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THE TACHIBANA OPERATOR AND TRANSFER OF LIFTS

Abdullah Mağden, Muhammet Kamali, Arif A. Salimov

Abstract

The main purpose of this paper is to investigate, using the Tachibana operator, transfer of the complete lifts of affinor structures along the cross-sections of the tangent and cotangent bundles.

1. Introduction

Let A_m be an associative commutative unital algebra of finite dimension m over the field $\mathbb{R}$ of real numbers and $z = x^\alpha e_\alpha$, $\alpha = 1, \ldots, m$, a variable in the algebra A_m, where e_α and x^α denote the basic units of A_m and real variables, respectively. Then, $w = f^\alpha(x^1, \ldots, x^m)e_\alpha$ is an algebraic function of z, where $f^\alpha(x^1, \ldots, x^m)$ are real functions of all x^α. We now define the differentials in A_m by

$$dw = df^\alpha e_\alpha = (\partial_\beta f^\alpha)dx^\beta e_\alpha, \quad dz = dx^\alpha e_\alpha.$$

If, for A-functions $w = w(z)$, the differential dw can be represented in the form $dw = w'(z)dz$, then f is said to be A-holomorphic ([1] p.85, [2]), and the A-function $w'(z)$ is called the derivative.

The necessary and sufficient condition for an A-function $w = w(z)$ to be A-holomorphic is that

$$S_\alpha \mathcal{D} = \mathcal{D} S_\alpha, \tag{1.1}$$

where $S_\alpha = (C^\gamma_{\alpha\beta})$, $\mathcal{D} = \left(\frac{\partial f^\alpha}{\partial x^\beta}\right)$ and $C^\gamma_{\alpha\beta}$ are the structure constants of the algebra A_m. The conditions (1.1) will be called the Scheffers conditions [3]. In particular, in case of the algebra of complex numbers $A_2 = \mathbb{C}(i)$, $i^2 = -1$, the Scheffers conditions coincide with the Cauchy-Riemann conditions.

On a differentiable manifold M_n of class $\mathcal{C}^\infty$ we consider a polyaffinor structure $\Pi = \{\underset{\alpha}{\varphi}{}^i_j\}$-a collection of tensor fields of type (1, 1) that represents the algebra A_m isomorphically, that is

$$\underset{\alpha}{\varphi}{}^m_i \underset{\beta}{\varphi}{}^j_m = C^\gamma_{\alpha\beta} \underset{\gamma}{\varphi}{}^j_i$$

where we indicate by $\underset{\alpha}{\varphi}{}^{i}_{j}$ $(\alpha = 1, \ldots, m)$ the affinors of Π-structure corresponded elements e_α $(\alpha = 1, \ldots, m)$ of the base of A_m under the isomorphism. If M_n admits a smooth atlas of local charts such that all the affinors of the Π-structure have constant components in any chart of this atlas, then the Π-structure is said to be integrable. Let Π-structure defined with the Frobenius algebra A_m be a r-regular structure $\underset{\alpha}{\varphi}{}^{j}_{i} = \delta^u_v \mathcal{C}^{\gamma}_{\alpha\beta}$, $i, j = 1, \ldots, n$; $\alpha, \beta = 1, \ldots, m$; $u, v = 1, \ldots, r$; and δ^u_v as the Kronecker symbol (for example, almost complex structure) [3]. If the structure is integrable, then it can be shown that the manifold M_n is transformed to the holomorphic manifold $X_r(A_m)$ over the algebra A_m, where the atlas determined the holomorphic manifold $X_r(A_m)$ is one for which every pair of charts is A-holomorphic related. In particular, if $A_m = \mathbb{C}(i)$, $i^2 = -1$ $(m = 2)$ then $X_r(\mathbb{C})$ is an analytic complex manifold [3, 6, 7].

