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SOME ASYMPTOTIC ESTIMATES FOR THE INSTABILITY INTERVALS OF HILL'S EQUATION

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Abstract

In this study, some asymptotic estimates for the length of the instability intervals of Hill's equation, $y''(t) + (\lambda - q(t))y(t) = 0$, are derived by means of an auxiliary eigenvalue problem under various assumptions on the Fourier coefficients of the potential q.

1. Introduction

In this study, we consider the Hill's equation

$$y''(t) + (\lambda - q(t))y(t) = 0. \tag{1}$$

We assume that $q(t) \in L_1[0,a]$, periodic with period a and $\int_0^a q(t)dt = 0$, that is, $q(t)$ has a zero mean value. The last condition is clearly not a restriction on $q(t)$. If $q(t)$ has a nonzero mean value, say c, we work on the following equation

$$y''(t) + \left[(\lambda - \frac{c}{a}) - (q(t) - \frac{c}{a})\right] y(t) = 0,$$

in which

$$q(t) - \frac{c}{a}$$

has a mean value zero. We consider two different types of boundary conditions:

$$\begin{aligned} y(0) &= y(a), \\ y'(0) &= y'(a), \end{aligned} \tag{2}$$

and

$$\begin{aligned} y(a) &= -y(0), \\ y'(a) &= -y'(0) \end{aligned} \tag{3}$$

which are known as the periodic and semi-periodic boundary conditions, respectively. If $\lambda_n\,(n = 0, 1, \ldots)$ denotes the periodic eigenvalues of the problem (1)-(2), and μ_n denotes the semi-periodic eigenvalues of the problem (1)-(3), then the intervals (λ_{2m}, μ_{2m}) and $(\mu_{2m+1}, \lambda_{2m+1})$ are called the stability intervals of (1) whereas the intervals $(-\infty, \lambda_0)$, (μ_{2m}, μ_{2m+1}) and $(\lambda_{2m+1}, \lambda_{2m+2})$ are called the instability intervals of (1) and referred to as the zero-th, $(2m+1)-th$ and $(2m+2)-th$ instability interval, respectively. The length of the $n-th$ instability interval of (1), whether it is absent or not, will be denoted by l_n. We note that the absence of an instability interval means that there is a value of λ for which all solutions of (1) have either period a or semi-period a.

We note that a more general second order periodic differential equation

$$\{p(x)z'(x)\}' + \{\lambda s(x) - q(x)\}z(x) = 0 \tag{4}$$

can be reduced to an equation of type (1) if $p''(x)$, $s''(x)$ exist and are integrable. Here $p(x)$, $q(x)$, $s(x)$ are real-valued piecewise continuous periodic functions with the same period a, $-\infty < x < \infty$ and λ is a real parameter. To see this, we apply to (4) the Liouville transformation

$$t = \int_0^x [s(u)/p(u)]^{1/2} du, \;\; y(t) = [p(x)s(x)]^{1/4} z(x).$$

We then have

$$y''(t) + [\lambda - Q(t)]y(t) = 0, \tag{5}$$

where

$$Q(t) = q(x) - p^{\frac{1}{4}}(x)s^{-\frac{3}{4}}(x)\frac{d}{dx}p(x)\frac{d}{dx}[p(x)s(x)]^{\frac{-1}{4}}. \tag{6}$$

We also note that λ_n and μ_n, $(n = 0, 1, \ldots)$ for (5) are the same as those of (1)[4].

The following results are known for the instability intervals of (4).

Theorem 1.1. *[4] As $n \to \infty$, l_n satisfies*

1. *(i)* $l_n = o(n^2)$*;*

2. *(ii)* $l_n = o(n)$ *if* $s'(x)$ *exists and is piecewise continuous;*

3. *(iii)* $l_n = o(n^{-r})$ *if* $s^{(r+2)}(x), p^{(r+2)}(x),$ *and* $q^{(r)}(x)$ *all exist and are piecewise continuous.*

Theorem 1.2. *[4] As* $n \to \infty$*,* l_n *satisfies*

1. *(i)* $l_n = O(n)$ *if* $s(x)$ *is piecewise smooth;*

2. *(ii)* $l_n = O(1)$ *if* $s'(x)$ *exists and is piecewise smooth and* $p'(x)$ *is piecewise smooth;*

3. *(iii)* $l_n = O(n^{-r-1})$ *if* $s^{(r+2)}(x), p^{(r+2)}(x),$ *and* $q^{(r)}(x)$ *all exist and are piecewise smooth.*

It is also known that if $p(x) = s(x) = 1$ and if $q(x)$ is piecewise continuous then

$$l_n = \sqrt{a_n^2 + b_n^2} + O(n^{-1/2}).$$

By considering the equations (1) and (4), we see that if $p''(x)$ and $s''(x)$ exist and are integrable, then the estimates for the periodic and semi-periodic eigenvalues of (4), thus the instability intervals of (4), can be obtained through the simpler equation (1). Therefore, we work on equation (1) to derive estimates for the length l_n of the $n-th$ instability interval of (1) as $n \to \infty$.

