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A STONE'S REPRESENTATION THEOREM AND SOME APPLICATIONS

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Abstract

In this paper, we prove the following form of Stone's representation theorem: Let $\sum$ be a σ-algebra of subsets of a set X. Then there exists a totally disconnected compact Hausdorff space $\mathcal{K}$ for which $(\sum, \cup, \cap)$ and $(\mathcal{C}(\mathcal{K}), \cup, \cap)$, where $\mathcal{C}(\mathcal{K})$ denotes the set of all clopen subsets of $\mathcal{K}$, are isomorphic as Boolean algebras. Furthermore, by defining appropriate joins and meets of countable families in $\mathcal{C}(\mathcal{K})$, we show that such an isomorphism preserves σ-completeness. Then, as a consequence of this result, we obtain the result that if ba$(X, \sum)$ (respectively, ca$(X, \sum)$) denotes the Banach space (under the variation norm) of all bounded, finitely additive (respectively, all countably additive) complex-valued set functions on $(X, \sum)$, then ca$(X, \sum)$=ba$(X, \sum)$ if and only if (1) $\mathcal{C}(\mathcal{K})$ is σ-complete; and if and only if (2) $\sum$ is finite. We also give another application of these results.

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1. Introduction

In 1936, M. H. Stone [16] developed a representation theory for Boolean rings and Boolean algebras. Among other things, M. H. Stone showed (see Theorem 2.3 below) that such Boolean structures can be represented by "spaces of continuous functions" on some "nice" topological spaces.

In this paper, we use Stone's original representation theorem (Theorem 2.3) to derive our own version of Stone's representation theorem (see Theorem 3.2), and we give in Section 4 some applications of this remarkable result to analysis.

For the most part the notation and symbols we use will be standard. The notation := will denote "equals by definition".

2. Definitions and Preliminaries

Following [4], a *Boolean ring* is a ring $(R,\ +,\ .)$ in which every element is idempotent; that is,

$$r.r = r \quad \text{for all} \quad r \epsilon R.$$

It follows from this definition that if R is a Boolean ring, then $r + r = 0$ for all $r \in R$, and R is commutative.

2.1 Examples. (i) An example of a Boolean ring is $(\mathbf{2}^x, \cap, \Delta)$ where the intersection $\cap$ plays the role of multiplication and the symmetric difference Δ plays the role of addition.

(ii) Let $(X, \sum)$ be a measurable space, and let

$$R := \{\chi_A : A \in \sum\},$$

where χ_A denotes the characteristic function of A. Define (Boolean) operations on R by

$$\chi_A \oplus \chi_B := \chi_{A \Delta B}, \quad \text{and} \quad \chi_A . \chi_B := \chi_{A \cap B}$$

for all $A, B \in \sum$. Evidently, R is a Boolean ring under these operations and χ_X is a unit for R.

Throughout this article, unless otherwise stated, we let R denote an arbitrary Boolean ring. Let $\mathbf{Z}_2$:={0,1} be the field of integers modulo 2, and let

$$\mathcal{K} := \{f \mid f : R \rightarrow \mathbf{Z}_2 \text{is a ring homomorphism}\} \setminus \{0\}.$$

Note that $\mathcal{K} \subseteq \{0,1\}^R$. The *Stone space* of the ring R is $\mathcal{K}$ with its relative topology. We assume that {0,1} is assigned the discrete topology so that $\{0,1\}^R$ is a compact Hausdorff space. A *Boolean space* is a Hausdorff space with base consisting of all sets which are both compact-and-open. Hence, a Boolean space is necessarily locally compact. The *characteristic ring* of a Boolean space X is the ring of all continuous functions $f : X \rightarrow \mathbf{Z}_2$ for which $f^{-1}(1)$ is compact. Hence, members of the characteristic ring are exactly all characteristic functions χ_A of sets A in X that are compact-and-open.

A sketch of the proofs (which are straight-forward) of the following two results can be found in [4, 5.S, p.168].

