| \left(\int_R \frac{\left| K(\frac{x_1}{y_1}, x_1) \right|^p}{y_1^2} dy_1 \right)^{\frac{1}{p}}$$

$$\left(\int_R \frac{\left| K(\frac{x_2}{y_2}, x_2) \right|^q}{y_2^2} dy_2 \right)^{\frac{1}{q}} dx_1 dx_2.$$

Let us use a change of variables $\frac{x_1}{y_1} = \zeta$ we have

$$\int_R \frac{\left| K(\frac{x_1}{y_1}, x_1) \right|^p}{y_1^2} dy_1 = \int_a^b \frac{|K(\zeta, x_1)^p}{|x_1|} d\zeta \leq \frac{c_1(x_1)}{|x_1|}$$

where $\rho_1 \in L^\infty(\mathbf{R})$.

By the analogy we get

$$\int_{\mathbf{R}} \frac{\left|K(\frac{x_2}{y_2}, x_2)\right|^q}{y_2^2} dy_2 \leq \frac{\rho_2(x_2)}{|x_2|},$$

where $\rho_2 \in L^\infty(R)$ by the condition of the theorem, since $q < 2$. Now by using Lemma 2.1 as above we arrive to proof of Theorem 1.2.

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Received 02.03.1998

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