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DIFFERENCE METHOD FOR A SINGULARLY PERTURBED INITIAL VALUE PROBLEM

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Abstract

In this paper we construct a completely exponentially fitted finite difference scheme for the initial value problem with small parameter by first and second derivatives. We prove the first order uniform convergence of the scheme in the sense of discrete maximum norm. Numerical results are presented.

1. Introduction

Singularly perturbed problems for differential equations arise very frequently in applications and have been extensively studied in recent years. It is known that these problems depend on a small positive parameter ϵ in such a way that the solution exhibits a multiscale character, i.e. there are thin transition layers where the solution varies rapidly, while away from layers it behaves regularly and varies slowly [9], [10], [11]. So the treatment of singularly perturbed problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions [7], [8], [12]. In this paper, we restrict our attention to the numerical solution by using finite-difference approximation of linear second order singularly perturbed problem of the form

$$Lu \equiv \epsilon^2 u'' + \epsilon a(t)u' + b(t)u = f(t), \quad 0 < t \le T, \tag{1}$$

$$u(0) = A, \tag{2}$$

$$u'(0) = B/\epsilon. \tag{3}$$

Here ϵ is a small positive parameter, A and B are given constans, $a(t) \ge \alpha > 0$, $b(t) \ge \beta > 0$, $f(t)$ are sufficiently smooth functions in the $[0, T]$. The solution $u(t)$ of problem (1)-(3) has in general an initial layer near $t = 0$ for small values of ϵ (see Section 2).

Singularly perturbed initial value problems arise in many fields of application-for instance, fluid mechanics [9], electrical networks [8], chemical reactions [14], control theory [9], [10] and other physical models. Various mathematical aspects of problems of this type, in particular of (1)-(3) have considered in [7]-[11].

It is known that, the use of classical difference methods for solving such problem may give rise to difficulties when the singular perturbation parameter ϵ is small [7], [8]. Therefore, it is important to develop suitable numerical methods to these problems. Several uniform difference methods for problems with initial layers have been proposed in [4]-[7]. To our knowledge, the only reference to a uniformly convergent scheme for the second order differential equations, which reduce for $\epsilon = 0$ to zero-order equations is [5], but this scheme correspond to the particular case when the second initial condition is regular and furthermore only is $O(\tau^{1/4})$ accurate uniformly in ϵ, where τ is the mesh size.

The proposed, in present paper, difference scheme as well its method of construction differ from those in [5] and the scheme has uniform $O(\tau)$ accuracy. The difference scheme is constructed by the method integral identities with the use exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form [1], [2], [3], which in mentioned papers this technique was applied to the another mathematical models.

In Section 2, the asymptotic estimations of the problem (1)-(3) are established. The difference scheme constructed on the uniform mesh for numerical solution (1)-(3) is presented in Section 3 and in Section 4 we prove $O(\tau)$ uniform convergence of the scheme. Some numerical examples confirming the theory are given in the Section 5.

We shall use the indexless notation of [13] for mesh functions.

2. The Continuous Problem

Lemma 2.1. *Let $u(t)$ be the solution of the problem (1)-(3) and $a \in C[0,T]$, $b, f \in C^1[0,T]$. Then there exist a positive constant C such that*

$$|u^{(K)}(t)| \le C\epsilon^{-K}\left\{\delta_* + \left(\int_0^t (f^2(\zeta) + f'^2(\zeta))d\zeta\right)^{1/2}\right\},$$

$$\left(\int_0^t |u^{(K)}(\zeta)|d\zeta\right)^{1/2} \le C\epsilon^{-\frac{K}{2}}\left\{\delta_* + \left(\int_0^t (f^2(\zeta) + f'^2(\zeta))d\zeta\right)^{1/2}\right\},\ K = 0,1,2 \quad (4)$$

where $\delta_*^2 = |B^2 + b(0)A^2 - 2f(0)A| + f^2(0)$. Throughout in the paper c, c_i, C, C_i $(i = 0,1,\ldots)$ denote the positive constants independent of ϵ (also τ, in our following discussion about numerical solution).