The procedure we follow is quite different from the existing ones in the sense that we derive the estimates through what we call an auxiliary eçgenvalue problem, whereas the classical procedures works directly on the problem. One of the advantages of our method is that it allows us to obtain the existing estimates by putting less restrictive assumptions on $q(t)$. More precisely, we will show that we can replace the assumption of piecewise continuity by integrability. Unlike the results reported in the literature, we put assumptions on the Fourier coefficients of $q(t)$, instead of $q(t)$ and its derivatives of various orders.

The paper is organized as follows: In §2 we introduce an auxiliary eigenvalue problem and the modified Prufer transformation, and define a sequence of approximating functions for (1) under the transformation. We then state some more results relevant to the subject of this study. In §3, we state and prove our main results. In §4, we compare our results with the reported results in the literature and highlight our contributions.

2. An auxiliary eigenvalue problem and the modified Prufer transformation

In this section we introduce another type of eigenvalue problem for (1) over $[\tau, \tau+a]$ with the following boundary condition

$$y(\tau) = y(\tau + a) = 0, \tag{7}$$

where $0 \leq \tau < a$. We refer to the problem (1) and (7) as the "auxiliary eigenvalue problem ", and we denote the eigenvalues of this problem by $\Lambda_n(\tau)\,(n = 0, 1, \ldots)$. Here, we note that (1) and (7) are equivalent to the following problem[6]

$$y''(t) + (\lambda - q(t + \tau))y(t) = 0, \tag{8}$$

$$y(0) = y(a) = 0. \tag{9}$$

We introduce the function $\Theta(x, \Lambda, \tau)$, the so-called modified Prufer transformation of [1], which is defined for any given solution of (1) by

$$tan\theta(t, \Lambda, \tau) = \frac{\Lambda^{1/2} y(t, \tau)}{y'(t, \tau)},$$

for $\tau \leq t \leq \tau + a$. This fixes Θ to within additive multiples of π. For definiteness we assume that $0 \leq \theta(\tau) < \pi$ and we observe that the boundary condition (7) corresponds to

$$\Theta(\tau) = 0, \quad \Theta(\tau + a) = (n + 1)\pi,$$

for integral n. From now on, we will be suppressing the dependence of Θ on Λ and simply write it as $\Theta(t, \tau)$.

Under the Prufer transformation, the differential equation corresponding to (1) can be written as

$$\Theta'(t, \tau) = \Lambda^{1/2} - \frac{1}{2}\Lambda^{-1/2}q(t) + \frac{1}{2}\Lambda^{-1/2}q(t)cos(2\Theta(t, \tau)) \tag{10}$$

which leads to

$$\Theta(x,\tau)=\Lambda^{1/2}(x-\tau)-\frac{1}{2}\Lambda^{-1/2}\int_{\tau}^{x}q(t)dt+\frac{1}{2}\Lambda^{-1/2}\int_{\tau}^{x}q(t)cos(2\theta(t,\tau))dt. \tag{11}$$

We define a sequence of approximating functions for (11) as follows

$$\begin{aligned}\Theta_1(t,\tau)&:=\Lambda^{1/2}(t,\tau)-\frac{1}{2}\Lambda^{-1/2}\int_{\tau}^{t}q(s)ds,\\ \Theta_{k+1}(t,\tau)&:=\Theta_1(t,\tau)+\frac{1}{2}\Lambda^{-1/2}\int_{\tau}^{t}q(s)cos(2\Theta_k(s,\tau))ds,\end{aligned} \tag{12}$$

for $k=1,2,\ldots,$ and $\tau\le t\le\tau+a$. In what follows, we give some results which will be used in this study.

