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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

The object of the present paper is to derive several interesting properties of the class $T_n(\lambda, \alpha)$ consisting of analytic and univalent functions with negative coefficients. Coefficient inequalities, distortion theorems and closure theorems of functions in the class $T_n(\lambda, \alpha)$ are determined. Also radii of close-to-convexity, starlikeness and convexity are determined. Furthermore, integral operators and modified Hadamard products of several functions belonging to the class $Tn(\lambda, \alpha)$ are studied here.

Key words and phrases. Analytic, univalent, modified Hadamard product.

1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. For a function $f(z)$ in S, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N = \{1, 2, ...\}). \tag{1.4}$$

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The differential operator D^n was introduced by Salagean [3]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to S is in the class $S_n(\lambda, \alpha)$ if and only if

$$Re\left\{\frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda\frac{D^{n+1}f(z)}{D^n f(z)}+(1-\lambda)}\right\} > \alpha \quad (n \in N_0 = N \cup \{0\}) \tag{1.5}$$

for some $\alpha(0 \leq \alpha < 1), \lambda(0 \leq \lambda < 1)$ and for all $z \in U$.

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{1.6}$$

Further, we define the class $T_n(\lambda, \alpha)$ by

$$T_n(\lambda, \alpha) = S_n(\lambda, \alpha) \cap T. \tag{1.7}$$

We note that by specializing the parameters n, λ, and α, we obtain the following subclasses studied by various authors:

(i) $T_0(\lambda, \alpha) = T(\lambda, \alpha)$ and $T_1(\lambda, \alpha) = C(\lambda, \alpha)$ (Altintas and Owa [1]);

(ii) $T_0(0, \alpha) = T^*(\alpha)$ and $T_1(0, \alpha) = C(\alpha)$ (Silverman [5]);

(iii) $T_n(0, \alpha) = T(n, \alpha)$ (Hur and Oh [2]).

2. Coefficient Estimates

Theorem 1. *Let the function $f(z)$ be defined by (1.6). Then $f(z) \in T_n(\lambda, \alpha)$ if and only if*

$$\sum_{k=2}^{\infty} k^n \{k - \alpha[1\lambda(k-1)]\} a_k \leq 1 - \alpha. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z| = 1$. Then we have

$$\left|\frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda\frac{D^{n+1}f(z)}{D^n f(z)}+(1-\lambda)} - 1\right| = \left|\frac{\sum_{k=2}^{\infty} 51-\lambda)k^n(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty} k^n[1+\lambda(k-1)]a_k z^{k-1}}\right| \tag{2.2}$$

$$\leq \frac{\sum_{k=2}^{\infty}(1-\lambda)k^n(k-1)k_a}{1-\sum_{k=2}^{\infty} k^n[1+\lambda(k-1)]k_k} \leq 1-\alpha$$

This shows that the values of $\frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda\frac{D^{n+1}f(z)}{D^n f(z)}+(1-\lambda)}$ lies in a circle centered at $w = 1$ whose radius is $1-\alpha$. Hence $f(z)$ satisfies the condition (1.5).

Conversely, assume that the function $f(z)$ defined by (1.6) is in the calss $T_n(\lambda, \alpha)$. Then

$$Re\left\{\frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda\frac{D^{n+1}f(z)}{D^n f(z)}+(1-\lambda)}\right\} = Re\left\{\frac{1-\sum_{k=2}^{\infty}k^{n+1}a_k z^{k-1}}{1-\sum_{k=2}^{\infty}k^n[1+\lambda(k-1)]a_k z^{k-1}}\right\} > \alpha \tag{2.3}$$

for $z \in U$. Choose values of z on the real axis so that $\frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda\frac{D^{n+1}f(z)}{D^n f(z)}+(1-\lambda)}$ is real. Upon clearing the denominator in (2.3) and letting $z+1^-$ through real values, we obtain

$$1-\sum_{k=2}^{\infty}k^{n+1}a_k \geq \alpha\left\{1-\sum_{k=2}^{\infty}k^n[1+\lambda(k-1)]a_k\right\} \tag{2.4}$$

which gives (2.1). Finally the result is sharp with the extremal function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{k^n\{k^n-\alpha[1+\lambda(k-1)]\}}z^k \quad (k \geq 2). \tag{2.5}$$

□

Corollary 1. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$. Then we have*

$$a_k \leq \frac{1-\alpha}{k^n\{k^n-\alpha[1+\lambda(k-1)]\}} \quad (k \geq 2). \tag{2.6}$$

The equality in (2.6) is attained for the function $f(z)$ given by (2.5).