Proof. Multiplying the Eq. (1) by $2u'(t)$, we obtain

$$[\epsilon^2 u'^2 + b(t)u^2 - 2f(t)u]' + 2\epsilon a(t)u'^2 = b'(t)u^2 - 2f'(t)u.$$

Integrating this from 0 to t, we have

$$\epsilon^2 u'^2(t) + b(t)u^2(t) - 2f(t)u(t) \quad + \quad 2\epsilon\alpha\int_0^t u'^2(\zeta)d\zeta \le B^2 + b(0)A^2 - 2f(0)A$$

$$+\int_0^t |b'(\zeta)||u(\zeta)|^2 d\zeta + 2\int_0^t |f'(\zeta)||u(\zeta)|d\zeta$$

and

$$\epsilon^2 u'^2(t) + \beta u^2(t) - 1/\mu f^2(t) - \mu u^2(t) + 2\epsilon\alpha \int_0^t u'^2(\zeta)d\zeta \le B^2 + b(0)A^2 - 2f(0)A + \int_0^t [(|b'(\zeta)| + 1)u^2(\zeta) + f'^2(\zeta)]d\zeta, \quad \mu > 0.$$

The choice $\mu < \beta$ yields inequality of the form

$$\epsilon^2 u'^2(t) + u^2(t) + \epsilon \int_0^t u'^2(\zeta)d\zeta \le C_0 \left\{ \delta_*^2 + \int_0^t [u^2(\zeta) + f^2(\zeta) + f'^2(\zeta)]d\zeta \right\}. \tag{5}$$

From (5), by Gronwall's inequality, we obtain

$$\epsilon^2 u'^2(t) + u^2(t) + \epsilon \int_0^t u'^2(\zeta)d\zeta \le C_0\delta_*^2 \exp(C_0 t) + C_0 \int_0^t \lceil f^2(\zeta) + f'^2(\zeta) \rceil \exp(C_0(t-\zeta))d\zeta.$$

Therefore, (4) holds for $K = 0, 1$. Validity of the (4) for $K = 2$ already is an immediate consequence of (1). Lemma 2.1 is proved. □

Lemma 2.2. *Suppose $u(t)$ is the solution of (1)-(3) and $a \in C^1[0,T]$, $b, f \in C^3[0,T]$. Then*

$$|u^{(K)}(t)| \le C\left(1 + \epsilon^{1-K} + \epsilon^{-K} e^{-\frac{c_0 t}{\epsilon}}\right), \quad K = 0, 1, 2. \tag{6}$$

Proof. The solution of the problem (1)-(3) has the following construction

$$u(t) = u_0(t) + v(t) + R_\epsilon(t), \tag{7}$$

where $u_0(t) = f(t)/b(t)$ is the solution of the "reduced" problem, and $v(t), R_\epsilon(t)$ satisfy the following problems respectively

$$\begin{aligned} Lv &= 0, \\ v(0) &= A - u_0(0), \\ v'(0) &= B/\epsilon - u_0'(0), \end{aligned} \tag{8}$$

$$\begin{aligned} LR_\epsilon(t) &= \psi_\epsilon(t) \equiv -\epsilon^2 u_0'' - \epsilon a(t) u_0', \\ R_\epsilon(0) &= 0, \\ R_\epsilon'(0) &= 0. \end{aligned} \tag{9}$$

Now, we prove that

$$|v^{(K)}(t)| \leq C\epsilon^{-K}e^{-\frac{c_0 t}{\epsilon}}, \quad K=0,1,2. \tag{10}$$

From the identity

$$Lv.(v' + \lambda\epsilon^{-1}v) = 0,$$

where λ is an arbitrary positive constant, it is easy to get

$$\{\epsilon^2 v'^2 + bv^2 + 2\lambda\epsilon v'v + \lambda a v^2\}' = -(2a\epsilon - 2\lambda\epsilon)v'^2 - (2\epsilon^{-1}\lambda b - b' - \lambda a')v^2. \tag{11}$$

For the function

$$\delta(t) = \epsilon^2 v'^2 + bv^2 + 2\lambda\epsilon v'v + \lambda a v^2$$

it follows:

$$\delta(t) \leq \epsilon^2 v'^2 + bv^2 + \lambda\epsilon^2 v'^2 + \lambda v^2 + \lambda a v^2 \leq \epsilon^2(1+\lambda)v'^2 + (b^* + \lambda + \lambda a^*)v^2, \tag{12}$$

where $b^* = \max_{[0,T]} b(t),\ a^* = \max_{[0,T]} a(t)$.

