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DERIVATION OF SEPARABLE AMPLITUDE EQUATIONS BY MULTIPLE SCALES METHOD

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Abstract

The method of multiple scales is used to derive separable nonlinear Schrödinger equations as amplitude equation from three component $2D$ nonlinear Klein-Gordon Equation. We further discuss the integrability of the derived separable amplitude equations and reduce them into finite dimensional Hamiltonian systems. Finally we give first integrals for the reduced systems.

Key Words: Multiple Scales Method, Three Component $2D$ Nonlinear Klein-Gordon Equation, Separable Amplitude NLS Equation, Spectral Problem, Separable Integrable Hamiltonian system.

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1. Introduction

In [1, 2] Fordy and Gibbons introduced a three component $2D$ nonlinear Klein-Gordon equations in laboratory coordinates as a system of equations:

$$\begin{aligned} u_{tt} - u_{xx} &= -e^{u}\cosh v + e^{-u}\cosh w, \\ v_{tt} - v_{xx} &= -e^{u}\sinh v, \\ w_{tt} - w_{xx} &= -e^{-u}\sinh w, \end{aligned} \tag{1}$$

where subscript indicate partial differentiation with respect to some variables. They derived a linear spectral problem for the above system and thus proved its complete integrability.

There are various restrictions that one can make to this system which make sense; either or both of v and w may be suppressed. If w is suppressed, we get

$$\begin{aligned} u_{tt} - u_{xx} &= e^{-u} - e^{u}\cosh v, \\ v_{tt} - v_{xx} &= -e^{u}\sinh v, \end{aligned} \tag{2}$$

or if w and v are suppressed, we get

$$u_{tt} - u_{xx} = e^{-u} - e^{u}. \tag{3}$$

It is well known that a multiple scales analysis of the Sinh-Gordon equation (and, indeed many other equations) leads to the NLS equation for the modulated amplitude [6, 9, 8]. In this paper, we use the method of multiple scales [5] to derive separable NLS equations from the system (1) and give the corresponding spectral problem to show their integrability. We also reduce the derived separable systems into a Hamiltonian system of six-degrees of freedom. Then we prove its complete integrability by a perturbation method [6, 7].

In section 2 we present the multiple scales method to derive separable NLS equations. We then discuss the integrability of the derived decoupled equations. Sections 3 and 4 are respectively concerned with the related integrable finite Hamiltonian systems and their integrals of motion.

2. A Multiple Scales Analysis

In this section the method of multiple scales is used to derive integrable decoupled nonlinear Schrödinger equations as amplitude equations from integrable systems of three component $2D$ nonlinear Klein-Gordon equation (1) in laboratory coordinates.

We first take the Taylor series expansion of right hand side functions at zero. We then seek a solution for system (1) in the series form

$$\mathbf{u}(x,t) = \sum_{n=1}^{\infty} \epsilon^{n}\,\mathbf{u}_n(t,\xi,\tau), \tag{4}$$

and define the scaling parameter ϵ dependence as

$$\xi = \epsilon x, \quad \tau = \epsilon^2 t, \tag{5}$$

in order to use the multiple scales method.

2.1. The Derivation

Inserting the expansion (4) with (5) and $\mathbf{u} = (u, v, w)$ into (1) and using the Taylor series expansions of the right hand side functions at zero, and then equating to zero coefficients of like powers of ϵ we find the following:

$$\begin{aligned} &u_{1tt} + 2u_1 = 0, \\ &v_{1tt} + v_1 = 0, \\ &w_{1tt} + w_1 = 0; \end{aligned} \tag{6}$$

$$\begin{aligned} &u_{2tt} + 2u_2 = \tfrac{1}{2}(-v_1^2 + w_1^2), \\ &v_{2tt} + v_2 = -u_1 v_1, \\ &w_{2tt} + w_2 = u_1 w_1; \end{aligned} \tag{7}$$

$$\begin{aligned} &u_{3tt} + 2u_3 = u_{1\xi\xi} - 2u_{1t\tau} - \tfrac{1}{3}u_1^3 - \tfrac{1}{2}(u_1 v_1^2 + u_1 w_1^2) - v_1 v_2 + w_1 w_2, \\ &v_{3tt} + v_3 = v_{1\xi\xi} - 2v_{1t\tau} - \tfrac{1}{6}v_1^3 - \tfrac{1}{2}v_1 u_1^2 - v_1 u_2 - u_1 v_2, \\ &w_{3tt} + w_3 = w_{1\xi\xi} - 2w_{1t\tau} - \tfrac{1}{6}w_1^3 - w_1 u_1^2 - w_1 u_2 - u_1 w_2, \end{aligned} \tag{8}$$

and so on. The general solutions of the system (6) are

$$\begin{aligned} &u_1 = e^{i\sqrt{2}\,t} A + e^{-i\sqrt{2}\,t} A^*, \\ &v_1 = e^{it} B + e^{-i\,t} B^*, \\ &w_1 = e^{it} C + e^{-i\,t} C^*, \end{aligned} \tag{9}$$

where $A \equiv A(\xi,\tau)$, $B \equiv B(\xi,\tau)$, $C \equiv C(\xi,\tau)$ and A^*, B^*, C^* are respectively the complex conjugates of A, B, C.