3. Some Properties of the Class $T_n(\lambda,\alpha)$

Theorem 2. *Let* $0 \le \alpha < 1, 0 \le \lambda_1 \le \lambda_2$ *and* $n \in N_0$. *Then*

$$T_n(\lambda_1,\alpha) \subseteq T_n(\lambda_2,\alpha)$$

Proof. If follows from Theorem 1 that

$$\sum_{k=2}^{\infty} k^n\{k-\alpha[1+\lambda_2(k-1)]\}a_k \le \sum_{k=2}^{\infty} k^n\{k-\alpha[1+\lambda_1(k-1)]\}a_k \le 1-\alpha$$

for $f(z) \in T_n(\lambda_1,\alpha)$. Hence $f(z)|inT_n(\lambda_2,\alpha)$. □

Theorem 3. *Let* $0 \le \alpha < 1, 0 \le \lambda < 1$ *and* $n \in N_0$. *Then*

$$T_{n+1}(\lambda,\alpha) \subset T_n(\lambda,\alpha).$$

The proof follows immediately from Theorem 1.

4. Distortion Theorems

Theorem 4. *Let the function* $f(z)$ *defined by (1.6) be in the class* $T_n(\lambda,\alpha)$. *Then we have*

$$|D^i f(z)| \ge |z| - \frac{1-\alpha}{2^{n-1}[2-\alpha(1+\lambda)]}|z|^2 \tag{4.1}$$

and

$$|D^i f(z)| \le |z| + \frac{1-\alpha}{2^{n-1}[2-\alpha(1+\lambda)]}|z|^2 \tag{4.2}$$

for $z \in U$, *where* $0 \le i \le n$. *Then equalities in (4.1) and (4.2) are attained for the function* $f(z)$ *given by*

$$D^i f(z) = z - \frac{1-\alpha}{2^{n-1}[2-\alpha(1+\lambda)]}z^2. \tag{4.3}$$

Proof. Note that $f(z) \in T_n(\lambda,\alpha)$ if and only if $D^i f(z) \in T_{n-i}(\lambda,\alpha)$, where

$$D^i f(z) = z - \sum_{k=2}^{\infty} k^i a_k z^k. \tag{4.4}$$

Using Theorem 1, we know that

$$z^{n-1}[2-\alpha(1+\lambda)] \sum_{k=2}^{\infty} k a_k \leq \sum_{k=2}^{\infty} k^n [k - \alpha(1+\lambda(k-1))] a_k \leq 1-\alpha, \tag{4.5}$$

that is, that

$$\sum_{k=2}^{\infty} k^i a_k \leq \frac{1-\alpha}{2^{n-1}[2-\alpha(1+\lambda)]}. \tag{4.6}$$

It follows from (4.4) and (4.6) that

$$|D^i f(z)| \geq |z| - |z|^2 \frac{1-\alpha}{2^{n-1}[2-\alpha(1+\lambda)]} k^i a_k \geq |z| - \frac{1-\alpha}{2^{n-1}[2-\alpha(1+\lambda)]} |z|^2 \tag{4.7}$$

ant

$$|D^i f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} k^i a_k \frac{1-\alpha}{2^{n-1}[2-\alpha(1+\lambda)]} |z|^2. \tag{4.8}$$

This completes the proof of Theorem 4. □

Corollary 2. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$. Then we have*

$$|f(z)| \geq |z| - \frac{1-\alpha}{2^n[2-\alpha(1+\lambda)]} |z|^2 \tag{4.9}$$

and

$$|f(z)| \leq |z| + \frac{1-\alpha}{2^n[2-\alpha(1+\lambda)]} |z|^2 \tag{4.10}$$

for $z \in U$. Then equalities in (4.9) and (4.10) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{2^n[2-\alpha(1+\lambda)]} z^2. \tag{4.11}$$

Proof. Taking $i = 0$ in Theorem 4, we can easily show (4.9) and (4.10). □

Corollary 3. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda,\alpha)$. Then we have*

$$|f'(z)| \geq 1 - \frac{1-\alpha}{2^{n-1}[2-\alpha(1+\lambda)]}|z| \tag{4.12}$$

and

$$|f'(z)| \leq 1 + \frac{1-\alpha}{2^{n-1}[2-\alpha(1+\lambda)]}|z| \tag{4.13}$$

for $z \in U$. The equalities in (4.12) and (4.13) are attained for the function $f(z)$ given by (4.11).

