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THEOREMS ON THREE-TERM RELATIONS FOR HARDY SUM

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Abstract

Some three-term and mixed three-term relations for Hardy sums were given by Goldberg [7]. His proofs are based on Bernd's transformation formulae for the logarithms of the classical Theat-functions. Pettet and Sitaramachandararo [9] proved elementary proofs for all of Goldberg's results and also proved some three-term relations of Dedekind sums. In this paper, some new theorems on three-term relations for hardy sums were found by applying derivative operator to three-term polynomial relation. Furthermore, proofs of the reciprocity relations for Hardy sums are presented in a more concise way from the original proofs of Berndt [2, 3, 4] and Goldberg [7].

1. Introduction

In the customary notation, we write

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer} \\ 0, & \text{otherwise,} \end{cases}$$

where $[x]$ denotes the largest integer $\leq x$.

If h and k are integers with $k > 0$, the Dedekind sum s(h,k), arising in the theory of the Dedekind Eta function, is defined by

$$s(h,k) = \sum_{r(modk)} ((\frac{r}{k}))((\frac{hr}{k})).$$

The most important property of Dedekind sums is the following reciprocity theorem.

If h and k are coprime, positive integers, then

$$s(h,k)+s(k,h)=-\frac{1}{4}+\frac{1}{12}(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}). \tag{1}$$

For various proofs of (1) which do not depend on the theory of the Dedekind Eta function, we refer to Rademacher and Grosswald [10], and Sitaramachandararo [11].

The first proof of (1) which does not depend on the theory of the Dedekind Eta function is given by Hardy [8]. By using contour integration, Hardy proved two reciprocity theorems in detail and stated, at the end of the paper with indications of proofs, eleven more reciprocity theorems.

In recent years, five of Hardy's reciprocity theorems have been found in an interesting way by Berndt [3] and Goldberg [7]. Berndt and Goldberg [5] deduced these from Bernds transformation formulae [3] for the logarithms of the classical theta function. Goldberg [7], and Pettet and Sitararomachanrarao [9] also discovered three-term and mixed three-term relations.

The main object of this paper is t give elementary proofs of three-term and mixed term relations for hardy sums. Our proofs are based on a tree-term relation for polynomials.

In defining Hardy sums and stating Hardy's reciprocity theorems, we will use the notation of Berndt and Goldberg [5]. If h and k are integer with $k>0$, the Hardy sums are defined by

$$S(h,k)=\sum_{j=1}^{k-1}(-1)^{j+1+[\frac{hj}{k}]}, \tag{2}$$

$$s_1(h,k)=\sum_{j=1}^{k}(-1)^{[\frac{hj}{k}]}((\frac{j}{k})), \tag{3}$$

$$s_2(h,k)=\sum_{j=1}^{k}(-1)^{j}((\frac{hj}{k}))((\frac{j}{k})), \tag{4}$$

$$s_3(h,k)=\sum_{j=1}^{k}(-1)^{j}((\frac{hj}{k})) \tag{5}$$

$$s_4(h,k)=\sum_{j=1}^{k-1}(-1)^{[\frac{hj}{k}]}, \tag{6}$$

$$s_5(h,k)=\sum_{j=1}^{k-1}(-1)^{j+[\frac{hj}{k}]}((\frac{j}{k})). \tag{7}$$

Theorem 1.1. *[9, Theorem 2.1] (Hardy's reciprocity theorems) Let h and k be coprime positive integers. Then*

$$S(h,K)+S(h,k)=1 \ \text{ if } h+k \text{ is odd,} \tag{8}$$

$$s_1(h,k)-2s_2(k,h)=\frac{1}{2}-\frac{1}{2}(\frac{1}{hk}+\frac{k}{h}) \text{ if } h \text{ is even,} \tag{9}$$

$$2s_3(h,k)-s_4(k,h)=1-\frac{h}{k} \text{ if is odd, and} \tag{10}$$

$$s_5(h,k)+s_5(h,k)=\frac{1}{2}-\frac{1}{2hk}h \text{ and } k \text{ are odd.} \tag{11}$$

It may be noted that Sitaramachandrarao [11] expressed the Hardy sums in terms of Dedekind sums using elementary arguments and deduced to Theorem 1.1 from (1).