Theorem 2.1. *[4] The ranges of $\Lambda_{2m}(\tau)$ and $\Lambda_{2m+1}(\tau)$, as functions of τ are $[\mu_{2m},\mu_{2m+1}]$ and $[\lambda_{2m+1},\lambda_{2m+2}]$, respectively.*

By Theorem 2.1 and the fact that $\Lambda_n(\tau)$ is a continuous function of τ, we observe that

$$\max_{\tau}\Lambda_{2m}(\tau)=\mu_{2m+1},\quad \min_{\tau}\Lambda_{2m}(\tau)=\mu_{2m}$$

and,

$$\max_{\tau}\Lambda_{2m+1}(\tau)=\lambda_{2m+2},\quad \min_{\tau}\Lambda_{2m+1}(\tau)=\lambda_{2m+1}.$$

Theorem 2.2. *[8] If $f(x)$ is integrable over (a,b), then as $\lambda\to\infty$*

$$\int_a^b f(x)cos(\lambda x)dx=o(1),\quad \int_a^b f(x)sin(\lambda x)dx=o(1).$$

Lemma 2.1. *[6] Let $q(z)$ be a periodic function with period π, integrable over $[0,\pi]$ and such that*

$$c_n=\frac{1}{\pi}\int_0^{\pi}q(z)e^{-2inz}dz=O(\frac{1}{n^2}).$$

Then $q(z)$ is absolutely continuous almost everywhere.

Lemma 2.2. *[5] For* $k = 1, 2, 3, \ldots,\ \tau \leq t \leq \tau + a$

$$\Theta(t, \tau) - \Theta_k(t, \tau) = o(\lambda^{-k/2})$$

as $\lambda \to \infty$.

Lemma 2.3. *[1] For* q *integrable and for any* x_1, x_2 *such that* $\tau \leq x_1 < x_2 \leq \tau + a$

$$\int_{x_1}^{x_2} q(t) sin(2\lambda^{1/2} t) dt = o(1)$$

as $\lambda \to \infty$.

Lemma 2.4. *[8] If* f *is of bounded variation, then*

$$a_n = O(n^{-1}),\ b_n = O(n^{-1})$$

where a_n , b_n are the real Fourier coefficients of $q(t)$ referred to the interval $[0, a]$ which are defined as

$$\begin{aligned} a_n &= \frac{2}{a} \int_0^a q(t) cos(\frac{2n\pi}{a} t) dt, \\ b_n &= \frac{2}{a} \int_0^a q(t) sin(\frac{2n\pi}{a} t) dt. \end{aligned}$$

Lemma 2.5. *[8] If q(t) is an integral, that is,*

$$q(t) = q(0) + \int_0^t f(x) dx, \quad x \geq 0,$$

and has the period a *then* $a_n = o(n^{-1}), b_n = o(n^{-1})$, *where* a_n *and* b_n *are the Fourier coefficients of* $q(t)$ *on* $[0, a]$.

3. Statement and Proof of Results

In this section, we state and prove our main results.

Theorem 3.1. *If $a_n = O(n^{-1})$, $b_n = O(n^{-1})$ then as $n \to \infty$ $l_n = O(n^{-1})$.*

Proof. We consider the differential equation (8) with the boundary condition (9). As noted before, the corresponding Θ equation is

$$\Theta'(x,\tau) = \Lambda^{1/2} - \frac{1}{2}\Lambda^{-1/2}q(x+\tau) + \frac{1}{2}\Lambda^{-1/2}q(x+\tau)cos(2\Theta(x,\tau)). \tag{13}$$

Integrating (13) over $[0,a]$ and using the boundary condition (9) with the assumption that $q(t)$ has a vanishing meanvalue, we get

$$(n+1)\pi = \Lambda^{1/2}a + \frac{1}{2}\lambda^{-1/2}\int_0^a q(t+\tau)cos(2\Theta(t,\tau))dt. \tag{14}$$

We know from Lemma 2.2 that

$$\Theta(t,\tau) - \Theta_1(t,\tau) = o(\lambda^{-1/2}),$$

so that

$$cos(2\Theta(t,\tau)) = cos(2\Theta_1(t,\tau)) + o(\Lambda^{-1/2}). \tag{15}$$

Substitution of (15) into (14) yields

$$(n+1)\pi = \Lambda^{1/2}a + \frac{1}{2}\Lambda^{-1/2}\int_0^a q(t+\tau)cos(2\Theta_1(t,\tau))dt + o(\Lambda^{-1}). \tag{16}$$

