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ON THE ACTION OF STEENROD OPERATIONS ON POLYNOMIAL ALGEBRAS

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Abstract

Let $\mathbb{A}$ be the mod-p Steenrod Algebra. Let p be an odd prime number and $Z_p = Z/pZ$. Let $P_s = Z_p[x_1, x_2, \ldots, x_1]$. A polynomial $N \in P_s$ is said to be hit if it is in the image of the action $A \otimes P_s \to P_s$. In [10] for $p = 2$, Wood showed that if $\alpha(d+s) > s$ then every polynomial of degree d in P_s is hit where $\alpha(d+s)$ denotes the number of ones in the binary expansion of $d+s$. Latter in [6] Monks extended a result of Wood to determine a new family of hit polynomials in P_s. In this paper we are interested in determining the image of the action $A \otimes P_s \to P_s$. So our results which determine a new family of hit polynomials in P_s for odd prime numbers generalize cononical antiautaomorphism of formulas of Davis [2], Gallant [3] and Monks [6].

1. Introduction

Let $\mathcal{A}$ be a mod-p Steenrod algebra. Let p be an odd prime number and $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$. Let $P_s = \mathbf{Z}_p[x_1, x_2, \ldots, x_s]$. A polynomial $N \in P_s$ is said to be **hit** if it is in the image of the action $A \otimes P_s \to P_s$, i.e. $N \in AP_s$ where A is the augmentation ideal of $\mathcal{A}$, i.e. $N = \sum_i P^i M_i$ for some $M_i \in P_s$.

We are interested in determining the image of the aciton $A \otimes P_s \to P_s$: the space of elements in P_s that are hit by positive dimensional Steenrod operations. In [10], when $p = 2$ Wood showed that if $\alpha(d+s) > s$ then every polynomial of degree d in P_s is hit where $\alpha(d+s)$ denotes the number of ones in the binary expansion of $d+s$. In [9] Singer generalized Wood's result conjectured by Peterson and identified a larger class of hit polynomials. In [8] Silverman generalized a result of Wood and proved a conjecture of Singer. In [6] Monks extended a result of Wood to determine a new family of hit polynomials in P_s.

In order to state our result we need to introduce some notation. For $m \geq 0$ and $t \geq 1$,

$$\gamma_t(m) = \sum_{i=0}^{m-1} p^{it}, \tag{1}$$

where $\gamma_t(0) = 0$. A sequence of nonnegative integers $L = (l_1, l_2, \ldots, l_n)$ is called a t-decomposition of a positive integer m if $m = \sum_{i=1}^{n} \gamma_t(l_i)$. We define $\mu_t(m)$ to be the number of terms in the shortest t-decomposition of m, i.e.

$$\mu_t(m) = \min\{n | m = \sum_{i=1}^{n} \gamma_t(l_i)\}. \tag{2}$$

The following results are odd-primary analogues of results of Monks [6].

Theorem 1.1. *Let H and K be polynomials of degree $2h$, $2k$ respectively. If $h < \mu_t(k)$, then $H\, P^{p^t}$ is hit.*

Let $P_t(r_1, r_2, \ldots, r_m)$ be the Milnor basis element $P(s_1, s_2, \ldots, s_{tm})$ where $s_{ti} = r_i$ and $s_j = 0$ if t does not divide j. In particular $P_t(p^s) = P_t^s$ and $P_1(n) = P(n)$.

If $R = (r_1, r_2, \ldots, r_m)$ is a sequence of nonnegative integers, we will write $P_t(R)$ for the corresponding Milnor basis element. The degree of $P_t(R)$ is $2|R|_t$ where $|R|_t = \sum_{i=1}^{\infty}(p^{it} - 1)r_i$ and the excess of $P_t(R)$ is $2e(R)$ where $e(R) = \sum_{i=1}^{\infty} r_i$. For a fixed t let B_t be the vector subspace of A with basis the set of all $P_t(R)$. For $P_t^s \in B_t$ write $\widehat{T}_t^s$ for $(-1)^s \chi(P_t^s)$ where χ denotes the canonical antiautomorphism of B_t.

Theorem 1.2. *For $s, t \geq 1, 0 \leq k < s,$ and $k \leq t$,*

$$\widehat{P}_t(p^s - p^k) = P_t((p-1)p^{s-1})P_t((p-1)p^{s-2}) \cdots P_t((p-1)p^k) \tag{3}$$

2. Some Tools

In this section we list some properties of the Steenrod algebra we need to prove Theorem 1.1 and Theorem 1.2

Lemma 2.1. *For $m \geq 0$,*

$$\mu_t(m) = \min\{e(P_t(R)).$$

Proof. These is a $1-1$ correspondence between Milnor basis elements $P_t(R)$ satisfying $|R|_t = (p^t - 1)$ and t-representations of m given by

$$P_t(R) \longleftrightarrow m = \sum_i r_i \gamma_t(i).$$

Under this correspondence, $e(P_t(R))$ corresponds to the number $\sum_i r_i$ which is used in determining $\mu_t(m)$. The lemma follows immediately from this observation. □

Following lemma is analoguos to [6, Lemma 2.1].