We define the Tachibana operators $\Phi_{\underset{\alpha}{\varphi}} g, \Phi_{\underset{\alpha}{\varphi}} t, \Phi_{\underset{\alpha}{\varphi}} w$ ([4], see also [5]) associated with an algebraic structure $\Pi = \{\underset{\alpha}{\varphi}\}$ and an arbitrary $X \in \mathcal{T}^1_0(M_n)$, and we apply to the arbitrary tensor fields $g \in \mathcal{T}^0_2(M_n)$, $t \in \mathcal{T}^1_0(M_n)$, $w \in \mathcal{T}^0_1(M_n)$ as follows:

$$\begin{aligned}(\Phi_{\underset{\alpha}{\varphi}} g)(X, Z_1, Z_2) &= L_{\underset{\alpha}{\varphi} X}\, g - L_X(go\underset{\alpha}{\varphi}))(Z_1, Z_2) + g(Z_1, \underset{\alpha}{\varphi}(L_X Z_2)) \\ &\quad - g(\underset{\alpha}{\varphi} Z_1, L_X Z_2) \end{aligned} \tag{1.2}$$

$$(\Phi_{\underset{\alpha}{\varphi}} t)(X) = -(L_t \underset{\alpha}{\varphi} t)(X), \tag{1.3}$$

$$(\Phi_{\underset{\alpha}{\varphi}} w)(X, Y) = (L_{\underset{\alpha}{\varphi} x}\, w - L_X(w \circ \underset{\alpha}{\varphi})(Y), \tag{1.4}$$

where L_X denotes the operator of Lie derivation with respect to X and

$$(go\underset{\alpha}{\varphi})(Z_1, Z_2) = g(\underset{\alpha}{\varphi} Z_1, Z_2),$$

$$(wo\underset{\alpha}{\varphi})(Y) = w(\underset{\alpha}{\varphi} Y).$$

The expression (2) define the tensor fields $\Phi_{\underset{\alpha}{\varphi}} g \in \mathcal{T}^0_3(M_n)$, if and only if g a pure tensor field [5], that is,

$$g(\underset{\alpha}{\varphi} Z_1, Z_2) = g(Z_1, \underset{\alpha}{\varphi} Z_2), \tag{$*$}$$

for all $Z_1, Z_2 \in \mathcal{T}^1_0(M_n)$, $\underset{\alpha}{\varphi} \in \Pi$. The expressions (3) and (4) always defines the tensor fields $\Phi_{\underset{\alpha}{\varphi}} t \in \mathcal{T}^1_1(M_n)$ and $\Phi_{\underset{\alpha}{\varphi}} w \in \mathcal{T}^0_2(M_n)$, respectively. The equality (*) is

$$g_{mj} \underset{\alpha}{\varphi}{}^{m}_{i} = g_{im} \underset{\alpha}{\varphi}{}^{m}_{j}, \;\; \forall \; \underset{\alpha}{\varphi}{}^{i}_{j} \in \Pi$$

with respect to a natural coordinate system in M_n. A tensor field $t^{i_1 \cdots m}_{j_1 \cdots j_q}$ is said to be pure with respect to the Π-structure if

$$t^{i_1 \cdots i_p}_{m j_2 \cdots j_q} \underset{\alpha}{\varphi}{}^{m}_{j_1} = \cdots = t^{i_1 \cdots i_p}_{j_1 j_2 \cdots m} \underset{\alpha}{\varphi}{}^{m}_{j_q} = t^{m i_1 \cdots i_p}_{j_1 \cdots j_q} \underset{\alpha}{\varphi}{}^{i_1}_{m} = \cdots = t^{i_1 \cdots m}_{j_1 \cdots j_q} \underset{\alpha}{\varphi}{}^{i_p}_{m}, \quad \forall\, t^{i_1 \cdots i_p}_{m j_2 \cdots j_q} \underset{\alpha}{\varphi}{}^{i}_{j} \in \Pi.$$

We consider for convenience the tensor fields of type (1, 0) and (0, 1) as pure tensor fields [12].