From (12) we find that

$$\begin{aligned} cos(2\Theta_1(t,\tau)) &= cos(2\Lambda^{1/2}t) + \Lambda^{-1/2}(\int_0^t q(x+\tau)dx)sin(2\Lambda^{1/2}t) \\ &\quad +O(\Lambda^{-1}). \end{aligned} \tag{17}$$

Substituting (17) in (16) and using Lemma 2.3 we find that

$$(n+1)\pi = \Lambda^{1/2}a + \frac{1}{2}\Lambda^{-1/2}\int_0^a q(t+\tau)cos(2\Lambda^{1/2}t)dt + o(\Lambda^{-1}). \tag{18}$$

Then reversion on (18) leads to

$$\begin{aligned}\Lambda_n^{1/2}(\tau) \quad = \quad & \frac{(n+1)\pi}{a} - \frac{1}{2(n+1)\pi}\int_0^a q(t+\tau)cos(\frac{2(n+1)\pi}{a}t)dt \\ & +o(n^{-2}).\end{aligned} \tag{19}$$

Upon introducing a change of variable $t+\tau = u$ and using the fact that for any periodic function f with the period a

$$\int_\tau^{\tau+a} f(t)dt = \int_0^a f(t)dt \tag{20}$$

we get

$$\begin{aligned}\Lambda_n^{1/2}(\tau) \quad &= \quad \frac{(n+1)\pi}{a} - \frac{a}{4(n+1)\pi}\left[cos(\frac{2(n+1)\pi}{a}\tau)a_{n+1} + sin(\frac{2(n+1)\pi}{a}t)b_{n+1}\right] \\ &+ \quad o(n^{-2}).\end{aligned} \tag{21}$$

The terms involving a_{n+1} and b_{n+1} in (21) included in the error term by the hypothesis. Hence

$$\Lambda_n^{1/2}(\tau) = \frac{(n+1)\pi}{a} + O(n^{-2}) \tag{22}$$

for any τ in $[0,a)$. From (22) and Theorem 2.1 we have

$$\mu_{2m}^{1/2} = \frac{(2m+1)\pi}{a} + O(m^{-2})$$
$$\mu_{2m+1}^{1/2} = \frac{(2m+1)\pi}{a} + O(m^{-2}).$$

Hence,

$$l_{2m+1} \quad = \quad \mu_{2m+1} - \mu_{2m}$$

$$
\begin{aligned}
&= (\mu_{2m+1}^{1/2} + \mu_{2m}^{1/2})O(m^{-2}) \\
&= O(m^{-1}).
\end{aligned}
$$

Similar result holds for l_{2m+2}.

We note that if $q(t)$ is of bounded variation then by Lemma 2.4, it satisfies the hypothesis of Theorem 3.1. □

Theorem 3.2. *if $a_n = o(n^{-1})$ and $b_n = o(n^{-1})$ then as $n \to \infty$, $l_n = o(n^{-1})$.*

Proof. Similar proof as in Theorem 3.1 goes through by replacing the O terms with o terms.

We observe that if $q(t)$ is an integral then the hypothesis of Theorem 3.2 are satisfied by Lemma 2.5. □

Theorem 3.3. *if $a_n = O(n^{-2})$ and $b_n = O(n^{-2})$ then as $n \to \infty$, $l_n = O(n^{-2})$.*

Proof. Taking $k = 2$ in Lemma 2.2 we get

$$\Theta(x,\tau) - \Theta_2(x,\tau) = o(\Lambda^{-1})$$

so that

$$cos(2\Theta(x,\tau)) = cos(2\Theta_2(x,\tau)) + o(\Lambda^{-1}). \tag{23}$$

Substituting (23) into (14) we get

$$(n+1)\pi = \Lambda^{1/2}a + \frac{1}{2}\Lambda^{-1/2}\int_0^a q(t+\tau)cos(2\Theta_2(t,\tau))dt + o(\Lambda^{-3/2}). \tag{24}$$

From (12) we find that

$$
\begin{aligned}
cos(2\Theta_2(t,\tau)) &= cos(2\Lambda^{1/2}t) + \Lambda^{-1/2}sin(2\Lambda^{1/2}t)\int_0^t q(x+\tau)dx \\
&- \Lambda^{-1/2}sin(2\Lambda^{1/2}t)\int_0^t q(x+\tau)cos(2\Lambda^{1/2}x)dx \\
&+ O(\Lambda^{-1}). \qquad (25)
\end{aligned}
$$