Proposition 2.2. *The Stone space $\mathcal{K}$ of a Boolean ring R is a Boolean space and is compact whenever R has a unit.*

Theorem 2.3. *(Stone Representation). Every Boolean ring R is isomorphic (under the evaluation map) to the characteristic ring of its Stone space.*

3. Results

Let $(X, \sum)$ be a measurable space, and let $R := \{\chi_A : A \in \sum\}$ be the Boolean ring of Example 2.1(ii). Let $\mathcal{K}$ be the Stone space of R. That is,

$$\mathcal{K} = \{k \mid k : R \rightarrow \mathbf{Z}_2 \text{is a nonzero homomorphism}\}.$$

By Proposition 2.2, $\mathcal{K}$ is compact Hausdorff with basis consisting of all compact-and-open (hence clopen) subsets. Let $\mathcal{F}$ be the characteristic ring of $\mathcal{K}$. That is, by definition,

$$\begin{aligned} \mathcal{F} &= \{f \mid f : \mathcal{K} \rightarrow \mathbf{Z}_2 \quad \text{is continuous with} f^{-1}(1) \quad \text{is compact}\} \\ &= \{\chi_K \mid K \text{is clopen in} \mathcal{K}\} (\subseteq C(\mathcal{K})). \end{aligned}$$

By Stone's Representation Theorem 2.3, R is isomorphic to $\mathcal{F}$ (as Boolean rings, where addition in $\mathcal{F}$ is pointwise addition mod 2) under the evaluation map $\chi_A \overset{e}{\longmapsto} e(\chi_A) =: e_A$ where

$$e_A(k) = k(\chi_A) \text{ for all } k \in \mathcal{A} \in \sum .$$

Thus, if we let

$$K(A) := \{k \in \mathcal{K} : e_A(k) = k(\chi_A) = 1\},$$

which is clopen in $\mathcal{K}$, then

$$e_A = \chi_{K(A)} \quad \text{for all } A \in \sum .$$

This induces a map

$$A \overset{K}{\longmapsto} K(A) : (\sum, \cup, \cap) \to (\mathcal{C}(\mathcal{K}), \cup, \cap)$$

where $\mathcal{C}(\mathcal{K})$ denotes the set of all clopen subsets of $\mathcal{K}$. Since the lattice operations Λ, V in R and $\mathcal{F}$ can be defined in terms of the (Boolean) operations . and $\oplus$, the ring homomorphism e is also a lattice homomorphism. It follows that the induced map K is a lattice isomorphism. Thus for all $n \in \mathbf{N}$, we have

$$\begin{aligned} K(\cup_{j=1}^n A_j) &= \cup_{j=1}^n K(A_j), \text{and} \\ K(\cap_{j=1}^n A_j) &= \cap_{j=1}^n K(A_j). \end{aligned}$$

The foregoing observations lead to the following result.

Theorem 3.1. *Let $(X, \sum)$ be a measurable space, $B(X, \sum)$ be the $C^\star$-algebra of all complex-valued, bounded measurable functions on X, and $C(\mathcal{K})$ be the $C^\star$-algebra of all continuous functions on $\mathcal{K}$, the Stone space of the Boolean ring $R = \{\chi_A : A \in \sum\}$. Then $B(X, \sum)$ is isometrically isomorphic to $C(\mathcal{K})$.*

Proof. Let $\mathcal{K}$, $\mathcal{F}$ be as above. Use Theorem 2.3 to get a Boolean ring isomorphism $e : R \to \mathcal{F}$ where

$$[e(\chi_A)](k) =: e_A(k) = k(\chi_A) = \chi_{K(A)}(k) \quad \text{for all } A \in \sum, k \in \mathcal{K},$$

and $K(A) := \{k \in \mathcal{K} : e_A(k) = 1\}$ which is clopen in $\mathcal{K}$.

Let $\mathcal{L}(R)$ (resp.,$\mathcal{L}(\mathcal{F})$) be the linear span of R (resp., of $\mathcal{F}$). Define $G : \mathcal{L}(R) \to \mathcal{L}(\mathcal{F})$ as follows: For each $f \in \mathcal{L}(R)$, let $a_1, \ldots, a_n$ be the distinct values of f and let $A_j := f^{-1}(a_j)$ for $1 \leq j \leq n$. Then

$$f = \sum_{j=1}^{n} a_j \chi_{A_j}$$

is the *canonical representation* of the simple function f where $\{A_j\}_{j=1}^n$ is disjoint. Set

$$G(f) := \sum_{j=1}^{n} a_j e_{A_j}.$$

Clearly, G is well-defined. It is routine to check that, since e is a ring isomorphism, G is linear and multiplicative; i.e.,

$$G(\alpha f + \beta g) = \alpha G(f) + \beta G(g) \text{ and } G(f.g) = G(f).G(g)$$

for all $f, g \in \mathcal{L}(R)$ and all $\alpha, \beta \in \mathbf{C}$. Also, G is surjective, since e is surjective.