The tensors $\Phi_{\underset{\alpha}{\varphi}} g \Phi_{\underset{\alpha}{\varphi}} t$ and $\Phi_{\underset{\alpha}{\varphi}} w$ have, respectively, components

$$\underset{\alpha}{\Phi}_k g_{ij} = \underset{\alpha}{\varphi}{}_k^m \partial_m g_{ij} - \partial_k (g_{mj} \underset{\alpha}{\varphi}{}_i^m) + g_{im} \partial_j \underset{\alpha}{\varphi}{}_k^m + g_{mj} \partial_i \underset{\alpha}{\varphi}{}_k^m, \tag{1.5}$$

$$\underset{\alpha}{\Phi}_k t^i = -L_t \underset{\alpha}{\varphi}{}_k^i = -t^m \partial_m \underset{\alpha}{\varphi}{}_k^i + \underset{\alpha}{\varphi}{}_k^m \partial_m t^i - \underset{\alpha}{\varphi}{}_m^i \partial_k t^m, \tag{1.6}$$

$$\underset{\alpha}{\Phi}_k w_i = \underset{\alpha}{\varphi}{}_k^m \partial_m w_i - \underset{\alpha}{\varphi}{}_i^m \partial_k w_m - w_m (\partial_k \underset{\alpha}{\varphi}{}_i^m - \partial_i \underset{\alpha}{\varphi}{}_k^m) \tag{1.7}$$

with respect to a natural coordinate system in M_n.

When

$$(\Phi_{\underset{\alpha}{\varphi}} g)(X, Z_1, Z_2) = 0 \tag{1.8}$$

for a pure tensor g and $X, Z_1, Z_2 \in \mathcal{T}_0^1(M_n), M_n$ being a manifold with integrable algebraic Frobenius r-regular Π-structure, g is said to be A-holomorphic. Actually, in case of the tensor $g^*_{uv} = g_{uv\sigma} e^\sigma$ in $X_r(A_m)$ corresponding the pure tensor g satisfies the A-holomorphic condition

$$\mathcal{C}^\mu_{\alpha\gamma} \partial_{w\mu} g_{uv\sigma} = \mathcal{C}^\mu_{\alpha\sigma} \partial_{w\gamma} g_{uv\mu}$$

(see [3]). If Π-structure is non-integrable, then the pure tensor g satisfying the equality (1.8) is called almost A-holomorphic [3] [4].

2. Complete Lifts on the Cross-Section

Let us consider the tensor bundle of $T_q^p(M_n)$ with a natural projection $\pi : T_q^p(M_n) \to M_n$. If a differentiable mapping $\sigma : M_n \to T_q^p(M_n)$ which satisfies $\pi o \sigma = id_{M_n}$, then σ is called a cross-section of $T_q^p(M_n)$, where id_{M_n} is the identity mapping on M_n. It is obvious that the cross-section of $T_q^p(M_n)$ on M_n defines a tensor field $t^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ of type (p, q). Since the rank of the differential of the mapping σ is n and σ injective, the cross-section of $T_q^p(M_n)$ is a submanifold of $T_q^p(M_n)$ with respect to induced topology, which is diffeomorphic to M_n. We will investigate the complete lift of a tensor φ^i_j along a pure submanifold defined by the pure cross-section (i.e., the pure tensor field $t^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ of type (p, q)).

The complete lift of a vector field $V = (v^i) \in \mathcal{T}_0^1(M_n)$ to the tensor bundle $T_q^p(M_n)$ with respect to the coordinate neighborhood $\pi^{-1}(U) \subset T_q^p(M_n)$ was defined in [6] as

$${}^cV = ({}^cV^i, {}^cV^{\bar{i}}) = (v^i, L_V \alpha), \tag{2.9}$$

$\forall\, \alpha \in T_p^q(U)$; $i = 1, \ldots, n$; $\bar{i} = n+1, \ldots, n + n^{p+q}$, where α can be considered as a differentiable function on the space $T_q^p(M_n)$ in the usual way by contraction $\alpha = \alpha(t)$.

In particularly, if we get $\alpha = -t^{i_1\cdots i_p}_{j_1\cdots j_q}$, then the complete lift of V to $T^p_q(M_n)$ in the coordinate neighborhood $\pi^{-1}(U)$ with respect to the natural frame $\{\partial_j, \partial_{\overline{j}}\}$, $x^{\overline{j}} = t^{i_1\cdots i_p}_{j_1\cdots j_q}$ is of the form

$$^cV = (\,^cV^j,\ ^cV^{\overline{j}}) = \left(v^j, \sum_{\lambda=1}^{p} t^{i_1\cdots m\cdots i_p}_{(j)} \partial_m v^{i_\lambda} - \sum_{\mu=1}^{q} t^{(i)}_{j_1\cdots m\cdots j_q} \partial_{j_\mu} v^m \right). \tag{2.10}$$