Theorem 1.2. *(Explicit formulae, cf. [11, theorem 5.1]) Let (h,k) = 1. Then*

$$S(h,k)=8s(h,2k)+8s(2h,k)-20s(h,k) \text{ if } h+k \text{ is odd,} \tag{12}$$

$$s_1(h,K)=2s(h,k),4s(2h,k) \text{ if } h \text{ is even,}$$

$$s_2(h,k)=s(h,k)+2s(2h,k) \text{ if } k \text{ is even,}$$

$$s_3(h,k)=2s(h,k)=4s(2h,k) \text{ if } k \text{ is odd,}$$

$$s_4(h,k)=-4s(h,k),8s(h,2k) \text{ if } h \text{ is odd,}$$

$$s_5(h,k)=-10s(s,k)+4s(2h,k)+4s(h,2k) \text{ if } h+k \text{ is even, and}$$

$$\text{Each of } S(h,k)(h+k \text{ even}), s_1(h,k)(h \text{ odd}), s_2(h,k)(k \text{ odd}),$$

$$s_3(h,k)(k \text{ even}), s_4(h,k)(h \text{ even}), \text{ and } s_5(h,k)(h+k \text{ odd) is zero.}$$

Reciprocity Theorem 1.2 appeared in Hardy's [8] list. Berndt [8] deduced (8), (9) and (10) and Goldberg [7] deduced (11) from Berndt's transformation formulae [3]. For other proofs which do not depend on transformation theory, wer refer to Apostol and Vu [1], Berndt and Golberg [5]. In this paper, a different technique is used in [1], [3] and [7] to prove (8) and (10).

Throughout this section, we assume that a, b, and c are pairwise coprime positive integers and a', b' and c' satisfy

$$aa' \equiv 1(\text{mod } b), bb' \equiv 1(\text{mod } c), \text{ and } cc' \equiv 1(\text{mod } a).$$

Corollary 1.3. *[9, Corollary 2.1] (Three-term polynomial relation) If a, b, and c are pairwise coprime positive integers, then*

$$(u-1)\sum_{x=1}^{a-1} u^{x-1} v^{[\frac{bx}{a}]} w^{[\frac{cx}{a}]} + (v-1)\sum_{y=1}^{b-1} v^{y-1} w^{[\frac{cy}{b}]} u^{[\frac{ay}{b}]} \tag{13}$$

$$+(w-1)\sum_{z=1}^{c-1} w^{z-1} u^{[\frac{az}{c}]} v^{[\frac{bz}{c}]} = u^{a-1}v^{b-1}w^{c-1} - 1,$$

$$(u-1)\sum_{x=1}^{a-1} u^{x-1} v^{[\frac{bx}{a}]} + (v-1)\sum_{y=1}^{b-1} v^{y-1} u^{[\frac{ay}{b}]} = u^{a-1}v^{b-1} - 1. \tag{14}$$

Identity (14) is originally due to Berndt and dieter [4]. The next Corollary, which is equivalent to (14), was first established by Carlitz [6].

Corollary 1.4. *[6] If a and b are coprime positive integers, then*

$$(u-1)\sum_{x=1}^{b-1} u^{b-x-1} v^{[\frac{ax}{b}]} - (v-1)\sum_{y=1}^{a-1} v^{a-y-1} u^{[\frac{by}{a}]} = u^{b-1} - v^{a-1}.$$

We need following relations which were proved by Pettet and Sitaramachandrarao [9].

$$s_1(ca',b)=\sum_{y=1}^{b-1}(-1)^{y+[\frac{cy}{b}]}((\frac{ay}{b})), \tag{15}$$

$$s_4(bc',a)=\sum_{x=1}^{a-1}(-1)^{[\frac{bx}{a}]+[\frac{cx}{a}]}, \tag{16}$$

$$s_3(ab',c)=\sum_{z=1}^{c-1}(-1)^{z+[\frac{bz}{c}]}((\frac{az}{c})), \tag{17}$$

$$s_2(ca',b)=\sum_{y=1}^{b-1}(-1)^{y}((\frac{cy}{b}))((\frac{ay}{b})), \tag{18}$$

$$s_1(bc',a)=\sum_{x=1}^{a-1}(-1)^{[\frac{bx}{a}]}((\frac{cx}{a})), \tag{19}$$

$$s_3(ab',c)=\sum_{z=1}^{c-1}(-1)^{[\frac{bz}{c}]}((\frac{az}{c})), \tag{20}$$

and also, we define,

$$s_5(cb',a)=\sum_{x=1}^{a-1}(-1)^{x+[\frac{cx}{a}]}((\frac{bx}{a})). \tag{21}$$

In the next section, we will give new proofs on the three-term relations for Hardy sums and reciprocity laws by applying derivative operator to Corollary 1.3 and Corollary 1.4

2. Main Theorems

Theorem 2.1. *Let a and c be odd. Then*

$$s_4(ac',b)-2s_5(cb',a)-2s_3(ab',c)=\frac{b-ac}{ac}.$$

Proof. We apply the operator $(v(\frac{\partial}{\partial v}))$ to both sides of (13) and set $u=w=-1$, $v=1$

to obtain

$$-2\sum_{x=1}^{a-1}(-1)^{x-1+[\frac{cx}{a}]}[\frac{bx}{a}]+\sum_{y=1}^{b-1}(-1)^{[\frac{ay}{b}]+[\frac{cy}{b}]}-2\sum_{z=1}^{c-1}(-1)^{z-1+[\frac{az}{c}]}[\frac{bz}{c}]=(b-1)(-1)^{a+c-2}.$$