We now show that G is an isometry with respect to the supremum-norm. Indeed, let $f \in \mathcal{L}(R)$ and let $f = \sum_{j=1}^{n} a_j \chi_{A_j}$ be its canonical representation. Then $\{A_j\}_{j=1}^n$ is disjoint and, since G is multiplicative, $\{K(A_j)\}_{j=1}^n$ is disjoint. Thus

$$\begin{aligned}
\| G(f) \|_u &= \sup\{| \sum_{j=1}^{n} a_j e_{A_j}(k) | : k \in \mathcal{K}\} \\
&= \sup\{| \sum_{j=1}^{n} a_j \chi_{K(A_j)}(k) | : k \in \mathcal{K}\} \\
&= \max_{1 \leq j \leq n} | a_j | = \| f \|_u,
\end{aligned}$$

and therefore G is an isometry, hence injective. Thus G is a linear bijection which is multiplicative and an isometry. Since $\mathcal{L}(R)$ is dense in $B(X, \sum), G$ extends to a linear isometry, still call it G, on $B(X, \sum)$. This extension is still multiplicative and so maps

$B(X,\sum)$ onto a closed subalgebra of $C(\mathcal{K})$, which by the Stone-Weierstrass theorem has to be $C(\mathcal{K})$ [=closure of $\mathcal{L}(\mathcal{F})$].

As remarked before in Theorem 3.1, the evaluation map

$$\chi_A \overset{e}{\longmapsto} e(\chi_A) = \chi_{K(A)} : R \to \mathcal{F}$$

of Stone's Theorem 2.3, where $\mathcal{F}$ is the characteristic ring of the Stone space $\mathcal{K}$ of the Boolean ring $R = \{\chi_A : A \in \sum\}$, induces a lattice homomorphism

$$A \overset{K}{\longmapsto} K(A) : \sum \to \mathcal{C}(\mathcal{K}).$$

Since e is bijective, it is easy to see that K is also bijective. Thus we have the following form of Stone's representation theorem. □

Theorem 3.2. *(Stone Representation).*

Let $(X,\sum)$ be a measurable space. Then $(\sum,\cup,\cap)$ is isomorphic to $(\mathcal{C}(\mathcal{K}),\cup,\cap)$ via the mapping K. Moreover, for all sequences $\{A_j\}_{j=1}^{\infty} \subseteq \sum, K$ satisfies

(a) $K(\cup_{j=1}^{\infty} A_j) = [\cup_{j=1}^{\infty} K(A_j)]^-$, and

(b) $K(\cap_{j=1}^{\infty} A_j) = [\cap_{j=1}^{\infty} K(A_j)]^\circ$,

where $^-$ and $^\circ$ denote respectively the closure and interior operations. Thus the σ-completeness of $\sum$ entails that of $\mathcal{C}(\mathcal{K})$ with

$$\vee_{j=1}^{\infty} C_j = (\cup_{j=1}^{\infty} C_j)^- \quad \text{and} \quad \wedge_{j=1}^{\infty} C_j = [\cap_{j=1}^{\infty} C_j]^\circ \quad \text{for all} \quad \{C_j\}_{j=1}^{\infty} \subseteq \mathcal{C}(\mathcal{K}),$$

and K is a σ-complete lattice isomorphism.

Proof. In light of the preceding remarks, we only need to prove (a) and (b). To prove (a), let $\{A_j\}_{j=1}^{\infty} \subseteq \sum$. Since the mapping K is monotone and $A_n \subseteq \cup_{j=1}^{\infty} A_j$ for all n, we have $K(A_n) \subseteq K(\bigcup_{j=1}^{\infty} A_j)$ for all n which implies that $\cup_{j=1}^{\infty} K(A_n) \subseteq K(\cup_{j=1}^{\infty} A_j)$. As $K(\cup_{j=1}^{\infty} A_j)$ is clopen, it follows that