Substitution of (25) into (24) yields

$$
\begin{aligned}
(n+1)\pi = \Lambda^{1/2}a &+ \frac{1}{2}\Lambda^{-1/2}\int_0^a q(t+\tau)cos(2\Lambda^{1/2}t)dt \\
&+ \frac{1}{2}\Lambda^{-1}\int_0^a q(t+\tau)(\int_0^t q(x+\tau)dx)sin(2\Lambda^{1/2}t)dt \\
&- \frac{1}{2}\Lambda^{-1}\int_0^a q(t+\tau)(\int_0^t q(x+\tau)cos(2\Lambda^{1/2}x)dx)sin(2\Lambda^{1/2}t)dt \\
&+ O(\Lambda^{-3/2}). \qquad (26)
\end{aligned}
$$

It follows from Lemma 2.1 that $q'(t)$ is integrable. Hence, integration by parts results in

$$
\begin{aligned}
&\int_0^a q(t+\tau)(\int_0^t q(x+\tau)cos(2\Lambda^{1/2}x)dx)sin(2\Lambda^{1/2}t)dt \\
&= -\frac{1}{2}\Lambda^{-1/2}cos(2\Lambda^{1/2}a)q(a+\tau)(\int_0^a q(t+\tau)cos(2\Lambda^{1/2}t)dt) \\
&+ \frac{1}{2}\Lambda^{-1/2}\int_0^a q'(t+\tau)(\int_0^t q(x+\tau)cos(2\Lambda^{1/2}x)dx)cos(2\Lambda^{1/2}t)dt \\
&+ \frac{1}{2}\Lambda^{-1/2}\int_0^a (q^2(t+\tau)cos^2(2\Lambda^{1/2}t)dt. \qquad (27)
\end{aligned}
$$

Substituting (27) in (26) and using Theorem 2.2, we observe that

$$
\begin{aligned}
(n+1)\pi = \Lambda^{1/2}a &+ \frac{1}{2}\Lambda^{-1/2}\int_0^a q(t+\tau)cos(2\Lambda^{1/2}t)dt \\
&+ \frac{1}{2}\Lambda^{-1}\int_0^a q(t+\tau)(\int_0^t q(x+\tau)dx)sin(2\Lambda^{1/2}t)dt \\
&+ O(\Lambda^{-3/2}). \qquad (28)
\end{aligned}
$$

Now, reversion on (28) leads to

$$\begin{aligned}
\Lambda_n^{1/2} &= \frac{(n+1)\pi}{a} \\
&- \frac{1}{2(n+1)\pi}\int_0^a q(t+\tau)cos(\frac{2(n+1)\pi}{a}t)dt \\
&- \frac{a}{2((n+1)\pi)^2}\int_0^a q(t+\tau)(\int_0^t q(x+\tau)dx)sin(\frac{2(n+1)\pi}{a}t)dt
\end{aligned}$$
$$+O(n^{-3}). \tag{29}$$

The second term on the right handside of (29) is included in the error term, $O(n^{-3})$, by the assumption on the Fourier coefficients of $q(t)$. The third term is also included in the same error term by Lemma 2.1 and Lemma 2.4. Hence,

$$\Lambda_n^{1/2}(\tau) = \frac{(n+1)\pi}{a} + O(n^{-3}) \tag{30}$$

for any τ in $[0,a)$. From this and Theorem 2.1 we have

$$\mu_{2m}^{1/2} = \frac{(2m+1)\pi}{a} + O(m^{-3}),$$
$$\mu_{2m+1}^{1/2} = \frac{(2m+1)\pi}{a} + O(m^{-3}).$$

Hence,

$$\begin{aligned}
l_{2m+1} &= \mu_{2m+1} - \mu_{2m} \\
&= (\mu_{2m+1}^{1/2} + \mu_{2m}^{1/2})O(m^{-3}) \\
&= O(m^{-2}).
\end{aligned}$$

Similar result holds for l_{2m+2}. □

Theorem 3.4. *As $m \to \infty$, l_{2m+1} and l_{2m+2} satisfy*

$$l_{2m+1} = (a_{2m+1}^2 + b_{2m+1}^2)^{1/2} + o(m^{-1}),$$
$$l_{2m+2} = (a_{2m+2}^2 + b_{2m+2}^2)^{1/2} + o(m^{-1}).$$

Proof. First, we rewrite (21) as

$$\Lambda_n^{1/2}(\tau) = \frac{(n+1)\pi}{a} - \frac{a}{4(n+1)\pi}F_1(n,\tau) + o(n^{-2}), \tag{31}$$

where

$$F_1(n,\tau) = a_{n+1}cos(\frac{2(n+1)\pi}{a}\tau) + b_{n+1}sin(\frac{2(n+1)\pi}{a}\tau).$$