Let us consider the cross-section of $T^p_q(M_n)$ defined by the tensor field $t^{i_1\cdots i_p}_{j_1\cdots j_q}(x^i)$. This cross-section equation is written as

$$\overline{x}^J = \overline{x}^J(x^j), \quad J = 1, \ldots, n + n^{p+q}$$

or

$$\left.\begin{array}{rcl} \overline{x}^j &=& x^j \\ \overline{x}^{\overline{j}} &=& t^{i_1\cdots i_p}_{j_1\cdots j_q}(x^j). \end{array}\right\}$$

It is obvious that the system

$$\left.\begin{array}{rcl} B_i &=& \{\partial_i \overline{x}^A\} = \{B^h_i, B^{\overline{h}}_i\} = \{\delta^h_i, \partial_i t^{i_1\cdots i_p}_{j_1\cdots j_q}\} = \delta^h_i \partial_h + \partial_i t^{i_1\cdots i_p}_{j_1\cdots j_q} \partial_{\overline{h}} \\ C_{\overline{i}} &=& \{\partial_{\overline{i}} \overline{x}^A\} = (C^h_{\overline{i}}, C^{\overline{h}}_{\overline{i}}) = (0, \delta^{\ell_1}_{j_1} \cdots \delta^{\ell_q}_{j_q} \delta^{i_1}_{h_1} \cdots \delta^{i_p}_{h_p}) = \delta^{\ell_1}_{j_1} \cdots \delta^{\ell_q}_{j_q} \delta^{i_1}_{h_1} \cdots \delta^{i_p}_{h_p} \partial_{\overline{h}} \end{array}\right\}$$

defined a frame along the cross-section. B_i and $C_{\overline{i}}$, $i = 1, \ldots, n$; $\overline{i} = n+1, \ldots, n+n^{p+q}$ span the tangent plane of $T^p_q(M_n)$ and are tangent to the cross-section and the fibre, respectively.

Using (2.10) and $^cV^A = \tilde{V}^i B^A_i + \tilde{V}^{\overline{i}} C^A_{\overline{i}}$, we have

$$\left.\begin{array}{rcl} v^i \partial_i x^{\overline{h}} &+& \left(\sum_{\lambda=1}^{p} t^{i_1\cdots m\cdots i_p}_{(j)} \partial_m v^{i_\lambda} - \sum_{\mu=1}^{q} t^{(i)}_{j_1\cdots m\cdots j_q} \partial_{j_\mu} v^m\right) \partial_{\overline{i}} x^{\bar{h}} = \tilde{V}^i B^{\overline{h}}_i + \tilde{V}^{\overline{i}} C^{\overline{h}}_{\overline{i}} \\ v^i \partial_i x^h &+& \left(\sum_{\lambda=1}^{p} t^{i_1\cdots m\cdots i_p}_{(j)} \partial_m v^{i_\lambda} - \sum_{\mu=1}^{q} t^{(i)}_{j_1\cdots m\cdots j_q} \partial_{j_\mu} v^m\right) \partial_{\overline{i}} x^{\bar{h}} = \tilde{V}^i B^h_i + \tilde{V}^{\overline{i}} C^h_{\overline{i}} \end{array}\right\}.$$

Therefore, we obtain

$$\begin{array}{rcl} \tilde{V}^i &=& v^i \\ \tilde{V}^{\overline{i}} &=& -L_V t^{i_1\cdots i_p}_{j_1\cdots j_q}, \end{array}$$

that is, the complete lift ^cV of V with respect to the frame (B, C) along the cross-section $t^{i_1\cdots i_p}_{j_1\cdots j_q}$, is written as

$$^cV = (\,^cV^j,\ ^cV^{\overline{j}}) = (v^j, -L_V t^{i_1\cdots i_p}_{j_1\cdots j_q}). \tag{2.11}$$

2.1. Complete Lifts of the Affinor to $T_0^1(M_n)$ Along a Pure Cross-Section

We will find a formula for a complete lift of affinor field φ_j^i along the pure cross-section $t \in \mathcal{T}_0^1(M_n)$ of tangent bundle $T_0^1(M_n)$.