Proof. Note that $D(f(z) = zf'(z))$. Hence taking $i = 1$ in Theorem 4, we have the corollary. □

Corollary 4. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda,\alpha)$. Then the unit disc U is mapped onto a domain that contains the disc*

$$|w| < \frac{2^n[2-\alpha(1+\lambda)]-(1-\alpha)}{2^n[2-\alpha(1+\lambda)]}. \tag{4.14}$$

The result is sharp with the extremal function $f(z)$ given by (4.11).

5. Closure Theorems

Let the functions $f_i(z)$ be defined, for $j = 1, 2, ..., m$, by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \ \ (a_{k,j} \geq 0) \tag{5.1}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the class $T_n(\lambda,\alpha)$.

Theorem 5. *Let the functions $f_j(z)$ defined by (5.1) be in the class $T_n(\lambda,\alpha)$ for every $j = 1, 2, ..., m$. Then the functions $h(z)$ defined by*

$$h(z) = \sum_{j=1}^{m} c_j f_j(z) \ \ (c_j \geq 0) \tag{5.2}$$

is also in the same class $T_n(\lambda,\alpha)$ where

$$\sum_{j=1}^{m} c_j = 1. \tag{5.3}$$

Proof. According to the definition of $h(z)$, we can write

$$h(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{m} c_j a_{k,j} \right) z^k. \tag{5.4}$$

Further, since $f_j(z)$ are in $T_n(\lambda, \alpha)$ for every $j = 1, 2, ..., m$ we get

$$\sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k-1)]\} a_{k,j} \leq 1 - \alpha \tag{5.5}$$

for every $j = 1, 2, ..., m$. Hence we can see that

$$\sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k-1)]\} \left(\sum_{j=1}^{m} c_j a_{k,j} \right) \tag{5.6}$$

$$\sum_{j=1}^{m} c_j \left(\sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k-1)]\} a_{k,j} \right)$$

$$\leq \left(\sum_{j=1}^{m} c_j \right) (1 - \alpha) = 1 - \alpha,$$

which implies that $h(z) 6 in T_n(\lambda, \alpha)$. Thus we have the theorem. □

Corollary 5. *The class $T_n(\lambda, \alpha)$ is closed under conveq linear combination.*

Proof. Let the function $f_j(z) (j = 1, 2)$ defined by (5.1) be in the class $T_n(\lambda, \alpha)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \mu f_1(z) + (1 - \mu) f_2(z) \quad (0 \leq \mu \leq 1) \tag{5.7}$$

is in the class $T_n(\lambda, \alpha)$. But, taking $m = 2, c_1 = \mu$, and $c_2 = 1 - \mu$ in Theorem 5, we have the corollary.

As a consequence of Corollary 5, there exists the extreme points of the class $T_n(\lambda, \alpha)$. □

Theorem 6. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{1-\alpha}{k^n\{k - \alpha[1+\lambda(k-1)]\}} z^k \quad (k \geq 2) \tag{5.8}$$

for $0 \leq \alpha < 1, 0 \leq \lambda < 1$ *and* $n \in N_0$. *Then* $f(z)$ *is in the class* $T_n(\lambda, \alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=2}^{\infty} \mu_k f_k(z) \tag{5.9}$$

where $\mu_k \geq 0 (k \geq 1)$ *and* $\sum_{k=2}^{\infty} \mu_k = 1$.

Proof. Soppose that

$$f(z) = \sum_{k=2}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{1-\alpha}{k^n\{k - \alpha[1+\lambda(k-1)]\}} \mu_k z^k. \tag{5.10}$$

Then it follows that

$$\sum_{k=2}^{\infty} \frac{k^n\{k - \alpha[1+\lambda(k-1)]\}}{1-\alpha} \frac{(1-\alpha)\mu_k}{k^n\{k - \alpha[1+\lambda(k-1)]\}} \tag{5.11}$$
$$= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1.$$

So by Theorem 1, $f(z) | in T_n(\lambda, \alpha)$.