On the other hand

$$\delta(t) \geq \epsilon^2 v'^2 + (b+\lambda a)v^2 - \lambda\epsilon^2 a^{-1} v'^2 - \lambda a v^2 = \epsilon^2(1-\lambda a^{-1})v'^2 + bv^2,$$

from which, choosing $\lambda < \alpha$ it is clear that

$$\delta(t) \geq c_1(\epsilon^2 v'^2 + v^2), \quad c_1 > 0. \tag{13}$$

For the right-hand side of the equality (11), we have

$$2\epsilon(a-\lambda)v'^2 + \epsilon^{-1}(2\lambda b - \epsilon b' - \epsilon\lambda a')v^2 \geq 2\epsilon(\alpha-\lambda)v'^2 + \epsilon^{-1}(2\lambda\beta - \epsilon\overline{b}^* - \epsilon\lambda\overline{a}^*)v^2,$$

where $\overline{b}^* = \max_{[0,T]} b'(t),\ \overline{a}^* = \max_{[0,T]} a'(t)$.

For $\lambda < \alpha$ and $2\lambda\beta - \epsilon\overline{b}^* - \epsilon\lambda\overline{a}^* > 0$, taking into account (12), this means that

$$2\epsilon(a-\lambda)v'^2 + \epsilon^{-1}(2\lambda b - \epsilon\overline{b}^* - \epsilon\lambda\overline{a}^*)v^2 \geq c_2\epsilon^{-1}\delta, \tag{14}$$

where c_2 can be chosen independently of ϵ

$$\left(0 < c_2 \leq \min\left\{\frac{2(\alpha-\lambda)}{1+\lambda},\ \frac{2\lambda\beta - \epsilon\overline{\beta}^* - \epsilon\lambda\overline{a}^*}{b^* + \lambda + \lambda a^*}\right\}\right).$$

From (11), taking into account (14) we get inequality of the form

$$\delta'(t) \leq -c_2\epsilon^{-1}\delta(t).$$

Therefore

$$\delta(t) \leq \delta(0)\exp(-c_2 t/\epsilon)$$

from which by (13), estimate (10) for $K=0,1$, follows immediately with $c_0=c_2/2$. The case $K=2$ now directly follows from (8_1).

Next, applying Lemma 1.1 for the problem (9) we have

$$|R_\epsilon^{(K)}(t)| \le C\epsilon^{1-K}, \quad K=0,1,2. \tag{15}$$

Now, (6) follows directly from (7) by using estimates (10), (15). Thus Lemma 2.2 is proved. □

Corollary 2.1. $||u^{(K)}(t)||_{L_1(0,T)} \le C(1+\epsilon^{1-K}), \quad K=0,1,2$.

Remark 2.1. Let the condition (3) has the form $u'(0)=B$ and let $b(0)A-f(0)=0$, in Lemma 2.2. Then

$$|u^{(K)}(t)| \le C\left(1+\epsilon^{1-K}e^{-\frac{c_0 t}{\epsilon}}\right), \quad K=0,1.$$

Validity of this inequality follows from (7), if taking into account that in this case $v(0)=0$ and the function $v(t)$ satisfies the estimate

$$|v(t)| \le C\epsilon^{1-K}e^{-\frac{c_0 t}{\epsilon}}.$$

3. Construction of the Difference Scheme

We shall assume that $a^2(t)-4b(t)>0$ for all $t\in[0,T]$.