We define the complete lift ${}^c\varphi$ of a tensor field $\varphi \in \mathcal{T}_1^1(M_n)$ along the pure cross-section $t \in \mathcal{T}_0^1(M_n)$ of $T_0^1(M_n)$ by

$$ {}^c(\varphi(V)) = {}^c\varphi({}^cV), \quad \forall\, V \in \mathcal{T}_0^1(M_n), \tag{2.12} $$

where ${}^cV$ is in the form (2.11). The equality (2.12) can be written as

$$ {}^c(\varphi(V))^K = {}^c\varphi_L^K\, {}^cV^L, \tag{2.13} $$

by using coordinates. If we take $K = k$ in (2.13), we have

$$ \varphi_\ell^k v^\ell = (\varphi(V))^k = {}^c\varphi_L^k\, {}^cV^L = {}^c\varphi_\ell^k\, {}^cV^\ell + {}^c\varphi_{\overline{\ell}}^k\, {}^cV^{\overline{\ell}}. $$

Then, we obtain

$$ {}^c\varphi_\ell^k = \varphi_\ell^k, \qquad {}^c\varphi_{\overline{\ell}}^k = 0. \tag{2.14} $$

If we take $K = \overline{k}$ in the equality (2.13), we have

$$ {}^c(\varphi(V))^{\overline{k}} = {}^c\varphi_L^{\overline{k}}\, {}^cV^L = {}^c\varphi_\ell^{\overline{k}}\, {}^cV^\ell + {}^c\varphi_{\overline{\ell}}^{\overline{k}}\, {}^cV^{\overline{\ell}}. \tag{2.15} $$

Now, let us find solutions which are ${}^c\varphi_\ell^{\overline{k}}$ and ${}^c\varphi_{\overline{\ell}}^{\overline{k}}$ in equation (2.15). For this purpose, taking account of (1.6), we have

$$ L_{\varphi V} t^k = v^\ell \Phi_\ell t^k + \varphi_\ell^k L_V t^\ell. \tag{2.16} $$

From (2.11) and (2.16), we get

$$ -\,{}^c(\varphi(V))^{\overline{k}} = {}^cV^\ell \Phi_\ell t^k - \varphi_\ell^k\, {}^cV^{\overline{\ell}}. \tag{2.17} $$

Then from (2.15) and (2.17) we obtain

$$ {}^c\varphi_\ell^{\overline{k}} = -\Phi_\ell t^k, \qquad {}^c\varphi_{\overline{\ell}}^{\overline{k}} = \varphi_i^k, \quad (x^{\overline{k}} = t^k). \tag{2.18} $$

Thus (2.14) and (2.18) are the complete lift of the tensor structure $\varphi \in \mathcal{T}_1^1(M_n)$ along the pure cross-section of $T_0^1(M_n)$. As a special case, this lift was obtained with respect to the natural frame $\{\partial_i, \partial_{\overline{i}}\}$ in [7] (see also [8]).

2.2. Complete Lifts of the Affinor to $T_1^0(M_n)$ Along a Pure Cross-Section

We will find a formula for complete lift of affinor field φ_j^i along the pure cross-section $w \in \mathcal{T}_1^0(M_n)$ of cotangent bundle $T_1^0(M_n)$.

If the Tachibana operator Φ is applied to the pure tensor field $w \in \mathcal{T}_1^0(M_n)$, then from (1.7) we have

$$v^j \Phi_j w_i = L_{\varphi V} w_i - \varphi_i^j L_V w_j - w_j L_V \varphi_i^j. \tag{2.19}$$

We define a complete lift ${}^c\varphi$ of the tensor $\varphi \in \mathcal{T}_1^1(M_n)$ along the pure cross-section w of $T_1^0(M_n)$ by

$${}^c(\varphi(V)) + {}^v(L_V \varphi) = {}^c\varphi({}^cV)$$

or

$${}^c(\varphi(V))^I + {}^v(L_V\varphi)^I = {}^c\varphi_J^I \, {}^cV^J \tag{2.20}$$

by using the coordinates, where ${}^v(L_V\varphi)$ denotes the vertical lift of Lie derivative.