Inserting these into (7), we respectively find the second solutions in the form:

$$\begin{aligned} &u_2 = \tfrac{1}{4}e^{2it}\left(B^2 - C^2\right) + c.c. - \tfrac{1}{2}(B\,B^* - C\,C^*), \\ &v_2 = -\tfrac{1}{2}\Big(\left(-1+\sqrt{2}\right) e^{it+i\sqrt{2}t} AB - \left(1+\sqrt{2}\right) e^{it-i\sqrt{2}t} A^* B + c.c.\Big), \\ &w_2 = \tfrac{1}{2}\Big(\left(1-\sqrt{2}\right) e^{it+i\sqrt{2}t} AC + \left(1+\sqrt{2}\right) e^{it-i\sqrt{2}t} A^* C + c.c.\Big), \end{aligned} \tag{10}$$

where $c.c.$ denotes complex conjugates. Inserting (9) and (10) into (8), the secular terms are eliminated by taking

$$\begin{aligned} A_\tau &= -\frac{i}{2^{\frac{3}{2}}}\left(-A^2 A^* + A_{\xi\xi}\right), \\ B_\tau &= -\frac{i}{8}\left(-B^2 B^* + B^* C^2 - 2BC\, C^* + 4B_{\xi\xi}\right), \\ C_\tau &= -\frac{i}{8}\left(-2B\, B^* C + B^2 C^* - C^2 C^* + 4C_{\xi\xi}\right), \end{aligned} \tag{11}$$

with their complex conjugate. Then we find the third order solutions as:

$$\begin{aligned} u_3 &= e^{3i\sqrt{2}t}\, u_{31} + e^{2it+i\sqrt{2}t}\, u_{33} + e^{2it-i\sqrt{2}t}\, u_{35} + c.c. \\ v_3 &= e^{3it}\, v_{31} + e^{it+i2^{\frac{3}{2}}t}\, v_{33} + e^{-it+i2^{\frac{3}{2}}t}\, v_{35} + c.c. \\ w_3 &= e^{3it}\, w_{31} + e^{it+i2^{\frac{3}{2}}t}\, w_{33} + e^{-it+i2^{\frac{3}{2}}t}\, w_{35} + c.c. \end{aligned} \tag{12}$$

where

$$\begin{aligned} u_{31} &= \frac{1}{48} A^3,\ u_{33} = \frac{1}{8}\left(2B^2 - \sqrt{2}B^2 + 2C^2 - \sqrt{2}C^2\right) A, \\ u_{35} &= \frac{1}{8}\left(2B^2 + \sqrt{2}B^2 + 2C^2 + \sqrt{2}C^2\right) A^*. \end{aligned} \tag{13}$$

$$\begin{aligned} &v_{31} = \frac{1}{96}(5B^2 - 3C^2)\, B, \qquad v_{33} = -\frac{1}{8}\left(1-\sqrt{2}\right) A^2 B, \\ &v_{35} = -\frac{1}{8}\left(1+\sqrt{2}\right) A^2\, B^*. \end{aligned} \tag{14}$$

$$\begin{aligned} &w_{31} = \frac{1}{96}(-3B^2 + 5C^2)\, C, \quad w_{33} = -\frac{1}{8}\left(1-\sqrt{2}\right) A^2 C, \\ &w_{35} = -\frac{1}{8}\left(1+\sqrt{2}\right) A^2\, C^*. \end{aligned} \tag{15}$$

Let us now define the functions

$$A = \sqrt{2}\, p, \qquad B = 2\sqrt{2}\, q, \qquad C = 2\sqrt{2}\, r, \tag{16}$$

in order to find a system of nonlinear Schrödinger equations from (11) as a system of following amplitude equations:

$$\begin{aligned} 2\sqrt{2}ip_\tau &= p_{\xi\xi} - 2p \mid p \mid^2, \\ 2iq_\tau &= q_{\xi\xi} - 2q(\mid q \mid^2 + 2 \mid r \mid^2) + 2r^2\, q^*, \\ 2ir_\tau &= r_{\xi\xi} - 2r(\mid r \mid^2 + 2 \mid q \mid^2) + 2q^2\, r^*. \end{aligned} \tag{17}$$

Note that the first component p has decoupled, so this system gives the scalar NLS equation

$$ip_\tau = p_{\xi\xi} - 2p \mid p \mid^2, \tag{18}$$

and the system of NLS equations

$$
\begin{aligned}
iq_\tau &= q_{\xi\xi} - 2q(\mid q\mid^2 + 2\mid r\mid^2) + 2r^2\, q^*,\\
ir_\tau &= r_{\xi\xi} - 2r(\mid r\mid^2 + 2\mid q\mid^2) + 2q^2\, r^*.
\end{aligned}
\tag{19}
$$

by choosing a suitable change of variable for the slow time τ. It is shown that these equations are integrable by inverse scattering method [4, 6, 7, 9].