Lemma 2.2. *For all $t, m \geq 1, \mu_t(m) \leq \frac{p^t-1}{p-1}\mu_1(m)$.*

Proof. There exists positive integers $l_1, l_2, \ldots, l_{\mu_1(m)}$ that such

$$m = \sum_{i=1}^{\mu_1(m)} \gamma_1(l_i). \tag{4}$$

For each l_i let $l_i = tq_i + r_i$ where q_i and r_i are non-negative integers and $x \leq r_i < t$.

$$\begin{aligned}
m &= \sum_{i=1}^{\mu_u(m)} \gamma_1(l_i) = \sum_{i=1}^{\mu_1(m)} \gamma_1(tq_i + r_i) = \sum_{i=1}^{\mu_1(m)} \sum_{i=1}^{tq_i+r_i-1} p^j = \sum_{i=1}^{\mu_1(m)} q^{tq_i+r_i} - 1 \\
&= \sum_{i=1}^{\mu_1(m)} \frac{p^t-1}{(p^t-1)(p-1)}(q^{tq_i+r_i} - 1) \\
&= \sum_{i=1}^{\mu_1(m)} \left[\frac{p^t - p^{r_i}}{p-1} \frac{p^{tq_i} - 1}{p^t - 1} + \frac{p^{r_i} - 1}{p-1} \frac{p^{t(q_i+1)} - 1}{p^t - 1} \right] \\
&= \sum_{j=1}^{\frac{p^t - p^{r_i}}{p-1}} \sum_{i=1}^{\mu_1(m)} \gamma_t(q_i) + \sum_{j=1}^{\frac{p^{r_i}-1}{p-1}} \sum_{i=1}^{\mu_1(m)} \gamma_t(q_i + 1)
\end{aligned}$$

This yields a t-decomposition of m with $\frac{p^t-1}{p-1}\mu_1(m)$ terms. This completes the proof. □

Lemma 2.3. *If $m \leq p^t$ then $\mu_t(m) = m$.*

Proof. Let $m \leq p^t$. Then $m \leq p^t < p^t + 1 = \gamma_t$ (2). The only possible t-decomposition of m is a sequence of m ones because γ_t is strictly increasing □

Let $L = (l_1, l_2, \ldots, l_n)$ be any sequence of nonnegative integers. Define

$$|L| = \sum_{i=1}^{n} l_i \tag{5}$$

$$\nu(L) = \max_i l_i \tag{6}$$

and

$$Y_t(L) = \sum_{i=1}^{n} \gamma_t(l_i). \tag{7}$$

Suppose that $l_1 \geq l_2 \geq \cdots \geq l_n$ and that $|L| \geq 1$. For this sequence, we can define

$$\delta(L) = (l'_1, l'_2, \ldots, l'_n), \tag{8}$$

where

$$l'_i = \begin{cases} l_i - 1 & \text{if } l_i = l_1 \text{ and } (l_{i+1} \neq l_1 \text{ or } i = n) \\ l_i & \text{if otherwise.} \end{cases}$$

It is easy to verify that

$$\begin{gathered} l'_1 \geq l'_2 \geq \cdots \geq l'_n \\ |\delta(L)| = |L| - 1 \\ \nu(\delta(L)) \geq \nu(L) \end{gathered}$$

and

$$Y_t(\delta(L)) = Y_t(L) - p^{t(\nu(L)-1)}.$$

We can define δ^r to be the r-fold composition of δ with itself (δ^0 is the identity function) for $0 \leq r \leq |L|$. Let $F_L = (f_1, f_2, \ldots, f_{|L|})$ be the sequence given by

$$f_i = Y_t(\delta^{i-1}(L)) - Y_t(\delta^i(L)). \tag{9}$$

Since $\delta^{|L|}(L) = (0, 0, \ldots, 0)$ and $Y_t(\delta^{|L|}(L)) = 0$,

$$\begin{aligned} |F_L| &= \sum_{i=1}^{|L|} [Y_t(\delta^{i-1}(L)) - Y_t(\delta^i(L))] = Y_t(\delta^0(L)) - Y_t(\delta^{|L|}(L)) \\ &= Y_t(L) \end{aligned} \tag{10}$$