In the equality (2.20), let $I = i$. Then we have ${}^v(L_V\varphi)^i = 0$ by the definition of the vertical lift. In this case, the equality (2.20) can be written

$${}^c(\varphi(V))^i = {}^c\varphi_j^i \, {}^cV^j + {}^c\varphi_{\overline{j}}^i \, {}^cV^{\overline{j}}. \tag{2.21}$$

Thus, from (2.21), we see that

$${}^c\varphi_j^i = \varphi_j^i, \qquad {}^c\varphi_{\overline{j}}^i = 0. \tag{2.22}$$

Now, let $I = \overline{i}$. From the definition of the vertical lift, we have ${}^v(L_V\varphi)^{\overline{i}} = w_j L_V \varphi_i^j$. Taking account of (2.20), we have

$${}^c(\varphi(V))^{\overline{i}} = {}^c\varphi_j^{\overline{i}} \, {}^cV^j + {}^c\varphi_{\overline{j}}^{\overline{i}} \, {}^cV^{\overline{j}} - w_j L_V \varphi_i^j. \tag{2.23}$$

From (2.11) and (2.19), we see that

$$\begin{aligned} L_{\varphi V} w_i &= v^j \Phi_j w_i + \varphi_i^j L_V w_j + w_j L_V \varphi_i^j, \\ -{}^c(\varphi(V))^{\overline{i}} &= {}^cV^j \Phi_j w_i - \varphi_i^j \, {}^cV^{\overline{j}} + w_j L_V \varphi_i^j. \end{aligned} \tag{2.24}$$

From (2.23) and (2.24), we have

$${}^c\varphi_j^{\overline{i}} = -\Phi_j w_i, \qquad {}^c\varphi_{\overline{j}}^{\overline{i}} = \varphi_i^j, \quad (x^{\overline{i}} = w_i).$$

3. Transfer of the Complete Lift of the Affinor Structure

Let M_n be *a* paracompact manifold with a Riemannian metric. We shall mean by the Riemannian metric a symmetric covariant tensor field g of degree 2 which is nondegenerate. If g is a pure tensor, then a manifold M_n with an algebraic Π-structure is called an almost B-manifold [1, p.31] and this will be denoted V_n.

Suppose that $T_0^1(V_n)$ and $T_1^0(V_n)$ are the tensor bundle of type (1, 0) and (0, 1) over V_n, respectively. Clearly $\dim T_0^1(V_n) = \dim T_1^0(V_n) = 2n$.

Let the diffeomorphism $f : T_0^1(V_n) \to T_1^0(V_n),\ \ y^I = y^I(x^J),\ I, J = 1, \ldots, 2n$ be defined by a local expression such that

$$\begin{aligned} y^i &= x^i \\ y^{\overline{i}} &= w_i = g_{im}t^m. \end{aligned} \tag{3.25}$$

Since

$$\begin{aligned} x^{\overline{k}} &= t^k, \\ \frac{\partial y^{\overline{i}}}{\partial x^{\overline{k}}} &= \frac{\partial}{\partial x^{\overline{k}}}(w_i) = \frac{\partial}{\partial x^{\overline{k}}}(g_{im}t^m) = \frac{\partial}{\partial x^{\overline{k}}}(g_{ik}t^k) = g_{ik}, \\ 0 &= \frac{\partial y^{\overline{i}}}{\partial x^k} = \frac{\partial w_i}{\partial x^k} = \frac{\partial}{\partial x^k}(g_{im}t^m) = (\partial_k g_{im})t^m, \end{aligned}$$

we have

$$A = \left(\frac{\partial y^I}{\partial x^K}\right) = \begin{pmatrix} \frac{\partial y^i}{\partial x^k} & \frac{\partial y^i}{\partial x^{\overline{k}}} \\ \frac{\partial y^{\overline{i}}}{\partial x^k} & \frac{\partial y^{\overline{i}}}{\partial x^{\overline{k}}} \end{pmatrix} = \begin{pmatrix} \delta_k^i & 0 \\ 0 & g_{ik} \end{pmatrix}.$$

The inverse of the mapping f is written as

$$x^\ell = y^\ell, \quad x^{\overline{\ell}} = t^\ell = g^{\ell m}w_m.$$

Suppose that $y^{\overline{j}} = w_j$, we have

$$A^{-1} = \left(\frac{\partial x^L}{\partial y^J}\right) = \begin{pmatrix} \delta_j^\ell & 0 \\ 0 & g^{\ell j} \end{pmatrix},$$

which is the Jacobian matrix of inverse mapping f^{-1}.