3. Related Finite Dimensional Hamiltonian Systems

We consider solutions of the above NLS equations of the form:

$$
q(\xi,\tau) = e^{i\omega_1^2\tau}U(\xi), \quad r(\xi,\tau) = e^{i\omega_2^2\tau}V(\xi), \ldots, \text{etc.} \tag{20}
$$

We then consider the system of real equations satisfied by the real and imaginary parts of U, V, ... We now present this reduction for the system (17) as an example.

3.1. Six Degrees of Freedom

We consider the system of NLS equations (17) and assume separable solutions in the following form:

$$
\begin{aligned}
&p(\xi,\tau) = e^{i(\omega_1^2/2\sqrt{2})\tau}U(\xi), \quad q(\xi,\tau) = e^{i(\omega_2^2/2)\tau}V(\xi),\\
&r(\xi,\tau) = e^{i(\omega_3^2/2)\tau}W(\xi),
\end{aligned}
\tag{21}
$$

to find :

$$
\begin{aligned}
&\left(U_{\xi\xi} + \omega_1^2 U - 2U\mid U\mid^2\right) e^{i\tau(\omega_1^2/2\sqrt{2})} = 0,\\
&\left(V_{\xi\xi} + \omega_2^2 V - 2V(\mid V\mid^2 + 2\mid W\mid^2)\right) e^{i\tau(\omega_2^2/2)}\\
&+2\,V^* W^2 e^{i\tau(2\omega_3^2-\omega_2^2)/2} = 0,\\
&\left(W_{\xi\xi} + \omega_3^2 W - 2W(\mid W\mid^2 + 2\mid V\mid^2)\right) e^{i\tau\omega_3^2/2}\\
&+2W^* V^2 e^{i\tau(2\omega_2^2-\omega_3^2)/2} = 0.
\end{aligned}
\tag{22}
$$

For nontrivial V and W we must have $\omega_2 = \omega_3$. Defining

$$
\begin{aligned}
&U(\xi) = q_1(\xi) + iq_2(\xi),\ V(\xi) = q_3(\xi) + iq_4(\xi),\\
&W(\xi) = q_5(\xi) + iq_6(\xi),
\end{aligned}
\tag{23}
$$

the real and imaginary parts are:

$$
\begin{aligned}
&\ddot{q}_1 + \omega_1^2 q_1 - 2q_1(q_1^2 + q_2^2) = 0,\\
&\ddot{q}_2 + \omega_1^2 q_2 - 2q_2(q_1^2 + q_2^2) = 0,\\
&\ddot{q}_3 + \omega_2^2 q_3 - 2q_3(q_3^2 + q_4^2 + q_5^2 + 3q_6^2) + 4q_5 q_6 q_4 = 0,\\
&\ddot{q}_4 + \omega_2^2 q_4 - 2q_4(q_3^2 + q_4^2 + 3q_5^2 + q_6^2) + 4q_5 q_6 q_3 = 0,\\
&\ddot{q}_5 + \omega_2^2 q_5 - 2q_5(q_5^2 + q_6^2 + q_3^2 + 3q_4^2) + 4q_3 q_6 q_4 = 0,\\
&\ddot{q}_6 + \omega_2^2 q_6 - 2q_6(q_6^2 + q_5^2 + 3q_3^2 + q_6^4) + 4q_5 q_4 q_3 = 0,
\end{aligned}
\tag{24}
$$

with the Hamiltonian $H = H_0 + H_2$:

$$
\begin{aligned}
H_0 = {}& \tfrac{1}{2}({p_1}^2 + {p_2}^2 + \omega_1^2({q_1}^2 + {q_2}^2) + {p_3}^2 + {p_4}^2 +\\
& {p_5}^2 + {p_6}^2 + \omega_2^2({q_3}^2 + {q_4}^2 + {q_5}^2 + {q_6}^2)),\\
H_2 = {}& -\tfrac{1}{2}\Big(({q_1}^2 + {q_2}^2)^2 + ({q_3}^2 + {q_4}^2 + {q_5}^2 + {q_6}^2)^2 +\\
& 4\,(q_3 q_6 - q_4\, q_5)^2\Big),
\end{aligned}
\tag{25}
$$