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $T_n(\lambda, \alpha)$. Then

$$a_k \leq \frac{(1-\alpha)\mu_k}{k^n\{k - \alpha[1+\lambda(k-1)]\}} \quad (k \geq 2). \tag{5.12}$$

Setting

$$\mu_k = \frac{k^n\{k - \alpha[1+\lambda(k-1)]\}}{1-\alpha} a_k \quad (k \geq 2), \tag{5.13}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \tag{5.14}$$

we can see that $f(z)$ can be expressed in the form (5.9). This completes the proof of Theorem 6. □

Corollary 6. *The extreme points of the class* $T_n(\lambda, \alpha)$ *are the functions* $f_k(z)(k \geq 1)$ *given by Theorem 6.*

6. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 7. *Let the function* $f(z)$ *defined by (1.6) be in the class* $T_n(\lambda, \alpha)$*. Then* $f(z)$ *is close-to-convex of order* $\delta(0 \leq \delta < 1)$ *in* $|z| < r_1(n, \lambda, \alpha, \delta)$*, where*

$$r_1(n, \lambda, \alpha, \delta) = \inf_k \left[\frac{(1-\delta)k^{n-1}\{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{6.1}$$

The result is sharp with the extremal function $f(z)$ *given by (2.5)*

Proof. It is sufficient to show that $f'(z) - 1| \leq 1 - \delta(0 \leq \delta < 1)|z| < r_1(n, \lambda, \alpha, \delta)$. We have

$$|f'(z) - 1| - \left|\sum_{k=2}^{\infty} k a_k z^{k-1}\right| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \leq 1 - \delta$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\delta}\right) a_k |z|^{k-1} \leq 1. \tag{6.2}$$

But Theorem 1 confirms that

$$\sum_{k=2}^{\infty} \frac{k^n\{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha} a_k \leq 1. \tag{6.3}$$

Hence (6.2) will be true if

$$\frac{k|z|^{k-1}}{(1-\delta)} \leq \frac{k^n\{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha}$$

or if

$$|z| \leq \left[\frac{(1-\delta)k^{n-1}\{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{6.4}$$

The theorem follows easily from (6.4). □

Theorem 8. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda,\alpha)$. Then $f(z)$ is starlike of order $\delta(0 \leq \delta < 1)$ in $|z| < r_2(n,\lambda,\alpha,\delta)$, where*

$$r_2(n,\lambda,\alpha,\delta) = \inf_k \left[\frac{(1-\delta)k^n\{k-\alpha[1+\lambda(k-1)]\}}{(k-\delta)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{6.5}$$

The result is sharp with the extremal function $f(z)$ given by (2.5).

Proof. We must show that $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\delta(0 \leq \delta < 1)$ for $|z| < r_n(n\lambda,\alpha,\delta)$. We have
Thus $|\frac{zf'(z)}{f(z)} - 1| \leq 1-\delta$ if

$$\sum_{k=2}^{\infty} \frac{(k-\delta)a_k|z|^{k-1}}{1-\delta} \leq 1. \tag{6.6}$$

Hence, by using (6.3), (6.6) will be true if

$$\frac{(k-\delta)|z|^{k-1}}{1-\delta} \leq \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}$$

or if

$$|z| \leq \left[\frac{(1-\delta)k^n\{k-\alpha[1+\lambda(k-1)]\}}{(k-\delta)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{6.7}$$

□

Corollary 7. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda,\alpha)$. Then $f(z)$ is convex of order $\delta(0 \leq \delta 1)$ in $|z| < r_3(n,\lambda,\alpha,\delta)$, where*

$$r_3(n,\lambda,\alpha,\delta) = \inf_k \left[\frac{(1-\delta)k^n\{k-\alpha[1+\lambda(k-1)]\}}{(k-\delta)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{6.8}$$

The result is sharp with the extremel function $f(z)$ given by (2.5).

7. Integral Operators

Theorem 9. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda,\alpha)$ and let c be a real number such that $c>-1$. Then the function $F(z)$ defined by*

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{7.1}$$

also belongs to the class $T_n(\lambda,\alpha)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z)=z-\sum_{k=2}^{\infty}b_kz^k, \tag{7.2}$$

where

$$b_k=\left(\frac{c+1}{c+k}\right)a_k. \tag{7.3}$$

therefore,

$$\sum_{k=2}^{\infty}k^n\{k-\alpha[1+\lambda(k-1)]\}b_k=\sum_{k=2}^{\infty}k^n\{k-\alpha[1+\lambda(k-1)]\}\left(\frac{c+1}{c+k}\right) \tag{7.4}$$
$$\leq\sum_{k=2}^{\infty}k^n\{k-[1+\lambda(k-1)]\}a_k\leq 1-a,$$

since $f(z)\in T_n(\lambda,\alpha)$. Hence, by Theorem 1, $F(z)\in T_n(\lambda,\alpha)$. □

