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DIRECT SUMS AND THE SCHUR PROPERTY

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Abstract

It is a known fact that ℓ^1, the dual space of the null sequences c_0, has the Schur property, that is, weakly convergent sequences in ℓ^1 are norm convergent. In this paper, we prove that if $(X_\alpha)_{\alpha\in I}$ are Banach spaces and $X = (\oplus_{\alpha\in I} X_\alpha)_1$ their l_1-sum, then the space X has the Schur property iff each factor X_α has it.

Key words and phrases: Schur property, Banach spaces, AMS Subject Classification: 46 B 20.

1. Introduction

A Banach space X is said to have the Schur property if every weakly convergent sequence in X is norm convergent; equivalently if weakly compact subsets of X are norm compact. It is a known fact that ℓ^1, the dual space of the null sequences c_0, has the Schur property. Many other results about the Schur property can be found in the litterature. In [3], the authors show that the dual space of the group C^*-algebra $C^*(G)$ of a compact group G has the Schur property. In [1], W. S. Brown shows under a very mild condition that the dual space of every commutative subalgebra of the operator algebra $\mathcal{K}(\mathcal{H})$ of the compact operators on a Hilbert space $\mathcal{H}$ has the Schur property. Continuing this work, in [8], A. Ülger characterizes the closed subspaces and subalgebras of $\mathcal{K}(\mathcal{H})$ whose duals have the Schur property. The characterizations of C^*-algebras whose duals have the Schur property has been given by A. Lau and A. Ülger [3], and of commutative C^*-algebras whose duals have the Schur property by A. Pelczynski and Z. Semadeni [5]. J. Diestel gave the connection of the Schur property with the Dunford-Pettis property [2].

If two Banach spaces have the Schur property, then so do their injective and projective tensor products [4], [6].

In this paper, we prove that if $(X_\alpha)_{\alpha\in I}$ are Banach spaces and $X = (\oplus_{\alpha\in I} X_\alpha)_1$ their l_1-sum, then the space X has the Schur property if each factor X_α has it. The technique used to prove that ℓ^1 has the Schur property is the inspiration for most of the results [7].

Our notation is quite standard. For any Banach space $\langle X, \| \, \| \rangle$, we denote its dual by X^* and its closed unit ball by X_1. The natural duality between X and X^* is denoted as $\langle u, f \rangle$ or as $\langle f, u \rangle$.

2. Direct sums of spaces with the Schur property

Proposition *Let $(X_\alpha)_{\alpha\in I}$ be Banach spaces and $X = (\oplus_{\alpha\in I} X_\alpha)_1$ be their l_1-sum. The space X has the Schur property iff each factor X_α has it.*

Proof. If a space has the Schur property, then any closed subspace of the space clearly has the Schur property. Hence the implication is true. For the reverse implication, first recall that a Banach space has the Schur property iff all of its closed separable subspaces have the Schur property. So we can assume that each X_α is separable and take $I = \mathbf{N}$. Since $X = (\oplus_{k\in\mathbf{N}} X_k)_1$ is separable, the closed unit ball of $X^* = (\oplus_{k\in\mathbf{N}} X_k^*)_\infty$ under its w^*-topology is metrizable ([9], II.A.15). It follows that a weakly null sequence $\{a_n\}$ in X is norm-null if any w^*-null sequence $\{f_n\}$ in the unit ball of X^* satisfies $\lim_{n\to\infty} \langle f_n, a_n \rangle = 0$.

Let then $\{a_n\}_{n\in\mathbf{N}}$ be a weakly null sequence in X. Then $a_n = \{b_{n,k}\}_{k\in\mathbf{N}}$ and for all $k \in \mathbf{N}$, the sequence $\{b_{n,k}\}_{n\in\mathbf{N}}$ is weakly null in X_k. As X_k has the Schur property, $\lim_{n\to\infty} \| b_{n,k} \| = 0$, for all $k \in \mathbf{N}$. On the other hand, let $\{f_n\}_{n\in\mathbf{N}}$ be a w^*-null sequence in the unit ball of X^*. Each f_n is of the form $f_n = \{g_{n,k}\}_{k\in\mathbf{N}}$, and for all $k \in \mathbf{N}$, the sequence $\{g_{n,k}\}$ is w^*-converging in X_k^* to zero. To prove that $\lim_{n\to\infty} \langle f_n, a_n \rangle = 0$, it is enough to show that the sum $\sum_{k\in\mathbf{N}} |\langle g_{n,k}, b_{n,k} \rangle|$ converges to zero uniformly in $n \in \mathbf{N}$, i.e. $sup_{n\in\mathbf{N}} \sum_{k>N} |\langle g_{n,k}, b_{n,k} \rangle| \to 0$, as $N \to \infty$. In other words, we have to show that,