where $p_j = q_{j\xi}$ for j=1, …, 6. Note that the six degrees of freedom system (24) and its Hamiltonian (25),which is a generalisation of Garnier's system, are trivially separable in (q_1, q_2) and (q_3, q_4, q_5, q_6) components. Hence following [6, 7], we can write the resulting first integrals for $\omega_1 = \omega_2 = 1$ as follows:

$$
\begin{aligned}
k_1 &= q_1\, p_2 - q_2\, p_1,\\
k_2 &= q_3\, p_4 + q_5\, p_6 - q_4\, p_3 - q_6\, p_5,\\
k_3 &= q_3\, p_5 + q_4\, p_6 - q_5\, p_3 - q_6\, p_4,\\
k_4 &= H_0(p_1, p_2, q_1, q_2) + H_2(p_1, p_2, q_1, q_2),\\
k_5 &= H_0(p_3, p_4, p_5, p_6, q_3, q_4, q_5, q_6) + H_2(p_3, p_4, p_5, p_6, q_3, q_4, q_5, q_6),\\
k_6 &= p_4\, p_5 + q_4\, q_5 - p_3\, p_6 - q_3\, q_6 - (p_3\, p_5 - p_4\, p_6)\,(q_3\, q_4 - q_5\, q_6) -\\
& (p_3\, p_4 - p_5\, p_6)\,(q_3\, q_5 - q_4\, q_6) - 2(q_4\, q_5 - q_3\, q_6)({q_3}^2 + {q_4}^2 + {q_5}^2 + {q_6}^2)) -\\
& q_3\, q_6({p_4}^2 + {p_5}^2) + q_4\, q_5({p_3}^2 + {p_6}^2) + p_4\, p_5({q_5}^2 + {q_6}^2) - p_3\, p_6({q_4}^2 + {q_5}^2),
\end{aligned}
\tag{26}
$$

where

$$\begin{array}{lcl}
H_0(p_1,p_2,q_1,q_2) & = & \frac{1}{2}({p_1}^2+{p_2}^2+{q_1}^2+{q_2}^2),\\
H_2(p_1,p_2,q_1,q_2) & = & -\frac{1}{2}({q_1}^2+{q_2}^2)^2,\\
H_0(p_3,p_4,p_5,p_6,q_3,q_4,q_5,q_6) & = & \frac{1}{2}\sum_{j=3}^{6}({p_j}^2+{q_j}^2),\\
H_2(p_3,p_4,p_5,p_6,q_3,q_4,q_5,q_6) & = & -\frac{1}{2}\left(\left(\sum_{j=3}^{6}{q_j}^2\right)^2+4(q_3\,q_6-q_4\,q_5)^2\right).
\end{array} \tag{27}$$

Here these integrals of motion are in involution with respect to the canonical Poisson bracket.

4. Real Reduction

We now use real reductions:

$$U^*=U=q_1,\quad V^*=V=q_2,\ etc. \tag{28}$$

in order to find simple finite dimensional Hamiltonian systems with quartic potentials. Thus from the system of equations (22) we derive the following system:

$$\begin{array}{l}
\ddot{q}_1+\omega_1^2 q_1-2{q_1}^3=0,\\
\ddot{q}_2+\omega_2^2 q_2-2q_2({q_2}^2+{q_3}^2)=0,\\
\ddot{q}_3+\omega_2^2 q_3-2q_3({q_3}^2+{q_2}^2)=0.
\end{array} \tag{29}$$

Note that the last system and its Hamiltonian separate into q_1 and (q_2,q_3) components. The first equation is known as Duffing equation, which is also a Hamiltonian system with the quartic potential:

$$H=\frac{1}{2}\left(p_1^2+\omega_1^2 q_1^2-q_1^4\right). \tag{30}$$

The remaining equations are also a Hamiltonian system with quartic potential:

$$H=\frac{1}{2}\left(p_2^2+p_3^2+\omega_2^2 q_2^2+\omega_2^2 q_3-(q_2^2+q_3^2)^2)\right), \tag{31}$$

where $p_j=q_{j\xi}$ for j=2,3. Here again this system has rotational symmetry, with Noether's constant

$$k=q_2\,p_3-q_3\,p_2, \tag{32}$$

which is the second integral.

5. Conclusion

We have used the multiple scales method to derive a separable system of NLS equations and discuss the integrability. The starting point was the three-component $2D$ nonlinear Klein-Gordon equation. We also reduce the separable system of NLS equations to a Hamiltonian system of six-degrees of freedom and give first integrals of this system [6, 7]. We finally consider the real reduction for the separable system.

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