Lemma 2.4. *If $m < (p-1)p^s$, then $\mu_t(m) \leq \mu_t(m + (p-1)p^s)$.*

Proof. Assume that $L = (l_1, l_2, \ldots, l_n)$ is a t-decomposition of $m + (p-1)p^s$. Without lass of generality we can also assume that $l_1 \geq l_2 \geq \cdots l_n$. By definiton we have $Y_t(L) = m + (p-1)p^s$, and so by (10)

$$\sum_{i=0}^{|L|} f_i = m + (p-1)p^s.$$

So F_K is a non-increasing sequence whose power is $m+(p-1)p^s$. Hence we need following lemma: □

Lemma 2.5. *If $(p-1)p^b \leq a < p^{b+1}, \sum_{i=1}^{r} p^{x_i} = a$, and $p^{x_i} \geq p^{x_2} \geq \cdot \geq p^{x_r}$ then there is a $q \in \{1, \ldots, r\}$ such that $\sum_{i=1}^{q} p^{x_i} = (p-1)p^b$.*

Proof. If $a = (p-1)p^b$ then we can take $q = r$ and we are done. Assume that $(p-1)p^b < a$. Since $p^{b+1} > a$, we have $p^b \geq p^{x_1} \geq p^{x_2} \cdots \geq p^{X_{q+1}}$. Let q be the largest integer such that $\sum_{i=1}^{q} p^{x_i} \leq (p-1)p^b$. Then $(p-1)p^b - \sum_{i=1}^{q} p^{x_i} \equiv 0 \bmod p^{X_{q+1}}$ and $(p-1)p^b < \sum_{i=1}^{q+1} p^{x_i}$ and hence $\sum_{i=1}^{q} p^{x_i} = (p-1)p^b$. □

For Lemma 2.5 there exists $q \in \{1, \ldots, |L|\}$ such that $\sum_{i=1}^{q} f_i = (p-1)p^s$. Thus

$$\begin{aligned}\sum_{i=1}^{q}[Y_t(\delta^{i-1}(L)) - Y_t(\delta^i(L))] &= Y_t(L) + Y_T(\delta^q(L)) \\ &= m + (p-1)p^s - Y_t(\delta_q(L)) = (p-1)p^s\end{aligned}$$

and hence $Y_t(\delta^q(L)) = m$. Therefore $\mu_t(m) \leq \mu_t(m + (p-1)p^s)$ □

Using this result we can prove the following lemma:

Lemma 2.6. *$\mu_t(p^s - p^k) \geq (p-1)p^k$ where s, t, and k are any integers such that $s, t \geq 1, 0 \leq k < s$, and $k < t$.*

Proof. We will prove this by induction on s. If $s = k+1$ then $\mu_t(p^s - p^k) = \mu_t((p-1)p^k) = (p-1)p^k$ by Lemma 2.3 Assume that it is true for $s-1$. Then by Lemma 2.4, $\mu_t(p^s - p^k) = \mu_t((p-1)p^{s-1} + p^{s-1} + p^k) \geq \mu_t(p^{s-1} - p^k)$. By inductive hypothesis, $\mu_t(p^{s-1} - p^k) \geq (p-1)p^k$. Hence $\mu_t(p^s - p^k) \geq (p-1)p^k$. □

The Proof of the Main results

The key idea in Wood's argument is that for any $u, w \in P_s$ and any $\theta \in \mathcal{A}$, we have $u \cdot \theta w \equiv \tilde{\theta} u \cdot m$ module hit elements. In particular if $e(\widehat{\theta}) > \deg(u)$, then $u \cdot \theta w$ is hit. Using this We will prove Theorem 1.1. We accomplish this with the aid of the following lemma.

Lemma 3.7. *If $N \in P_s$ is any element of degree $2k$, then for any $t \geq 1$,*

$$P_t(k) \cdot N = N^{p^t} \tag{11}$$

Proof. We will prove this by induction on the number of variables in N. Suppose

$$N = q_{i_1}^{h_1} x_{i_2}^{h_2} \cdots x_{i_n}^{h_n}$$

Let $n = 1$. Then

$$P_t(k)x^k = (X^k)p^t. \tag{12}$$

So the result holds for $n = 1$. Assume that the result holds for all monomials comprised of less than n variables. Let $N_1 = x_{i_1}^{h_1} x_{i_2}^{h_2} \cdots x_{i_{n-1}}^{h_{n-1}}$ so that $N = N_1 x_{i_n}^{h_n}$. Let $\psi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be the diagonal map of $\mathcal{A}$. Then $\psi(P_t(k)) = \sum_{i=0}^{k} P_t(k-i) \otimes P_t(i)$. So

$$\begin{aligned} P_k(k) \cdot N &= \sum_{i=0}^{k} P_t(k-i)N_1 \cdot P_t(i)x_{i_n}^{h_n} \\ &= \sum_{i=0}^{h_n-1} P_t(k-i)N_1 \cdot P_t(i)x_{i_n}^{h_n} + P_t(k-h_n)N_1 \cdot P_t(h_n)x_{i_n}^{h_n} \\ &+ \sum_{i=h_n+1}^{k} P_t(k-i)N_1 \cdot P_t(i)x_{i_n}^{h_n}. \end{aligned}$$

