

ON HILL'S EQUATION WITH PIECEWISE CONSTANT COEFFICIENT

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Abstract

In this paper the eigenvalues of the periodic and the semi-periodic boundary value problems associated with Hill's equation are investigated in the case of piecewise constant coefficient. As a corollary the asymptotic formula for the lengths of the instability intervals of Hill's equation is derived and it is shown that they increase beyond all bounds. Also, the conditions for coexistence of periodic and semi-periodic solutions are indicated.

1. Introduction

Following [1], [2] and [4] we first present some needed facts about Hill's equation. We consider the second-order differential equation

$$-y'' = \lambda\rho(x)y \quad (-\infty < x < \infty), \quad (1.1)$$

where λ is a complex parameter and $\rho(x)$ is a real-valued function defined on the axis $-\infty < x < \infty$ and periodic with period $\omega > 0$:

$$\rho(x + \omega) = \rho(x).$$

In addition we assume that

$$\rho(x) \geq \rho_0 > 0, \quad \int_0^\omega \rho(x)dx < \infty.$$

Further, we consider the, so-called periodic

$$\begin{aligned} -y'' &= \lambda\rho(x)y & (0 \leq x \leq \omega) \\ y(0) &= y(\omega), & y'(0) = y'(\omega) \end{aligned} \quad (1.2)$$

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and the semi-periodic (or anti-periodic)

$$\begin{aligned} -y'' &= \lambda\rho(x)y & (0 \leq x \leq \omega) \\ y(0) &= y(\omega), & y'(0) = -y'(\omega) \end{aligned} \tag{1.3}$$

boundary value problems associated with equation (1.1).

Those values of complex parameter λ for which problem (1.2) or (1.3) has a non-trivial solution $y(x, \lambda)$ are called eigenvalues.

Let $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ denote the solutions of the equation (1.1) satisfying the initial conditions

$$\theta(0, \lambda) = 1, \quad \theta'(0, \lambda) = 0; \quad \varphi(0, \lambda) = 0, \quad \varphi'(0, \lambda) = 1. \tag{1.4}$$

Define the function

$$F(\lambda) = \theta(\omega, \lambda) + \varphi'(\omega, \lambda), \tag{1.5}$$

as the Hill discriminant of equation (1.1). Then the eigenvalues of the periodic problem (1.2) coincide with the roots λ of $F(\lambda) - 2 = 0$ (characteristic equation of the periodic problem) and the eigenvalues of the semi-periodic problem (1.3) coincide with the roots of $F(\lambda) + 2 = 0$.

Each of the problems (1.2) and (1.3) has a countably infinity real eigenvalues with the accumulation point at $+\infty$. It is clear that $\lambda = 0$ is a simple eigenvalue with eigenfunction $y \equiv 1$ for the periodic problem (1.2). Denote by

$$0 = \mu_0 < \mu_2^- \leq \mu_2^+ < \mu_4^- \leq \mu_4^+ < \dots < \mu_{2n}^- \leq \mu_{2n}^+ < \dots$$

the eigenvalues of periodic problem (1.2), and by

$$\mu_1^- \leq \mu_1^+ < \mu_3^- \leq \mu_3^+ < \dots < \mu_{2n+1}^- \leq \mu_{2n+1}^+ < \dots$$

the eigenvalues of semi-periodic problem (1.3) (the equality holds in the case of double eigenvalue). These values occur in the order

$$0 = \mu_0 < \mu_1^- \leq \mu_1^+ < \mu_2^- \leq \mu_2^+ < \mu_3^- \leq \mu_3^+ < \mu_4^- \leq \mu_4^+ \dots \tag{1.6}$$

If λ lies in any of the open intervals $(-\infty, 0)$ and (μ_n^-, μ_n^+) ($n = 1, 2, 3, \dots$), then all non-trivial solutions of (1.1) are unbounded in $(-\infty, \infty)$. These intervals are called the instability intervals of the equation (1.1). Apart from $(-\infty, 0)$, some or all the instability intervals will be absent in the case of double eigenvalues. If λ lies in any of the open complementary intervals (μ_{n-1}^+, μ_n^-) ($n = 1, 2, 3, \dots, \mu_0^+ = 0$), then all solutions of (1.1) are bounded in $(-\infty, \infty)$, and these intervals are called the stability intervals of (1.1).