$$[\cup_{j=1}^{\infty} K(A_n)]^- \subseteq K(\cup_{j=1}^{\infty} A_j).$$

To prove the reverse inclusion, assume that there exists $k \in K(\cup_{j=1}^{\infty} A_j \setminus [\cup_{j=1}^{\infty} K(A_j)]^-$. Since the space $\mathcal{K}$ is zero-dimensional, there is a clopen set B such that $k \in B$ and $B \cap [\cup_{j=1}^{\infty} K(A_j)]^- = \phi$. Hence, $B \cap [\cup_{j=1}^{\infty} K(A_j)] = \phi$ which implies $B \cap K(A_j) = \phi$ for all j. Write $B = K(A)$ for some $A \in \sum$ (that is, $e(A) = \chi_B$). Then we have $K(A \cap A_j) = \phi$ for all j, which implies $A \cap A_j = \phi$ for all j, hence $A \cap (\cup_{j=1}^{\infty} A_j) = \phi$, and therefore $K(A) \cap K(\cup_{j=1}^{\infty} A_j) = \phi$, a contradiction, since k lies in this intersection.

Similarly, the proof of (b) follows from (a) by writing $\cap_{j=1}^{\infty} A_j = X \setminus (\cup_{j=1}^{\infty} (X \setminus A_j))$ and by using the facts that $A \subseteq B$ implies $K(B \setminus A) = K(B) \setminus K(A)$ for all $A, B \in \sum$ and that $(C^-)' = (C')^\circ$ for all $C \subseteq \mathcal{K}$. $\square$

Applications

Let $(X, \sum)$ be a measurable space. By Theorem 3.2, there exists a totally disconnected compact Hausdorff space $\mathcal{K}$ for which $(\sum, \cup, \cap)$ and $(\mathcal{C}(\mathcal{K}), \cup, \cap)$, where $\mathcal{C}(\mathcal{K})$ denotes the set of all clopen subsets of $\mathcal{K}$, are isomorphic as Boolean algebras. Recall that

$$ba(X, \sum) = B(X, \sum)^\star$$

(cf. [2, p.77]), and by Riesz Representation Theorem (cf. [5]) we have

$$C(\mathcal{K})^\star = rca(\mathcal{B}(\mathcal{K}))[= M(\mathcal{K})].$$

Here $ba(X, \sum)$ denotes the Banach space of all finitely additive, bounded set functions on $\sum$ and $rca(\mathcal{B}(\mathcal{K}))$ denotes the Banach space of all regular, countably additive, bounded set functions on the Borel algebra $\mathcal{B}(\mathcal{K})$. By Theorem 3.1, $B(X, \sum)$ is isometrically algebra isomorphic to $C(\mathcal{K})$ via the "Gelfand" map

$$B(X, \sum) \xrightarrow{G} C(\mathcal{K})$$

(itself induced from or inducing the Boolean algebra isomorphism $\sum \leftrightarrow \mathcal{C}(\mathcal{K})$). This induces an isometric isomorphism $G^\star$,

$$ba(X, \sum) \simeq B(X, \sum)^\star \overset{G\star}{\leftarrow} C(\mathcal{K})^\star \simeq rca(\mathcal{B}(\mathcal{K})),$$

such that

$$G^\star(\phi) = \phi \circ G \quad \text{for all } \phi \in C(\mathcal{K})^\star.$$

Thus, upon identifying $B(X, \sum)^\star$ with $ba(X, \sum)$ and $C(\mathcal{K})^\star$ with $rca(\mathcal{B}(\mathcal{K}))$, we see that there exists an isometric isomorphism

$$G^{\star -1} : ba(X, \sum) \to rca(\mathcal{B}(\mathcal{K}))$$

such that

$$G^{\star -1}(\mu) = \mu \circ G^{-1} = \hat{\mu} \quad \text{for all} \quad \mu \in ba(X, \sum).$$

It follows from the definitions of the mappings G and $\hat{}$ that

$$\hat{E} \in \mathcal{C}(\mathcal{K}) \quad \text{implies} \quad \hat{u}(\hat{E}) = \mu(G^{-1}(\chi_{\hat{E}})) = \mu(\chi_E) = \mu(E),$$

where E is the pre-image in $\sum$ of $\hat{E}$ under the Boolean isomorphism $K = \hat{}: \sum \to \mathcal{C}(\mathcal{K})$.

With these remarks we are now ready to give the first application of our results.