On replacing $[\frac{bx}{a}]$ and $[\frac{bz}{c}]$ respectively with $\frac{bx}{a}-((\frac{bx}{a}))-\frac{1}{2}$ and $\frac{bz}{c}-((\frac{bz}{c}))-\frac{1}{2}$ and note that $b+c$ is even, this reduces to

$$2b\sum_{x=1}^{a-1}(-1)^{x+[\frac{cx}{a}]}(\frac{x}{a})+s_4(ac',b)+2b\sum_{x=1}^{c-1}(-1)^{z+[\frac{az}{c}]}(\frac{z}{c})-2s_3(ab',c)-2s_5(cb',a)$$
$$+S(a,c)+S(c,a)=b-1.$$

In the above, we used (16), (17) and (21); this, in return reduces to

$$2b(s_5(a,c)+s_5(c,a))+(s_4(ac',b)-2s_3(ab',c)-2s_5(cb',a))$$
$$-b(S(a,c)+S(c,a))+(S(a,c)+S(c,a))=b-1,$$

in view of $\sum_{x=1}^{a-1}(-1)^{x+[\frac{cx}{a}]}(\frac{x}{a})=S_5(c,a)-\frac{1}{2}(c,a)$ and $\sum_{z=1}^{c-1}(-1)^{z+[\frac{az}{c}]}(\frac{z}{c})=S_5(a,c)-\frac{1}{2}S(a,c)$. Now Theorem 2.1 follows from (12), (11), and (8) □

Theorem 2.2. *Let a be even. Then*

$$2s_2(ab',c)-s_1(cb',a)-s_3(ca',b)=-\frac{1}{2}+\frac{1}{2c}(\frac{a}{c}+\frac{b}{c}).$$

Proof. The proof is similar to the proof of Theorem 2.1. Now we apply the operator $(u(\frac{\partial}{\partial u}))(u(\frac{\partial}{\partial u}))$ to both sides of the identity in (13) and set $u=v=1$, $w=-1$ to obtain

$$\sum_{x=1}^{a-1}(-1)^{[\frac{cx}{a}]}[\frac{bx}{a}]+\sum_{y=1}^{b-1}(-1)^{[\frac{cy}{b}]}[\frac{ay}{b}]-2\sum_{z=1}^{c-1}(-1)^{z-1}[\frac{az}{c}][\frac{bz}{c}]$$
$$=(a-1)(b-1)(-1)^{c-1}.$$

On replacing $[\frac{bx}{a}]$, $[\frac{ay}{b}]$, $[\frac{az}{c}]$ and $[\frac{bz}{c}]$ respectively with $\frac{bx}{a}-((\frac{bx}{a}))-\frac{1}{2}$, $\frac{ay}{b}-((\frac{ay}{b}))-\frac{1}{2}$, $\frac{az}{c}-((\frac{az}{c}))-\frac{1}{2}$ and $\frac{bz}{c}-((\frac{bz}{c}))-\frac{1}{2}$, and by using (9), (12), (17), (18) and (19) we can obtain proof of Theorem 2.2. □

Theorem 2.3. *Let a be odd. Then*

$$2s_3(b,a)-s_4(a,b)=1-\frac{b}{a}.$$

Proof. Various proof of this theorem were given by Apostol and Vu [1] and Berndt [3] [4]. We apply the operator $(v(\frac{\partial}{\partial v}))$ to both sides of (13) and set $u=w=-1, v=1$ to obtain

$$-2\sum_{x=1}^{a-1}(-1)^{x-1}[\frac{bx}{a}]+\sum_{y=1}^{b-1}(-1)^{[\frac{ay}{b}]}=(b-1)(-1)^{a-1}.$$

On replacing $[\frac{bx}{a}]$ with $\frac{bx}{a}-((\frac{bx}{a}))-\frac{1}{2}$ and note that a is odd, and by using (5) and (6) we can obtain proof of Theorem 2.3 □

Theorem 2.4. *Let a be odd. Then*

$$s_4(ab',c)+2s_5(bc',a)-s_1(ac',b)=-1+\frac{c}{ab}.$$

Proof. We apply the operator $(w(\frac{\partial}{\partial w}))$ to both sides on (13) and set $u=v=-1$, $w=1$ to obtain

$$-2\sum_{x=1}^{a-1}(-1)^{x-1+[\frac{bx}{a}]}[\frac{cx}{a}]-2\sum_{y=1}^{b-1}(-1)^{y-1+[\frac{ay}{b}]}[\frac{cy}{b}]$$
$$+\sum_{z=1}^{c-1}(-1)^{[\frac{az}{c}]+[\frac{bz}{c}]}=(c-1)(-1)^{a+b-2}.$$