Theorem 10. *Let c be a real number such that $c>-1$. If $F(z)\in T_n(\lambda,\alpha)$, then the function $f(z)$ defined by (7.1) is univalent in $|z|<R^*$, where*

$$R^2=\inf_k\left[\frac{(c+1)k^{n-1}\{k-\alpha[1+\lambda(k-1)]\}}{(c+k)(1-\alpha)}\right]^{\frac{1}{k-1}}\quad(k\geq 2). \tag{7.5}$$

The result is sharp.

Proof. Let $F(z)=z-\sum_{k=2}^{\infty}a_kz^k(a_k\geq 0)$. If follows from (7.1) that

$$f(z)=\frac{z^{1-c}[z^cF(z)]'}{(c+1)}=z-\sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right)a_kz^k\quad(c>-1). \tag{7.6}$$

In order to obtain the required result, it stuffices to show that $|f'(z)-1|<1$ in $|z|<R^*$. Now

$$|f'(z)-1| \leq \sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}.$$

Thus $|f'(z)-1|<1$ if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1} < 1. \tag{7.7}$$

Hence, by using (6.3), (7.7) will be satisfied if

$$\frac{k(c+k)|z|^{k-1}}{c+1} \leq \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha} \quad (k \geq 2)$$

or if

$$|z| \leq \left[\frac{(c+1)k^{n-1}\{k-\alpha[1+\lambda(k-1)]\}}{(c+k)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.8}$$

Therefore $f(z)$ is univalent in $|z|<R^*$. Sharpness follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{k^n\{k-\alpha[1+\lambda(k-1)]\}(c+1)} z^k \quad (k \geq 2). \tag{7.9}$$

□

8. Modified Hadamard Products

Let the functions $f_j(z)(j=1,2)$ be defined by (5.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined here by

$$f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k. \tag{8.1}$$

Theorem 11. *Let the functions $f_j(z)(j=1,2)$ defined by (5.1) be in the class $T_n(\lambda,\alpha)$. Then $f_1 * f_2(z)$ belongs to theclass $T_n(\lambda,\beta(n,\lambda,\alpha))$ where*

$$\beta(n,\lambda,\alpha) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^n\{2-\alpha(1+\lambda)\}^2-(1+\lambda)(1-\alpha)^2}. \tag{8.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [4], we need to find the largest $\beta = \beta(n, \lambda, \alpha)$ such that

$$\sum_{k=2}^{\infty} \frac{k^n\{k - \beta[1 + \lambda(k-1)]\}}{1-\beta} a_{k,1}a_{k,2} \leq 1 \tag{8.3}$$

Since

$$\sum_{k=2}^{\infty} \frac{k^n\{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha} a_{k,1} \leq 1 \tag{8.4}$$

and

$$\sum_{k=2}^{\infty} \frac{k^n\{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha} a_{k,2} \leq 1, \tag{8.5}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=2}^{\infty} \frac{k^n\{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha} a\sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{8.6}$$

Thus it is sufficient to show that

$$\begin{aligned} &\frac{k^n\{k - \beta[1 + \lambda(k-1)]\}}{1-\beta} a_{k,1}a_{k,2} \\ &\leq \frac{k^n\{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha} \sqrt{a_{k,1}a_{k,2}} \quad (k \geq 2), \end{aligned} \tag{8.7}$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(1-\beta)\{k - \alpha[1 + \lambda(k-1)]\}}{(1-\alpha)\{k - \beta[1 + \lambda(k-1)]\}}. \tag{8.8}$$

Not that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{1-\alpha}{k^n\{k - \alpha[1 + \lambda(k-1)]\}} \quad (k \geq 2). \tag{8.9}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{k^n\{k - \alpha[1 + \lambda(k-1)]\}} \leq \frac{(1-\beta)\{k - \alpha[1 + \lambda(k-1)]\}}{(1-\alpha)\{k - \beta[1 + \lambda(k-1)]\}} (k \geq 2), \tag{8.10}$$

or, equivalently, that

$$\beta \leq -1\frac{(k-1)(1-\lambda)(1-\alpha)^2}{k^n\{k-\alpha[1+\lambda(k-1)]\}^2-[1+\lambda(k-1)](1-\alpha)^2}(k \geq 2). \tag{8.11}$$