$$\forall\varepsilon > 0\ \exists N \in \mathbf{N}\ \forall \mathbf{n} \in \mathbf{N} \sum_{\mathbf{k}>\mathbf{N}} |\langle \mathbf{g_{n,k}}, \mathbf{b_{n,k}}\rangle| < \varepsilon \quad (*)$$

Assume $(*)$ is false. So, there is an $\epsilon > 0$ such that,

$$\forall N\epsilon\mathbf{N}\ \exists \mathbf{n} \in \mathbf{N} \sum_{\mathbf{k}>\mathbf{N}} |\langle \mathbf{g_{n,k}}, \mathbf{b_{n,k}}\rangle| \geq \epsilon \quad (**)$$

Consider a sequence of positive numbers $\{\delta_k\}$ such that $\sum_{k=1}^{\infty} \delta_k < \frac{\epsilon}{4}$. We are going to construct two strictly increasing sequences $\{n_k\}_{k\geq 1}$ and $\{N_k\}_{k\geq 0}$ such that

1) $\sum_{p>N_k} \| b_{n_{k,p}} \| \leq \delta_k$ for each $k \geq 1$,
2) $\sum_{p=1}^{N_{k-1}} |\langle g_{n,p}, b_{n_{k-1},p}\rangle| \leq \delta_k$ for each $n \geq n_k$,
3) $\sum_{p>N_{k-1}} |\langle g_{n_k,p}, b_{n_k,p}\rangle| \geq \epsilon$.

Start with $n_1 = 1$ and $N_0 = 0$. Since $a_{n_1} \in X$, i.e. $\sum_{p\in\mathbf{N}} \| b_{n_1,p} \| < \infty$, we can choose N_1 such that $\sum_{p>N_1} \| b_{n_1,p} \| \leq \delta_1$. Since $\{g_{n,1}\}_{n\in\mathbf{N}},\ldots,\{g_{n,N_1}\}_{n\in\mathbf{N}}$ converge to zero in $(X_1^*, w^*), \ldots, (X_{N_1}^*, w^*)$ respectively, there exists $\bar{n}_2 > n_1$ such that

$$\forall n \geq \bar{n}_2, \sum_{p=1}^{N_1} |\langle g_{n,p}, b_{n_1,p}\rangle| \leq \delta_2.$$

By $(**)$, there exists $n_2 \geq \bar{n}_2$ such that

$$\sum_{p>N_1} |\langle g_{n_2,p}, b_{n_2,p}\rangle| \geq \epsilon.$$

Now let $N_2 > N_1$ such that $\sum_{p>N_2} \| b_{n_2,p} \| \leq \delta_2$.
Since $\{g_{n,1}\}_{n\in\mathbf{N}},\ldots,\{g_{n,N_2}\}_{n\in\mathbf{N}}$ converge to zero in $(X_1^*, w^*), \ldots, (X_{N_2}^*, w^*)$ respectively, there exists $\bar{n}_3 > n_2$ such that

$$\forall n \geq \bar{n}_3, \sum_{p=1}^{N_2} |\langle g_{n,p}, b_{n_2,p}\rangle| \leq \delta_3.$$

Let $n_3 \geq \bar{n}_3$ and $N_3 > N_2$ be such that, $\sum_{p=1}^{N_2} |\langle g_{n,p}, b_{n_2,p}\rangle| \leq \delta_3$ and $\sum_{p>N_3} \| b_{n_3,p} \| \leq \delta_3$ and so on...