In what follows, we denote by ω_τ the uniform mesh on $[0,T]$:

$$\omega_\tau=\{t_j=j\tau, \quad j=1,2,\dots,M-1;\ M\tau=T\}, \quad \overline{\omega}_\tau=\omega_\tau\cup\{t=0,T\}.$$

The difference scheme we shall construct as following from the identity

$$\xi_j^{-1}\tau^{-1}\int_0^T Lu.\varphi_j(t)dt=\xi_j^{-1}\tau^{-1}\int_0^T f(t)\varphi_j(t)dt \quad (j=1,2,\dots,M-1), \tag{16}$$

where basis functions $\{\varphi_j(t)\}_{j=1}^{M-1}$ have the form

$$\varphi_j(t)=\begin{cases} \frac{e^{\lambda_{1,j}(t-t_{j-1})}-e^{\lambda_{2,j}(t-t_{j-1})}}{e^{\lambda_{1,j}\tau}-e^{\lambda_{2,j}\tau}} \equiv \varphi_j^{(1)}(t), & t_{j-1}<t<t_j \\ \frac{e^{-\lambda_{2,j}(t_{j+1}-t)}-e^{-\lambda_{1,j}(t_{j+1}-t)}}{e^{-\lambda_{2,j}\tau}-e^{\lambda_{1,j}\tau}} \equiv \varphi_j^{(2)}(t), & t_j<t<t_{j+1} \\ 0, & t\notin(t_{j-1},t_{j+1}) \end{cases}$$

$$\lambda_{1,j}=0.5\epsilon^{-1}[a_j+\sqrt{a_j^2-4b_j}],\ \lambda_{2,j}=0.5\epsilon^{-1}[a_j-\sqrt{a_j^2-4b_j}],$$

$$\xi_j = \tau^{-1} \int_{t_{j-1}}^{t_{j+1}} \varphi_j(t)dt = \frac{2\tau^{-1}(\lambda_{1,j} - \lambda_{2,j})}{\lambda_{1,j}\lambda_{2,j} sh(\lambda_{1,j} - \lambda_{2,j})\tau/2} sh(\lambda_{1j}\tau/2).sh(\lambda_{2j}\tau/2).$$

We note that functions $\varphi_i^{(1)}(t)$ and $\varphi_i^{(2)}(t)$ are the solutions of the following problems, respectively,

$$\begin{aligned} &\epsilon^2\varphi'' - a_j\epsilon\varphi' + b_j\varphi = 0, \ \ t_{j-1} < t < t_j, \\ &\qquad \varphi(t_{j-1}) = 0, \qquad \varphi(t_j) = 1, \end{aligned} \tag{17}$$

$$\begin{aligned} &\epsilon^2\varphi'' - a_j\epsilon\varphi' + b_j\varphi = 0, \ \ t_j < t < t_{j+1}, \\ &\qquad \varphi(t_j) = 1, \qquad \varphi(t_{j+1}) = 0. \end{aligned} \tag{18}$$

The relation (16) can be rewritten as

$$\xi_j^{-1}\left[-\epsilon^2\tau^{-1}\int_{t_{j-1}}^{t_{j+1}} \varphi_j'(t)u'(t)dt + \epsilon a_j\tau^{-1}\int_{t_{j-1}}^{t_{j+1}} \varphi_j(t)u'(t)dt + b_j\tau^{-1} \int_{t_{j-1}}^{t_{j+1}} \varphi_j(t)u(t)dt\right] = f_j - R_j \tag{19}$$

with

$$\begin{aligned} R_j \ &= \ \xi_j^{-1}\epsilon\tau^{-1}\int_{t_{j-1}}^{t_{j+1}}[a(t) - a(t_j)]\varphi_j(t)u'(t)dt + \xi_j^{-1}\tau^{-1}\int_{t_{j-1}}^{t_{j+1}}[b(t) - b(t_j)]\varphi_j(t)u(t)dt \\ &+\xi_j^{-1}\tau^{-1}\int_{t_{j-1}}^{t_{j+1}}[f(t_j) - f(t)]\varphi_j(t)dt. \end{aligned} \tag{20}$$