Theorem 4.1 *Let $(X, \sum), \mathcal{K}, \mathcal{C}(\mathcal{K})$ be as above. Then the following statements are equivalent:*

(i) $\mathcal{C}(\mathcal{K})$ is a σ-algebra; that is, $\mathcal{C}(\mathcal{B}) = \mathcal{B}_0(\mathcal{K})$, the Baire σ-algebra on $\mathcal{K}$, defined as the σ-algebra generated by all compact G_δ sets.

(ii) $ba(X, \sum) = ca(X, \sum)$.

(iii) $\sum$ satisfies the finite chain condition (f.c.c) [3]: No infinite subcollection of $\sum$ can be pairwise disjoint.

Proof. (i)$\to$(ii). Assume (i) holds. Then for all countable $\{C_n\}_{n\in N} \subseteq \mathcal{C}(\mathcal{K}), \cup_{n\in N} C_n \in \mathcal{C}(\mathcal{K})$. Thus $\cup_{n\in N} C_n$ is clopen and

$$\cup_{n\in N}C_n = (\cup_{n\in N}C_n)^-. \tag{0.1}$$

Let $\mu \in ba(X,\sum)$. Consider its image $\hat{u}$ under the mapping $G^{\star-1}$ defined above. Then $\hat{u} \in rca(\mathcal{B}(\mathcal{K}))$ and it satisfies

$$\hat{\mu}(C) = \mu(A)$$

for all $C \in \mathcal{C}(\mathcal{K})$ with $C = K(A), A \in \sum$.

Let $\{A_n\}_{n\in N} \subseteq \sum$ be disjoint and let $A := \cup_{n\in N}A_n$. Then, by Theorem 3.2, there exist $C_n, C \in \mathcal{C}(\mathcal{K})$ such that $C_n = K(A_n), C = K(A)$ and $\{C_n\}_{n\in N}$ is a disjoint family. Since K is σ-complete lattice isomorphism (Theorem 3.2), we have

$$\begin{aligned} C &= K(A) = K(\vee_{n\in N}A_n) = \vee_{n\in N}K(A_n) \\ &= \vee_{n\in N}C_n = (\bigcup_{n\in N} C_n)^{-} \overset{(0.1)}{=\!=} \cup_{n\in N} C_n, \end{aligned}$$

and so

$$\begin{aligned} \mu(\cup_{n\in N}A_n) &= \mu(A) = \hat{\mu}(C) = \hat{\mu}(\bigcup_{n\in N} C_n) \\ &= \sum_{n=1}^{\infty}\hat{\mu}(C_n) = \sum_{n=1}^{\infty}\mu(A_n). \end{aligned}$$

Thus μ is countably additive, and (ii) holds.

(ii)$\rightarrow$(i): Suppose $ba(X,\sum) = ca(X,\sum)$. By the remarks preceding the theorem, for each $v \in rca(\mathcal{B}(\mathcal{K}))$, there exists $\mu \in ba(X,\sum)$ such that

$$v = G^{\star-1}(\mu)(= \hat{\mu}).$$

Then for all $v \in rca(\mathcal{B}(\mathcal{K}))$ and for disjoint $\{C_n\}_{n\in N} \subseteq \mathcal{C}(\mathcal{K})$, letting $A_n := K^{-1}(C_n) \in \sum$, we have

$$v((\cup C_n)^- \setminus \cup C_n) = v((\cup C_n)^-) - v(\cup C_n)$$

$$
\begin{aligned}
&= v(\vee C_n) - \sum_{n\in N} v(C_n) \\
&= \mu(\vee A_n) - \sum_{n\in N} \mu(A_n)[\text{here we used the fact that} \\
&\quad \vee C_n = \vee K(A_n) = K(\vee A_n)\text{by Theorem 3.2.}] \\
&= \mu(\cup_{n\in N} A_n) - \sum_{n\in N} \mu(A_n) \\
&= \sum_{n\in N} \mu(A_n) - \sum_{n\in N} \mu(A_n) \\
&= 0
\end{aligned}
$$

[the penultimate equality being due to the fact that $\{A_n\}_{n\in N}$ is disjoint, since $\{C_n\}_{n\in N}$ is disjoint and K is an isomorphism]. Thus we have shown that

$$v((\cup_{n\in N} C_n)^- \setminus \cup_{n\in N} C_n) = 0 \ \forall \text{ disjoint } \{C_n\}_{n\in N} \subseteq \mathcal{C}(\mathcal{K}), \tag{0.2}$$

for all $v \in rca(\mathcal{B}(\mathcal{K}))$.