Now let us choose a sequence $\{\gamma_p\}$ such that

for $N_0 + 1 \leq p \leq N_1, \gamma_p \langle g_{n_1,p}, b_{n_1,p} \rangle = |\langle g_{n_1,p}, b_{n_1,p} \rangle|$
for $N_1 + 1 \leq p \leq N_2, \gamma_p \langle g_{n_2,p}, b_{n_2,p} \rangle = |\langle g_{n_2,p}, b_{n_2,p} \rangle|$
for $N_{k-1} + 1 \leq p \leq N_k, \gamma_p \langle g_{n_k,p}, b_{n_k,p} \rangle = |\langle g_{n_k,p}, b_{n_k,p} \rangle|$

We shall define an element h in X^* as

$$\begin{aligned} h &= (\gamma_1 g_{n_1,1}, \gamma_2 g_{n_1,2}, \ldots, \gamma_{N_1} g_{n_1,N_1}, \gamma_{N_1+1} g_{n_2,N_1+1}, \gamma_{N_1+2} g_{n_2,N_1+2}, \ldots \\ &\quad \gamma_{N_2} g_{n_2,N_2}, \gamma_{N_2+1} g_{n_3,N_2+1}, \gamma_{N_2+2} g_{n_3,N_2+2}, \ldots) \end{aligned}$$

If we denote h as $h = (h_p)_{p \geq 1}$, then we have $\| h \| = \sup_{p \geq 1} \| h_p \| \leq 1$ and

$$\langle h, a_{n_k} \rangle = \sum_{p=1}^{\infty} \langle h_p, b_{n_k p} \rangle =$$

$$\sum_{j=1}^{k-1} \sum_{p=N_{j-1}+1}^{N_j} \gamma_p \langle g_{n_j,p} b_{n_j}, p \rangle + \sum_{p=N_{k-1}+1}^{N_k} |\langle g_{n_k,p}, b_{n_k,p} \rangle| + \sum_{p=N_k}^{\infty} \gamma_p \langle g_{n_k,p}, b_{n_k,p} \rangle.$$

Using the inequalities 1), 2), 3), and $\| g_{n_k,p} \| \leq 1$ we get:

$$|\langle h, a_{n_k} \rangle| \geq -\sum_{j=1}^{k-1} \delta_j + \sum_{p=N_{k-1}+1}^{N_k} |\langle g_{n_k,p}, b_{n_k,p} \rangle| - \delta_k$$

$$\begin{aligned} &\geq -\sum_{j=1}^{k-1} \delta_j + \sum_{p > N_{k-1}} |\langle g_{n_k,p}, b_{n_k,p} \rangle| - \sum_{p \leq N_k+1} |\langle g_{n_k,p}, b_{n_k,p} \rangle| - \delta_k \\ &\geq -\sum_{j=1}^{k-1} \delta_j + \sum_{p > N_{k-1}} |\langle g_{n_k,p}, b_{n_k,p} \rangle| - 2\delta_k \\ &\geq \epsilon - \sum_{j=1}^{k-1} \delta_j - 2\delta_k \\ &\geq \epsilon - 2\sum_{j=1}^{\infty} \delta_j > 2\frac{\epsilon}{2}. \end{aligned}$$

This contradicts the fact that $\{a_n\}$ converges weakly to zero. Consequently, $(*)$ holds, that is

$$\lim_{K\to\infty}\sup_{n\in\mathbf{N}}\sum_{k=1}^{K} |\langle g_{n,k}, b_{n,k}\rangle| = 0.$$

It follows that

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} |\langle g_{n,k}, b_{n,k}\rangle| = \sum_{k=1}^{\infty}\lim_{n\to\infty} |\langle g_{n,k}, b_{n,k}\rangle|.$$

As $\| b_{n,k} \|\to 0$, $|\langle g_{n_k}, b_{n_k}\rangle| \to 0$, so that,

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} |\langle g_{n,k}, b_{n,k}\rangle| = 0.$$

As $|\langle f_n, a_n\rangle| \leq \sum_{k=1}^{\infty} |\langle g_{n,k} b_{n,k}\rangle|$, we conclude that $\lim_{n\to\infty}\langle f_n, a_n\rangle = 0$, and so $\| a_n \|$ converges to zero. $\square$

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