Using the formulas (2.1), (2.2) from [1] on each intervals (t_{j-1}, t_j) and (t_j, t_{j+1}), and taking into account (17), (18) we have the following precise relation

$$\begin{aligned} &\epsilon^2\tau^{-1}\int_{t_{j-1}}^{t_{j+1}} u''(t)\varphi_j(t)dt + \epsilon a_j\tau^{-1}\int_{t_{j-1}}^{t_{j+1}} \varphi_j(t)u'(t)dt + b_j\tau^{-1}\int_{t_{j-1}}^{t_{j+1}} \varphi_j(t)u(t)dt \\ &\quad = \epsilon^2 u_{\bar{t}t,j} + \epsilon a_j(\xi_{1j}u_{\bar{t},j} + \xi_{2j}u_{t,j}) + b_j\xi_j u_j + b_j\mu_{1j}u_{\bar{t},j} + b_j\mu_{2j}u_{t,j} \\ &\quad = \epsilon^2\{1 + 0.5\tau\epsilon^{-1}a_j(\xi_{2,j} - \xi_{1,j}) + 0.5\tau\epsilon^{-2}b_j(\mu_{2j} - \mu_{1j})\}u_{\bar{t}t,j} \\ &\qquad +\epsilon a_j(\xi_j + \epsilon^{-1}a_j^{-1}b_j\mu_j)u_{t^0,j} + b_j\xi_j u_j \end{aligned}$$

where

$$\begin{aligned} \xi_{1j} \ &= \ \tau^{-1}\int_{t_{j-1}}^{t_j} \varphi_j^{(1)}(t)dt, \quad \xi_{2j} = \tau^{-1}\int_{t_j}^{t_{j+1}} \varphi_j^{(2)}(t)dt, \\ \mu_{1j} \ &= \ \tau^{-1}\int_{t_{j-1}}^{t_j}(t - t_j)\varphi_j^{(1)}(t)dt, \quad \mu_{2j} = \tau^{-1}\int_{t_j}^{t_{j+1}}(t - t_j)\varphi_j^{(2)}(t)d, \ \ \mu_j = \mu_{1j} + \mu_{2j}, \end{aligned}$$

$$u_j = u(t_j),\ \ u_{\bar{t},j} = (u_j - u_{j-1})/\tau,\ \ u_{t,j} = (u_{j+1} - u_j)/\tau,\ \ u_{t^0,j} = (u_{j+1} - u_{j-1})/(2\tau),$$
$$u_{\bar{t}t,j} = (u_{j+1} - 2u_j + u_{j-1})/\tau^2.$$

Simple calculation shows that

$$1 + 0.5\tau\epsilon^{-1}a_j(\xi_{2,j} - \xi_{1,j}) + 0.5\tau\epsilon^{-2}b_j(\mu_{2j} - \mu_{1j}) = 0.5\tau(\lambda_{1,j} - \lambda_{2,j})\frac{ch((\lambda_{1,j} + \lambda_{2,j})\tau/2)}{sh((\lambda_{1,j} - \lambda_{2,j})\tau/2)},$$

$$\xi_j + \epsilon^{-1}a_j^{-1}b_j\mu_j = \frac{(\lambda_{1,j} - \lambda_{2,j})sh((\lambda_{1,j} + \lambda_{2,j})\tau/2)}{(\lambda_{1,j} + \lambda_{2,j})sh((\lambda_{1,j} - \lambda_{2,j})\tau/2)}.$$

Further returning to the (19), we obtain the identity

$$lu_j \equiv \epsilon^2\theta_{1,j}u_{\bar{t}t,j} + \epsilon a_j\theta_{2,j}u_{t^0,j} + b_ju_j = f_j - R_j, \quad j = 1, 2, \ldots, M-1 \tag{21}$$

where

$$\theta_{1,j} = \frac{\tau^2}{4}\lambda_{1,j}\lambda_{2,j}\frac{ch((\lambda_{1,j} + \lambda_{2,j})\tau/2)}{sh(\lambda_{1,j}\tau/2).sh(\lambda_{2,j}\tau/2)} \equiv \frac{b_j\tau^2}{4\epsilon^2}(1 + cth(\lambda_{1j}\tau/2).cth(\lambda_{2j}\tau/2)), \tag{22}$$

$$\theta_{2,j} = \frac{\tau}{2}\frac{\lambda_{1,j}\lambda_{2,j}}{\lambda_{1,j} + \lambda_{2,j}}\frac{sh((\lambda_{1,j} + \lambda_{2,j})\tau/2)}{sh(\lambda_{1,j}\tau/2).sh(\lambda_{2,j}\tau/2)} \equiv \frac{b_j\tau}{2a_j\epsilon}(cth(\lambda_{1j}\tau/2) + cth(\lambda_{2j}\tau/2)), \tag{23}$$