Now if there exists $y \in (\cup C_n)^- \setminus \cup C_n$, we apply (2) to the point mass $v := \delta_y \in rca(\mathcal{B}(\mathcal{K}))$ to get

$$1 = \delta_y((\cup C_n)^- \setminus \cup C_n) \overset{(0.2)}{=\!=} 0,$$

a contradiction. Thus $(\cup C_n)^- = \cup C_n$ for all disjoint $\{C_n\} \subseteq \mathcal{C}(\mathcal{K})$. Hence, by the use of the disjointification trick [15, Prop. 2, p. 17], $(\cup C_n)^- = \cup C_n$ for all sequences $\{C_n\} \subseteq \mathcal{C}(\mathcal{K})$. Therefore, by Theorem 3.2, $\cup C_n \subseteq \mathcal{C}(\mathcal{K})$ for all sequences $\{C_n\} \subseteq \mathcal{C}(\mathcal{K})$ and $\mathcal{C}(\mathcal{K})$ is a σ-algebra.

(i)$\rightarrow$(iii): Let $\mathcal{E}$ be an infinite subcollection of $\sum$. If the members of $\mathcal{E}$ were pairwise disjoint, then there would exist a countable subcollection $\{A_n\}_{n\in N} \subseteq \mathcal{E}$ with $A_n \neq \phi \forall n \in \mathbf{N}$ and $A_i \cap A_j = \phi \forall i \neq j$. Now $C_n = K(A_n) \neq \phi$ for all $n \in \mathbf{N}$ and C_n is pairwise disjoint, since K is an isomorphism. Thus, by (i), $\{C_n\}$ is an open cover of the compact set $(\bigcup_{n\in N} C_n)^- \overset{(0.1)}{=\!=} \cup_{n\in N} C_n$ that cannot have a finite subcover, a contradiction. It follows that $\sum$ satisfies the f.c.c.

(iii)$\rightarrow$(ii): Suppose $\sum$ satisfies the f.c.c. If $\mu \in ba(X, \sum)$ and $\{A_n\}_{n\in N} \subseteq \sum$ is pairwise disjoint, then $A_n = \phi$ for all large n, say for all $n > N, N \in \mathbf{N}$. Hence, by the finite additivity of μ,

$$\mu(\cup_{n=1}^{\infty} A_n) = \mu(\cup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu(A_n) = \sum_{n=1}^{N} \mu(A_n),$$

and μ is countably additive. $\square$

A Boolean algebra B is said to satisfy the *countable chain condition* (c.c.c.) [12] if every collection $\{b_\alpha : \alpha \in \Delta\}$ of pairwise disjoint elements in B is at most countable.

Theorem 4.2. *Let $\sum$ be a σ-algebra of subsets of a set X that satisfies the c.c.c. Then $\mathcal{A} \subseteq \sum$ implies $\cup\mathcal{A} \in \sum$.*

Proof. See [1, Theorem 1.4.8].

Now as a consequence of Theorem 4.1 and Theorem 4.2 we obtain the following theorem: $\square$

Theorem 4.3 *Let X be a set and $\sum$ be a σ-algebra of subsets of X. Then $ba(X, \sum) = ca(X, \sum)$ if and only if $\sum$ is finite.*

Proof. If $\sum$ is finite, then $ba(X, \sum) = ca(X, \sum)$ is obvious. Conversely, suppose $ba(X, \sum) = ca(X, \sum)$. Then by Theorem 4.1, $\sum$ satisfies the f.c.c.; hence the c.c.c. Thus, by Theorem 4.2, $\sum$ is closed under arbitrary unions and hence under arbitrary intersections. Therefore, the *atoms*

$$A_x := \cap\{B \in \sum : x \in B\} \in \sum$$

for every $x \in X$. Note that for all $x, y \in X$, either $A_x = A_y$ or $A_x \cap A_y = \phi$. Indeed, $A_x \neq A_y$ implies either (i) $A_x \neq A_x \cap A_y$ or (ii) $A_y \neq A_x \cap A_y$. Now (i) and the definition of A_x implies that $x \in X \setminus A_y$, hence $A_x \subseteq X \setminus A_y$, and therefore $A_x \cap A_y = \phi$. By symmetry the same conclusion follows from (ii).