Then we write $F_1(n,\tau)$ in the following form

$$F_1(n,\tau) = \sqrt{a_{n+1}^2 + b_{n+1}^2}\,sin(\frac{2(n+1)\pi}{a}\tau + \Phi), \tag{32}$$

where Φ is chosen so that

$$sin\Phi = \frac{a_{n+1}}{\sqrt{a_{n+1}^2 + b_{n+1}^2}}, \quad cos\Phi = \frac{b_{n+1}}{\sqrt{a_{n+1}^2 + b_{n+1}^2}}. \tag{33}$$

From (32), we can find a τ with $0 \le \tau < a$ at which $F_1(n,\tau)$ assumes its minimum, which is given by

$$\tau_{1,min}(n) = \frac{a}{2(n+1)\pi}(\frac{3\pi}{2} - \Phi). \tag{34}$$

Similarly, we can find a τ with $0 \le \tau < a$ at which $F_1(n,\tau)$ assumes its maximum which is given by

$$\tau_{1,max}(n) = \frac{a}{2(n+1)\pi}(\frac{\pi}{2} - \Phi). \tag{35}$$

Replacing τ in (31) by $\tau_{1,max}(n)$ and using (33) we observe that

$$\overset{\min}{\tau}\Lambda_n^{1/2}(\tau) = \frac{(n+1)\pi}{a} - \frac{a}{4(n+1)\pi}[a_{n+1}cos(\frac{\pi}{2}-\Phi)+b_{n+1}sin(\frac{\pi}{2}-\Phi)]+o(n^{-2})$$
$$= \frac{(n+1)\pi}{a} - \frac{a}{4(n+1)\pi}(a_{n+1}^2+b_{n+1}^2)^{1/2}+o(n^{-2}). \quad (36)$$

Similar calculations yield

$$\overset{\max}{\tau}\Lambda_n^{1/2}(\tau) = \frac{(n+1)\pi}{a} + \frac{a}{4(n+1)\pi}(a_{n+1}^2+b_{n+1}^2)^{1/2}+o(n^{-2}). \quad (37)$$

From (36), (37) and Theorem 2.1

$$l_{2m+1} = \overset{\max}{\tau}\Lambda_{2m}(\tau) - \overset{\min}{\tau}\Lambda_{2m}(\tau)$$
$$= (\overset{\max}{\tau}\Lambda_{2m}^{1/2}(\tau) + \overset{\min}{\tau}\Lambda_{2m}^{1/2}(\tau))(\overset{\max}{\tau}\Lambda_{2m}^{1/2}(\tau) - \overset{\min}{\tau}\Lambda_{2m}^{1/2}(\tau))$$
$$= (a_{2m+1}^2+b_{2m+1}^2)^{1/2}+o(m^{-1}). \quad (38)$$

Similar result holds for l_{2m+2}. □

Conclusions

We summarize the contributions reported in this study as follows:

- The piecewise continuity assumptions on $q(t)$ is weakend and replaced by integrability in Theorem 1.1 and Theorem 1.2.
- The estimate $l_n = \sqrt{(a_n^2+b_n^2)} + O(n^{-1/2})$ when $q(t)$ is piecewise continuous is improved to the estimate $l_n = \sqrt{(a_n^2+b_n^2)} + o(n^{-2})$ when $q(t)$ is integrable.
- The estimate $l_n = o(n^{-1})$ when $p(x) = s(x) = 1$, $q'(x)$ exists and piecewise continuous in Theorem 1.1(iii) is preserved under the weaker assumptions $a_n = o(n^{-1})$, $b_n = o(n^{-1})$. In particular, the latter assumption holds for an integral or an absolutely continuous function $q(t)$.

- The estimate $l_n = o(n^{-1})$ when $p(x) = s(x) = 1$, $q'(x)$ exists and piecewise smooth in Theorem 1.2(iii) is preserved under the weaker assumptions $a_n = O(n^{-2})$, $b_n = O(n^{-2})$.

- The estimate $l_n = O(n^{-1})$ when $p(x) = s(x) = 1$, $q(x)$ piecewise smooth in Theorem 1.2(iii) is preserved under weaker assumptions $a_n = O(n^{-1})$, $b_n = O(n^{-1})$. In particular, the latter assumptions hold when $q(t)$ is of bounded variation on $[0, a]$.

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