Since $e(P_t(k-i)) > \frac{1}{2}\deg(N_1)$, $\sum_{i=0}^{h_n-1} P_t(k-i)N_1 \cdot P_t(i)x_{i_n}^{h_n} = 0$. Similarly $\sum_{i=h_n+1}^{k} P_t(k-i)N_1 \cdot P_t(i)x_{i_n}^{h_n} = 0$ because $e(P_t(i)) > \frac{1}{2}\deg(x_{i_n}^{h_n})$. By induction, we have

$$P_t(k) \cdot N = P_t(h_n - i)N_1 \cdot P_t(i)x_{i_n}^{h_n} = N_1^{p^t}(x_{i_n}^{h_n})p^t.$$

Hence we obtain $P_t(k) \cdot N = N^{p_t}$. □

Wood's argument shows that $HK^{p_t} \equiv \widehat{P}_t(k)H \cdot K$ module hit elements. Hence if $e(\widehat{P}_t(k)) > h$, then $\widehat{P}_t(k)H = 0$ and hence HK^{p^t} is hit. Therefore it remains to show that $e(\widehat{P}_t(k)) = \mu_t(k)$. The following limma was prowed by Gallant [3, Proposition 1].

Lemma 3.8.

$$\widehat{P}_t(k) = \sum_R P_t(R),$$

where $|R|_t = (p^t - 1)k$.

By Lemma 2.1, $\mu_t(k)$ is exactly the minimum excess of the element $P_t(R)$ where $|R|_t = (p^t - 1)k$. On the other hand, $\widehat{P}_t(k)$ is the summand of all $P_t(R)$ where $|R|_t = (p^t - 1)k$, By Lemma 3.8. Hence $e(\widehat{P}_t(k)) = \mu_t(k)$. This completes the proof of Theorem 1.1

Proof of Theorem 1.2. We will prove this by induction on s. Suppose that $s = k+1$. Then since for $k < t$ the only nonzero element $P_t(R)$ of B_t with $|R|_t = (p^t - 1)(p-1) \cdot p^k$ is $P((p-1)p^k)$, $\widehat{P}_t(p^s - p^k) = \widehat{P}_t((p-1)p^k) = P_t((P-1) \cdot p^k)$. This proves theorem for $s = k+1$.

Assume that it is true for $s-1$. Using induction hypothesis and [3, Corollary 1.a], we have

$$\begin{aligned} P_t((p-1)p^{s-1})P_t((p-1)p^{s-2}) \cdot P_t((p-1)p^k &= P_t((p-1)p^{s-1})\widehat{P}_t(p^{s-1} - p^k) \\ &= \sum_R \binom{\sum_i p^{it} r_i}{(p-1)p^{t+s-1}} P_t(R). \end{aligned}$$

where the sum is taken over all R such that $|R|_t = (p^t - 1)(p^s - p^k)$. Since $\widehat{P}_t(p^s - p^k)$ is the sum of all $P_t(R)$ where $|R|_t = (p^t - 1)(p^s - p^k)$, it is sufficient to show that

$$\binom{\sum_i p^{it} r_i}{(p-1)p^{t+s-1}} \equiv 1 \bmod p.$$

By Lemma 2.1 and Lemma 2.6, $\sum_i r_i \geq \mu_t(p^s - p^k) \geq (p-1)p^k$. For $s > k$ and $t \geq 1$ we have

$$(p-1)(p^k - p^{s+t-1}) + (p^t - 1)(p^s - p^k) = (p^s - p^{k+1})(p^{t-1} - 1) \geq 0.$$

Hence

$$\sum_i p^{it} r_i = \sum_i (p^{it} - 1) r_i + \sum_i r_i \geq (p^t - 1)(p^s - p^k) + (p-1)p^k \geq (p-1)p^{s+t-1}$$

On the other hand,

$$(p^t - 1)\sum_i r_i \leq \sum_i (p^{it} - 1)r_i = (p^t - 1)(p^s - p^k).$$

So $\sum_i r_i \leq p^s - p^k$. Using this inequality, we have

$$\sum_i p^{it} r_i = \sum_{\substack{i \\ \leq p^{s+t}}} (p^{it} - 1)r_i + \sum_i r_i \leq (p^t - 1)(p^s - p^k) + p^s - p^k$$

Hence $\binom{\sum_i p^{it} r_i}{(p-1)p^{t+s-1}} \equiv 1 \bmod p$ by Lucas's theorem [4]. This completes the proof.

□

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