Now every $A \in \sum$ is covered by atoms: $A = \cup_{x \in A} A_x$. Hence,

$$X = \cup_{x \in X} A_x \quad \text{(disjoint union)}.$$

Since $\{A_x : x \in X\} \subseteq \sum$ and every $A_x \neq \phi$, the fact that $\sum$ satisfies the f.c.c. ensures that only finitely many distinct atoms A_x exist. Let them be $A_{x_1}, \ldots, A_{x_n}$ for some $x_1, \ldots, x_n \in X$. Since each $A \in \sum$ is a union of atoms, it follows that there can be only finitely many, in fact $2^n, A \in \sum$. Therefore $\sum$ is finite. $\square$

Theorem 4.4. *Let $\sum$ be a σ-algebra on a set X. Suppose $\sum$ satisfies $ba(X, \sum) = ca(X, \sum)$. Then a sequence $(\mu_n) \subseteq ba(X, \sum)$ converges in the weak topology $\sigma(ba(X, \sum), (ba(X, \sum))^\star)$ to $\mu \in ba(X, \sum)$ if and only if*

$$\mu(E) = lim_{n \to \infty} \mu_n(E) \quad \textit{for each } E \in \sum. \tag{0.3}$$

Proof. ($\mapsto$): Since for each $E \in \sum$ the mapping $v \mapsto v(E) : ba(X, \sum) \to \mathbf{C}$ (where $\mathbf{C}$ denotes the complex numbers) defines a linear functional in $(ba(X, \sum))^\star$, the necessity of (0.3) follows.

($\leftarrow$): Suppose (0.3) holds. Since, by Theorem 4.3, $\sum$ is finite, it follows that $B(\sum) = B(X, \sum)$ is finite dimensional: every $f \in B(\sum)$ has only finitely many values because $\{f^{-1}(x) : x \in \mathbf{C}\}$ is a disjoint family is $\sum$, hence is finite. So $f^{-1}(x)$ is void except for finitely many x's, say $x_1, \ldots, x_m$. Set $A_j := f^{-1}(x_j) \in \sum, j = 1, \ldots, m$. Then

$$f = \sum_{j=1}^{m} f(x_j) \chi_{A_j}. \tag{0.4}$$

Thus $B(\sum) \subseteq$ span $\{\chi_A : A \in \sum\} =: L$. As $L \subseteq B(\sum)$ is clear, $B\sum = L$. Thus dim $B(\sum)=$ dim $L \leq \mid \sum \mid < \infty$. It follows that $B(\sum)$ is reflexive; i.e, $B(\sum) = (B(\sum))^{\star\star} = (ba(X, \sum))^\star$, where $ba(X, \sum) \simeq (B(\sum))^\star$ via isomorphism $\mu \to x_\mu^\star$ which is defined by $x_\mu^\star(f) = \int f d\mu \ (f \in B(\sum))$.

Now let $f \in B(\sum)$. Then it has the form (0.4). So

$$x_\mu^\star(f) = \sum_{j=1}^{m} f(x_j) \int \chi_{A_j} d\mu = \sum_{j=1}^{m} f(x_j) \mu(A_j),$$

hence (0.3) implies

$$x^{\star}_{\mu_n}(f) = \sum_{j=1}^{m} f(x_j)\mu_n(A_j) \to \sum_{j=1}^{m} f(x_j)\mu(A_j) = x^{\star}_{\mu}(f).$$

Since $f \in B(\sum)$ was arbitrary, this says that $x^{\star}_{\mu_n} \to x^{\star}_{\mu}$ in the $\sigma((B(\sum))^{\star}, B(\sum))$ topology. Hence $\mu_n \to \mu$ in the $\sigma(ba(X,\sum), (ba(X,\sum))^{\star})$ topology. $\square$

References

[1] Bhaskara Rao, K.P.S. and Bhaskara Rao, M. *Theory of Charges,* Academic Press, London and New York, 1983.

[2] Diestel, J. *Sequences and Series in Banach Spaces*, Springer Verlag, New York, 1984.

[3] Halmos, P. R. *Lectures on Boolean Algebras,* Van Nostrand, London, 1963.

[4] Kelly J. L. *General Topology,* Springer-Verlag, New York, 1955.

[5] Royden H. L. *Real Analysis*, Macmillan, New York, 1988.

[6] Stone M. H. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.* **40** (1936), 37-111.

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