Since

$$A(k) = 1-\frac{(k-1)(1-\lambda)(1-\alpha)^2}{k^n\{k-\alpha[1+\lambda(k-1)]\}^2-[1+\lambda(k-1)](1-\alpha)^2} \tag{8.12}$$

is an increasing function of $k(k \geq 2)$, letting $k = 2$ in (8.12), we obtain

$$\beta \leq A(2) = 1-\frac{(1-\lambda)(1-\alpha)^2}{2^n\{2-\alpha(1+\lambda)\}^2-(1+\lambda)(1-\alpha)^2}, \tag{8.13}$$

which completes the proofof Theorem 11.

Finally, by taking the functions $f_j(z)$ given by

$$f_j(z) = z-\frac{1-\alpha}{2^n[2-\alpha(1+\lambda)]}z^2 \quad (j = 1, 2), \tag{8.14}$$

we can see that the result is sharp. □

Corollary 8. *For $f_1(z)$ and $f_2(z)$ as in Theorem 11, the function*

$$h(z) = z-\sum_{k=2}^{\infty}\sqrt{a_{k,1}a_{k,2}}z^k \tag{8.15}$$

belongs to the class $T_n(\lambda, \alpha)$.

This result follows from the Cauchy-Schwarz inequality (8.6). It is sharp for the same functions as in Theorem 11.

Theorem 12. *Let the function $f_1(z)$ defined by (5.1) be in the class $T_n(\lambda, \alpha)$ and the function $f_2(z)$ defined by (5.1) be in the class $T_n(\lambda, \alpha)$. Then $f_1 * f_2(z)$ belongs to the class $T_n\eta(n, \lambda, \alpha, \gamma))$, where*

$$\eta(n, \lambda, \alpha, \gamma) = 1-\frac{(1-\lambda)(1-\alpha)(1-\gamma)}{2^n\{2-\alpha(1+\lambda)\}\{2-\gamma(1+\lambda)\}-(1+\lambda)(1-\alpha)(1-\gamma)}. \tag{8.16}$$

The result is best possible for the functions

$$f_1(z) = z-\frac{1-\alpha}{2^n[2-\alpha(1+\lambda)]}z^2 \tag{8.17}$$

and

$$f_g(z) = z - \frac{1-\gamma}{2^n[2-\gamma(1+\lambda)]}z^2. \tag{8.18}$$

Proof. Proceeding as in the proof of Theorem 11, we get

$$\eta \leq B(k) = 1 - \tag{8.19}$$
$$\frac{(k-1)(1-\lambda)(1-\alpha)(1-\gamma)}{k^n\{k-\alpha[1+\lambda(k-1)]\}\{k-\gamma[1+\lambda(k-1)]\} - [1+\lambda(k-1)](1-\alpha)(1-\gamma)}(k \geq 2)$$

Since the function $B(k)$ is an increasing function of $k(k \geq 2)$, setting $k = 2$ in (8.19), we get

$$\eta \geq B(2) = 1 - \frac{(1-\lambda)(1-\alpha)(1-\gamma)}{2^n\{2-\alpha(1+\lambda)\}\{2-\gamma(1+\lambda)\} - (1+\lambda)(1-\alpha)(1-\gamma)}. \tag{8.20}$$

This completes the proof of Theorem 12. □

Corollary 9. *Let the functions $f_j(z)(j = 1,2,3)$ defined by (5.1) be in the class $T_n(\lambda,\alpha)$. Then $f_1 * f_2 * f_3(z)$ belongs to the class $T_n(\lambda, \zeta(n,\lambda,\alpha))$, where*

$$\zeta(n,\lambda,\alpha) = 1 - \frac{(l-\lambda)(1-\alpha)^3}{4^n\{2-\alpha(1+\lambda)\}^3 - (1+\lambda)(1-\alpha)^3}. \tag{8.21}$$

The result is best possible for the functions

$$f_j(z) = z - \frac{1-\alpha}{2^n[2-\alpha(1+\lambda)]}z^2 \quad (j = 1,2,3). \tag{8.22}$$