Now, it remains to define an approximation for the initial condition (3). Here we start with the identity

$$\int_{t_0}^{t_1} Lu\varphi_0(t)dt = \int_{t_0}^{t_1} f(t)\varphi_0(t)dt,$$

where

$$\varphi_0(t) = \begin{cases} \frac{e^{-\lambda_{2,0}(t_1-t)} - e^{-\lambda_{1,0}(t_1-t)}}{e^{-\lambda_{2,0}\tau} - e^{-\lambda_{1,0}\tau}}, & t \in (t_0, t_1), \\ 0, & t \notin (t_0, t_1) \end{cases}$$

By argument similar to those as in the proof of (21), we have

$$\epsilon^2\theta_0u_{t,0} - \epsilon B + I_0(b_0A - f_0) + r = 0, \tag{24}$$

where

$$I_0 = \int_0^{t_1}\varphi_0(t)dt = \epsilon^2b_0^{-1}[\lambda_{1,0}(1 - e^{-\lambda_{2,0}\tau}) - \lambda_{2,0}(1 - e^{-\lambda_{1,0}\tau})]/(e^{-\lambda_{2,0}\tau} - e^{-\lambda_{1,0}\tau}),$$

$$\begin{aligned}\theta_0 &= 1 + \epsilon^{-1}a_0\int_0^{t_1}\varphi_0(t)dt + \epsilon^{-2}b_0\int_0^{t_1}t\varphi_0(t)dt \\ &= \frac{\tau(\lambda_{1,0} - \lambda_{2,0})}{2sh((\lambda_{1,0} - \lambda_{2,0})\tau/2)}e^{(\lambda_{1,0}+\lambda_{2,0})\tau/2},\end{aligned} \tag{25}$$

$$r = \int_0^{t_1}[f(0)-f(t)]\varphi_0(t)dt + \epsilon\int_0^{t_1}[a(t)-a(0)]u'(t)\varphi_0(t)dt + \int_0^{t_1}[b(t)-b(0)]u(t)\varphi_0(t)dt. \quad (26)$$

Neglecting R_j and r in (21) and (24), we may propose the difference scheme for (1)-(3) as follows

$$ly \equiv \epsilon^2\theta_1 y_{\overline{t}t} + \epsilon a\theta_2 y_{t^0} + by = f, \quad t \in \omega_\tau, \quad (27)$$

$$y(0) = A, \quad (28)$$

$$\epsilon^2\theta_0 y_{t,0} - \epsilon B + I_0(b_0 A - f_0) = 0, \quad (29)$$

where $\theta_1, \theta_2, \theta_0$ are given by (22), (23), (25) respectively.

4. Uniform Error Estimates

Let $z = y - u$. Then for the error of the difference scheme (27)-(29) we get

$$lz = R, \quad t \in \omega_\tau, \quad (30)$$

$$z(0) = 0, \quad (31)$$

$$\epsilon^2\theta_0 z_{t,0} = r, \quad (32)$$

where R and r are defined by (20) and (26).

Lemma 4.1. *Let z be the solution (30)-(32). Then the estimate*

$$\Delta_0|z_{t,j}| + |z_{j+1}+z_j| \le C\{\Delta_0|z_{t,0}| + \max_{1\le i\le M-1}|R_i| + \tau\sum_{i=1}^{M-2}|R_{t,i}|\} \quad (33)$$

holds for $j = 0, 1, \ldots, M-1$, where $\Delta_0 = \max(\epsilon, \tau)$.