On replacing $[\frac{cx}{a}]$ and $[\frac{cy}{b}]$ respectively with $\frac{cx}{a}-((\frac{cx}{a}))-\frac{1}{2}$ and $\frac{cy}{b}-((\frac{cy}{b}))-\frac{1}{2}$ and nothing that $a+b$ is even, where in the above, we also used (2), (7), (15), (16) and (21), this reduces to

$$\begin{gathered}(S(a,c)+S(c,a))+2c(s_5(b,a)+s_5(a,b))-c(S(b,a)+S(a,b)) \\ +s_4(ab',c)+2s_5(bc',a)-s_1(ac',b)=c-1.\end{gathered} \tag{22}$$

In view of $\sum_{x=1}^{a-1}(-1)^{x+[\frac{bx}{a}]}(\frac{x}{a})=s_5(b,a)-\frac{1}{2}S(b,a)$, and $\sum_{y=1}^{b-1}(-1)^{y+[\frac{ay}{b}]}(\frac{y}{b})=s_5(a,b)-\frac{1}{2}S(a,b)$ And, by using (8), (11), and (12) in (22) Theorem 2.4 follows. □

Theorem 2.5. *Let a and c be even and b be odd. Then*

$$2s_2(cb',a)-s_1(ca',b)-s_3(ca',b)=-\frac{1}{2}+\frac{1}{2a}(\frac{c}{b}+\frac{b}{c}).$$

Proof. The proof is similar to the proof of Theorem 2.2. Now we apply the operator $(w(\frac{\partial}{\partial w}))(v(\frac{\partial}{\partial v}))$ to both sides of (13) and set $w=v=1$ and $u=-1$ to obtain

$$\begin{gathered}-2\sum_{x=1}^{a-1}(-1)^{x-1}[\frac{bx}{a}][\frac{cx}{a}]+\sum_{y=1}^{b-1}(-1)^{[\frac{ay}{b}]}[\frac{cy}{b}]+\sum_{z=1}^{c-1}(-1)^{[\frac{az}{c}]}[\frac{bz}{c}] \\ =(b-1)(c-1)(-1)^{a-1}.\end{gathered} \tag{23}$$

On replacing $[\frac{bx}{a}],[\frac{cx}{a}],[\frac{cy}{b}]$ and $[\frac{bz}{c}]$ respectively with $\frac{bx}{a}-((\frac{bx}{a}))-\frac{1}{2}$, $\frac{cx}{a}-((\frac{cx}{a}))-\frac{1}{2}$, $\frac{cy}{b}-((\frac{cy}{b}))-\frac{1}{2}$ and $\frac{bz}{c}-((\frac{bz}{c}))-\frac{1}{2}$, and by using (3), (4), (5), (6), (9), (10), (15), (18) and (20) in (23) (note that a and c are odd and b is given) by (12), Theorem 2.5 follows. □

Corollary 2.6. *Let a and b be coprime positive integres. If $a+b$ is odd, then*

$$S(a,b)+S(b,a)=1.$$

Proof. The proof of this Corollary is presented in a more concise was from the proof of Berndt [3,5]. Let u and v be equal to 1 in (14). Using the fact that $a+b$ is odd, we find that

$$2\sum_{x=1}^{a-1}(-1)^{x+1+[\frac{bx}{a}]}-2\sum_{y=1}^{b-1}(-1)^{y+1+[\frac{ay}{b}]}=-2 \tag{24}$$

and using (2) in (24), we complete the proof. □

Remark 2.1. *We note that the method used to prove Theorem 2.3 is different than other proofs. On differentiating both sides of (14) with respect to v, multiplying by v and setting $u=-1, v=1$, we obtain (10) by a straightforward calculation (5), (6) and (12).*

Remark 2.2. *During the mineteenth century, the sum involving $[x]$ played a prominent part in number theory. The most well-known proof of Gauss's law of quadradratic reciprocity depends upon the relation*

$$\sum_{x=1}^{\frac{1}{2}(b-1)}[\frac{ax}{b}]+\sum_{x=1}^{\frac{1}{2}(a-1)}[\frac{bx}{a}]=(a-1)(b-1). \tag{25}$$

where a and b are odd, distinct primes, as shown by Berndt and diter [4. Eq. (1.1)]. We note that the proof of (25) can be given in a different way from Berndt and Diter [4, Eq. (1.1)]. In fact, we apply the operator $(v(\frac{\partial}{\partial v}))(u(\frac{\partial}{\partial u}))$ to both sides of (14) and set $u=v=1$, to obtain (25) by a straightforward calculation.

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