Theorem 3.1. *Suppose that* ${}^c\overset{1}{\varphi}$ *and* ${}^c\overset{2}{\varphi}$ *denote the complete lift of the affinor* φ *of the* Π*-structure to* $T_0^1(V_n)$ *and* $T_1^0(V_n)$ *along the pure cross-sections* t^i *and* w_i*, respectively. If* $\Phi_\varphi(g) = 0$*, then* ${}^c\overset{2}{\varphi}$ *is transferred from* ${}^c\overset{1}{\varphi}$ *by means of the diffeomorphism* f*, where* Φ_φ *denotes the Tachibana operator.*

Proof. Suppose that $\Phi_\varphi(g) = 0$. Then, if we write ${}^c\overset{2}{\varphi}$ along the pure cross-section $w_i(y)$, we obtain

$$\begin{aligned}
{}^c\overset{2}{\varphi} &= \left({}^{c^2}\varphi^I_J \right) \\
&= \begin{pmatrix} \varphi^i_j & 0 \\ -\underset{\varphi}{\Phi}{}_j w_i & \varphi^j_i \end{pmatrix} \\
&= \begin{pmatrix} \varphi^i_j & 0 \\ -g_{im}\Phi_j t^m - (\Phi_j g_{im})t^m & \varphi^j_i \end{pmatrix} \\
&= \begin{pmatrix} \varphi^i_j & 0 \\ -g_{im}\Phi_j t^m & \varphi^j_i \end{pmatrix} \\
&= \begin{pmatrix} \delta^i_k & 0 \\ 0 & g_{ik} \end{pmatrix} \begin{pmatrix} \varphi^k_\ell & 0 \\ -\Phi_\ell t^k & \varphi^k_\ell \end{pmatrix} \begin{pmatrix} \delta^\ell_j & 0 \\ 0 & g^{\ell j} \end{pmatrix} \\
&= A\, {}^c\overset{1}{\varphi}\, A^{-1}. \qquad (3.26)
\end{aligned}$$

To show (3.26), we have taken account of

$$g_{ik}\varphi^k_\ell g^{\ell j} = g_{k\ell}\varphi^k_i g^{\ell j} = \varphi^k_i \delta^j_k = \varphi^j_i$$

and used that g_{ij} is the pure tensor field. □

We introduce in some coordinate neighborhood $U \subset M_n$ a connection in which all the affinors of the Π-structure are covariantly constant. Such connections are called Π-connection. A Π-structure will be said to be almost integrable [3] if in a coordinate neighborhood of each point $x \in M_n$ there exists at least one Π-connection without torsion. The Π-structure is almost integrable on the Riemann connection if and only if $\Phi_{\underset{\alpha}{\varphi}}(g) = 0$, for all $\underset{\alpha}{\varphi} \in \Pi$ [9] (see also [10]). Further, it has been shown that if the algebraic Π-structure is almost integrable, then the structure ${}^c\Pi = \{ {}^c\underset{\alpha}{\varphi} \}$ determines the algebraic structure along the pure subbundle of the tensor bundle $T^p_q(M_n)$ [11]. Using these facts, we have the following result:

Theorem 3.2. *If the metric g of the B-manifold is almost A-holomorphic, then the algebraic*

$\Pi_2 = \{ {}^c\underset{\alpha}{\overset{2}{\varphi}} \}$ *-structure is transferred from the algebraic* $\Pi_1 = \{ {}^c\underset{\alpha}{\overset{1}{\varphi}} \}$ *-structure by means of the diffeomorphism f.*

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Abdullah MAĞDEN,
Muhammet KAMALİ,
Arif A. SALİMOV
Department of Mathematics
Faculty of Sciences and Arts
Atatürk University
25240 Erzurum - TURKEY

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