Proof. Multiplying (30) by z_{t^0} and taking into consideration the relations

$$\theta_1 z_{\overline{t}t} z_{t^0} = \frac{1}{2}\theta_1(z_t^2)_{\overline{t}} = \frac{1}{2}(\theta_1 z_t^2)_{\overline{t}} - \frac{1}{2}\theta_{1\overline{t}} z_{\overline{t}}^2,$$

$$\begin{aligned} bzz_{t^0} &= \frac{1}{8}b((\hat{z}+z)^2)_{\overline{t}} - \frac{\tau^2}{8}b(z_t^2)_{\overline{t}} \\ &= \frac{1}{8}(b(\hat{z}+z)^2)_{\overline{t}} - \frac{\tau^2}{8}(bz_t^2)_{\overline{t}} - \frac{1}{8}b_{\overline{t}}(z+\check{z})^2 + \frac{\tau^2}{8}b_{\overline{t}} z_{\overline{t}}^2 \end{aligned}$$

we have

$$\frac{1}{2}\epsilon^2(pz_t^2)_{\overline{t}} + \epsilon\theta_2 a z_{t^\circ}^2 + \frac{1}{8}(b(\hat{z}+z)^2)_{\overline{t}} = \frac{1}{2}\epsilon^2 q z_{\overline{t}}^2 + \frac{1}{8}b_{\overline{t}}(z+\check{z})^2 + Rz_{t^0}, \quad (34)$$

where

$$p = \theta_1 - \frac{\tau^2}{4\epsilon^2} b, \quad q = \theta_{1\bar{t}} - \frac{\tau^2}{4\epsilon^2} b_{\bar{t}}, \quad \hat{z} = z(t_{j+1}), \check{z} = z(t_{j-1}).$$

Multiplying (34) by 2τ and summing up it with respect to j (from 1 to s) and taking into account

$$2\tau \sum_{j=1}^{s} R_j z_{t^0,j} = R_s(z_{s+1} + z_s) - R_1(z_1 + z_0) - \tau \sum_{j=1}^{s-1} R_{t,j}(z_{j+1} + z_j)$$

we obtain

$$\begin{aligned} \epsilon^2 p_s z_{t,s}^2 \ &+ \ 2\epsilon\tau \sum_{j=1}^{s} \theta_{2j} a_j z_{t^0,j}^2 + \frac{1}{4} b_s (z_{s+1} + z_s)^2 \\ &= \ \epsilon^2 p_0 z_{t,0}^2 + \frac{1}{4} b_0 (z_1 + z_0)^2 - R_1(z_1 + z_0) + R_s(z_{s+1} + z_s) \\ &+ \ \tau \sum_{j=1}^{s} \left\{ \epsilon^2 q_j z_{\bar{t},j}^2 + \frac{1}{4} b_{\bar{t},j} (z_j + z_{j-1})^2 \right\} - \tau \sum_{j=1}^{s-1} R_{t,j}(z_{j+1} + z_j) \text{ for } s \le M-1. \quad (35) \end{aligned}$$

It is easy to verify that

$$0 < c_0 \le p \le c_1, \quad |q| \le C_0, \quad \text{when} \quad \tau \le \epsilon, \tag{36}$$

$$c_0 \tau^2 \le \epsilon^2 p \le c_1 \tau^2, \quad \epsilon^2 |q| \le C_0 \tau^2 \quad \text{when} \quad \tau \ge \epsilon. \tag{37}$$

Now, from the relation (35), using (36), (37), we have the following inequality

$$\delta_s \le \delta_* + \tau \sum_{j=1}^{s} \{ d_j \delta_{j-1} + \rho_j \}, \quad s \le M-1, \tag{38}$$

where

$$\begin{aligned} \delta_j &= \epsilon^2 p_j z_{t,j}^2 + \frac{1}{4} b_j (z_{j+1} + z_j)^2, \\ \delta_* &= c_0 \delta_0 + C \max_{1 \le j \le M-1} |R_j|^2, \quad c_0 > 1, \end{aligned}$$

$$|\rho_j| \le C |R_{tj}| \, |z_{j+1} + z_j| \quad \text{for} \quad j = 1, \ldots, s-1 \quad \text{and} \quad \rho_s = 0,$$
$$0 \le d_j \le c \quad \text{for} \quad j = 1, 2, \ldots, s.$$