Proof. From Theorem 11, we have $f_1 * f82(z) \in T_n(\lambda, \beta(n,\lambda,\alpha))$, where β is given by (8.2). By using Theorem 12, we get $f_1 * f_2 * f_3 * (z) \in T_n(\lambda, \zeta(n,\lambda,\alpha))$, where

$$\begin{aligned}\zeta(n,\lambda,\alpha) &= 1 - \frac{(1-\lambda)(1-\alpha)(1-\beta)}{2^n\{2-\alpha(1+\lambda)\}\{2-\beta(1+\lambda)\} - (1+\lambda)(1-\alpha)(1-\beta)} \\ &= \frac{(1-\lambda)(1-\alpha)^3}{4^n\{2-\alpha(1+\lambda)\}^3 - (1+\lambda)(1-\alpha)^3}\end{aligned}$$

This completes the proof of Corollary 9. □

Theorem 13. *Let the functions $f_j(z)(j=1,2)$ defined by (5.1) be in the class $T_n(\lambda,\alpha)$. Then the function*

$$h(z) = z - \sum_{k=2}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k \tag{8.23}$$

belong to the class $T_n(\phi(n,\lambda,\alpha))$ where

$$\phi(n,\lambda,\alpha) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^{n-1}\{2-\alpha(1+\lambda)\}^2 - (1+\lambda)(1-\alpha)^2}. \tag{8.24}$$

The result is sharp for the functions $f_j(z)(j=1,2)$ defined by (8.14).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=2}^{\infty}\left[\frac{k^n\{k^n\{k-\alpha[1+\lambda(k-1)]\}\}1-\alpha}{}\right]^2 a_{k,1}^2 \tag{8.25}$$
$$\leq \left[\sum_{k=2}^{\infty}\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}a_{k,1}^2\right] \leq 1$$

and

$$\sum_{k=2}^{\infty}\left[\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right]^2 a_{k,2}^2 \tag{8.26}$$
$$\leq \left[\sum_{k=2}^{\infty}\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}a_{k,2}\right]^2 \leq 1.$$

It follows from (8.25) and (8.26) that

$$|su\frac{1}{2}\left[\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right]^2 (a_{k,1}^2 + a_{k,2}^2 \leq 1. \tag{8.27}$$

Therefore, we need to find the largest $\phi = \phi(n,\lambda,\alpha)$ such that

$$\frac{k^n\{k-\phi[1+\lambda(k-1)]\}}{1-\phi} \leq \frac{1}{2}\left[\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{1-\alpha}\right]^2 \quad (k \geq 2), \tag{8.28}$$

that is,

$$\phi \leq 1 - \frac{2(k-1)(1-\lambda)(1-\alpha)^2}{k^n\{k-\alpha[1+\lambda(k-1)]\}^2 - 2[1+\lambda(k-1)](1-\alpha)^2} \quad (k \geq 2). \tag{8.29}$$

Since

$$D(k) = 1\frac{2(k-1)(1-\lambda)(1-\alpha)^2}{k^n\{k-\alpha[1+\lambda(k-1)]\}^2 - 2[1+\lambda(k-1)](1-\alpha)^2} \tag{8.30}$$

is an increasing function of $k (k \geq 2)$, we readily have

$$\phi \leq D(2) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^{n-1}\{2-\alpha(1+\lambda)\}^2 - (1+\lambda)(1-\alpha)^2}, \tag{8.31}$$

and Theorem 13 follows at once. □

Theorem 14. *Let the function* $f_1(z) = z - \sum_{k=2}^{\infty} a_{k,1} z^k (a_{k,1} \geq 0)$ *be in the class* $T_n(\lambda, \alpha)$ *and* $f_2(z) = z - \sum_{k=2}^{\infty} |a_{k,2}| z^k$, *with* $|a_{k,2}| \leq 1, k = 2, 3, \ldots$. *Then* $f_1 * f_2(z) \in T_n(\lambda, \alpha)$.

Proof. Since

$$\begin{aligned}
\sum_{k=2}^{\infty} k^n\{k-\alpha[1+\lambda(k-1)]\}|a_{k,1}a_{k,2}| &= \sum_{k=2}^{\infty} k^n\{k-\alpha[1+\lambda(k-1)]\}a_{k,1}|a_{k,2}| \\
&\leq \sum_{k=2}^{\infty} k^n\{k-\alpha[1+\lambda(k-1)]\}a_{k,1} \\
&\leq 1-\alpha,
\end{aligned}$$

by Theorem 1, it follows that $f_1 * f_2(z) \in T_n(\lambda, \alpha)$. □

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