In general, when λ is an eigenvalue of the periodic problem (1.2) only one of the two independent solutions of the differential equation (1.1) is periodic (the other linearly independent solution may grow only as a linear function), but if λ is a double

eigenvalue (it is all the same that λ is a double root of the characteristic equation) two and therefore all solutions will be periodic. In this case two periodic solutions are said to coexist. Similarly for semi-periodic solutions. The coexistence problem is the problem of finding a condition for such coexistence.

It is of considerable interest to study the asymptotic behaviour of the lengths

$$I_n = \mu_n^+ - \mu_n^- \quad (n = 1, 2, 3, \dots) \tag{1.7}$$

of the instability intervals (μ_n^-, μ_n^+) as $n \rightarrow \infty$. For this purpose the eigenvalues μ_n^\pm of the periodic and semi-periodic boundary value problems (1.2) and (1.3) must be investigated.

It is known that $I_n \rightarrow 0$ as $n \rightarrow \infty$ if $\rho(x)$ is twice differentiable function and $\rho''(x) \in L^1[0, \omega]$. Besides, the velocity of I_n tending to zero grows as the differentiability condition on $\rho(x)$ grows. However, if $\rho(x)$ is a discontinuous function, then I_n may increase beyond all bounds as $n \rightarrow \infty$.

In this paper we study the case in which the coefficient $\rho(x)$ of (1.1) is a discontinuous but piecewise constant function (a step-function):

$$\rho(x) = \begin{cases} \alpha^2, & 0 < x \leq a, \\ \beta^2, & a < x \leq \omega, \end{cases} \tag{1.8}$$

$\alpha > 0, \beta > 0$ are constants and a is a fixed point between 0 and ω . Over the whole axis $-\infty < x < \infty$, $\rho(x)$ is to be continued as a periodic function of period ω .

In the case of an even piecewise constant function $\rho(x)$ the equation (1.1) was studied earlier in [3]. In our case the function $\rho(x)$ is not even. Besides, our method of investigation in this paper is different from that applied in [3], and is based on using Rouché theorem about the roots of the analytical functions.

2. Hill's Discriminant

In equation (1.1) we put $\lambda = s^2$ and assume that the coefficient $\rho(x)$ has the form (1.8). Then it is easy to show that the solutions $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ of (1.1) satisfying the initial conditions (1.4) have the form

$$\theta(x, \lambda) = \begin{cases} \cos s\alpha x, & 0 \leq x \leq a, \\ \cos s\alpha a \cos s\beta(x - a) - \frac{\alpha}{\beta} \sin s\alpha a \sin s\beta(x - a), & a < x \leq \omega, \end{cases}$$

$$\varphi(x, \lambda) = \begin{cases} \frac{\sin s\alpha x}{s\alpha}, & 0 \leq x \leq a \\ \frac{\sin s\alpha a}{s\alpha} \cos s\beta(x - a) + \cos s\alpha a \frac{\sin s\beta(x - a)}{s\beta}, & a < x \leq \omega. \end{cases}$$

Therefore, according to (1.5) we find the following explicit formula for the Hill discriminant:

$$F(\lambda) = A \cos s\delta + B \cos s\gamma, \tag{2.1}$$

where

$$\begin{aligned} A &= 1 + \frac{1}{2} \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \right), & B &= 1 - \frac{1}{2} \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \right), \\ \delta &= \alpha a + \beta(\omega - a), & \gamma &= \alpha a - \beta(\omega - a). \end{aligned} \tag{2.2}$$

3. Some Particular Cases

In this section we point out some particular cases in which the eigenvalues of the periodic and the semi-periodic boundary value problems can be found explicitly.

1. If $\alpha = \beta$, then according to (2.2) and (2.1) we have $A = 2$, $B = 0$, $\delta = \alpha\omega$, $\gamma = \alpha(2a - \omega)$, and $F(\lambda) = 2 \cos s\alpha\omega$. Therefore,

$$\begin{aligned} F(\lambda) - 2 &= 4 \sin^2 \frac{s\alpha\omega}{2} = 0 \Leftrightarrow s = \frac{2n\pi}{\alpha\omega}, \quad n = 0, \pm 1, \pm 2, \dots; \\ F(\lambda) + 2 &= 4 \cos^2 \frac{s\alpha\omega}{2} = 0 \Leftrightarrow s = \frac{(2n+1)\pi}{\alpha\omega}, \quad n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

Besides, each λ -root, except for $\lambda = 0$, is double. Consequently,

$$\mu_0 = 0, \quad \mu_n^- = \mu_n^+ = \left(\frac{n\pi}{\alpha\omega} \right)^2, \quad n = 1, 2, 3, \dots,$$

and by (1.7) $I_n = 0$, $n = 1, 2, 3, \dots$. The single instability interval is $(-\infty, 0)$.