From (38), by difference analogue of integral inequality, we have

$$\begin{aligned} \delta_s \quad &\le \quad \delta_* \exp \left[\tau \sum_{i=1}^{s} d_i \right] + \tau \sum_{i=1}^{s} |\rho_i| \exp \left[\tau \sum_{j=i+1}^{s} d_j \right] \\ &\le \quad C \left(\Delta_0^2 z_{t,0}^2 + |z_1|^2 + \max_{1 \le j \le M-1} |R_j|^2 + \tau \sum_{i=1}^{s-1} |R_{t,i}| |z_{i+1} + z_i| \right) \end{aligned}$$

which leads to (33). Thus the Lemma is proved. □

Lemma 4.2. *Let R and r be defined as in (20) and (26) respectively. Then*

$$\Delta_0|z_{t,0}| \le C\tau,$$

$$||R||_{C(\omega_\tau)} \le C\tau,$$

$$\tau\sum_{i=1}^{M-2}|R_{t,i}| \le C\tau.$$

The proof evidently from explicit expressions for R and r, using (6).

Finally, we give the main result of this paper:

Theorem 4.1. *Suppose u is the solution (1)-(3), y_j is the solution (27)-(29). Then*

$$\max_{1\le j\le M}|y_j - u_j| \le C\tau. \tag{39}$$

Proof. Because

$$z_{j+1} = (z_{j+1} + z_j)/2 + (\tau z_{tj})/2$$

we have

$$|z_{j+1}| \le |z_{j+1} + z_j|/2 + \Delta_0|z_{tj}|/2.$$

Now, (39) immediately follows from Lemma 4.1 and Lemma 4.2. □

5. Numerical Results

Example 1. Consider the following problem:

$$\epsilon^2 u'' + 2\epsilon(t+2)u' + (t^2+4t+3+\epsilon)u = f(t), \quad 0 < t \le 1,$$

$$u(0) = 1,$$

$$u'(0) = 1/\epsilon + 1 - \epsilon,$$

where $f(t)$ is chosen so that the exact solution is

$$u(t) = t(1-\epsilon) + e^{-\frac{t(t+2)}{2\epsilon}}\left\{2 - e^{-\frac{2t}{\epsilon}}\right\}.$$

Due to uniformly boundness of $b^{(k)}(t), f^{(k)}(t)(k = 0, 1, 2)$ in ϵ, the scheme (27)-(29) is applicable to implement and some of values of $E = \max_{\overline{\omega}_\tau}|y-u|$ are given in Table 1.

Table 1.

ϵ / τ	0.5	10^{-2}	10^{-4}	10^{-6}
0.1	2.004 E-2	5.069 E-2	6.205 E-2	7.036 E-2
0.05	8.421 E-4	5.259 E-3	1.278 E-2	1.301 E-2
0.02	7.201 E-5	3.440 E-4	7.101 E-4	7.212 E-4

Example 2. Now consider the following problem

$$\epsilon^2 u'' + \epsilon 3e^{t^2} u' + (2-t)u = t^2 e^t,$$

$$u(0) = 1, \quad u'(0) = 1/\epsilon.$$

We have chosen to use an asymptotic approximation as our exact solution

$$u_A(t) = \frac{t^2 e^t}{2-t} + 3e^{-\frac{t}{\epsilon}} - 2e^{-\frac{2t}{\epsilon}}.$$

The computational results for $E_A = \max_{\overline{\omega}_\tau} |y - u_A|$ are presented in Table 2.

Table 2.

ϵ / τ	10^{-2}	10^{-4}	10^{-6}
0.1	9.350 E-2	8.241 E-2	8.035 E-2
0.05	8.402 E-2	7.391 E-3	7.420 E-3
0.02	3.047 E-2	2.489 E-3	2.471 E-3

The results show that the convergence rate of the considered schemes is essentially in accord with theoretical analysis.

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