2. Let us now set $\alpha \neq \beta$. Then from (2.2) it follows that

$$A > 2, \quad B < 0, \quad A + B = 2, \quad \delta > 0, \quad -\infty < \gamma < \infty, \quad \left| \frac{B}{A} \right| < 1, \quad \left| \frac{\gamma}{\delta} \right| < 1. \tag{3.1}$$

(i) Let $\gamma = 0$, that is $\alpha = \beta \frac{\omega - a}{a}$ (and $a \neq \frac{\omega}{2}$, since $\alpha \neq \beta$). According to (2.1) in this case $F(\lambda) = A \cos s\delta + B$ and therefore

$$F(\lambda) - 2 = -2A \sin^2 \frac{s\delta}{2} = 0 \Leftrightarrow s = \frac{2n\pi}{\delta}, \quad n = 0, \pm 1, \pm 2, \dots,$$

$$F(\lambda) + 2 = A \cos s\delta + 4 - A = 0 \Leftrightarrow s = \frac{1}{\delta} \left(\pm \cos^{-1} \frac{A-4}{A} + 2n\pi \right), \quad n = 0, \pm 1, \pm 2, \dots$$

Consequently, taking into account the inequalities (1.6), we have

$$\begin{aligned} \mu_0 &= 0, \quad \mu_{2n}^- = \mu_{2n}^+ = \left(\frac{2n\pi}{\delta} \right)^2, \quad n = 1, 2, 3, \dots; \\ \mu_{2n+1}^- &= \left(\frac{2n\pi + c}{\delta} \right)^2, \quad \mu_{2n+1}^+ = \left(\frac{(2n+2)\pi - c}{\delta} \right)^2, \quad n = 0, 1, 2, \dots, \end{aligned}$$

where $c = \cos^{-1} \frac{A-4}{A}$ and $0 < c < \pi$. Hence

$$I_{2n} = 0, \quad n = 1, 2, 3, \dots; \quad I_{2n+1} = \frac{4(\pi - c)}{\delta^2} \cdot (2n + 1)\pi, \quad n = 0, 1, 2, \dots$$

We see that $I_{2n+1} \rightarrow \infty$ as $n \rightarrow \infty$.

(ii) Let $\gamma = \frac{\delta}{2}$, that is $\alpha = \beta \cdot \frac{3(\omega-a)}{a}$ (and $a \neq \frac{3}{4}\omega$, since $\alpha \neq \beta$). In this case we have, from (2.1),

$$F(\lambda) = 2A \cos^2 \frac{s\delta}{2} + B \cos \frac{s\delta}{2} - A.$$

Therefore,

$$F(\lambda) - 2 = 0 \Leftrightarrow \cos \frac{s\delta}{2} = 1 \quad \text{or} \quad \cos \frac{s\delta}{2} = -\frac{2+A}{2A}.$$

Hence

$$\frac{s\delta}{2} = 2n\pi \quad \text{or} \quad \frac{s\delta}{2} = \pm d + (2n + 1)\pi \quad (n = 0, \pm 1, \pm 2, \dots),$$

where $d = \cos^{-1} \frac{2+A}{2A}$ and $0 < d < \frac{\pi}{2}$. Consequently,

$$\mu_0 = 0, \quad \mu_{4n}^- = \mu_{4n}^+ = \left(\frac{4n\pi}{\delta}\right)^2, \quad n = 1, 2, 3, \dots, \tag{3.2}$$

$$\mu_{4n+2}^- = \left(\frac{(4n+2)\pi + 2d}{\delta}\right)^2, \quad \mu_{4n+2}^+ = \left(\frac{(4n+2)\pi - 2d}{\delta}\right)^2, \quad n = 0, 1, 2, \dots \tag{3.3}$$

Further,

$$F(\lambda) + 2 = 0 \Leftrightarrow \cos \frac{s\delta}{2} = \frac{A - 2 - \sqrt{(A - 2)^2 + 8A(A - 2)}}{4A}$$

or

$$\cos \frac{s\delta}{2} = \frac{A - 2 + \sqrt{(A - 2)^2 + 8A(A - 2)}}{4A}.$$

Hence

$$\frac{s\delta}{2} = \pm d_1 + (2n + 1)\pi \quad \text{or} \quad \frac{s\delta}{2} = \pm d_2 + 2n\pi \quad (n = 0, \pm 1, \pm 2, \dots),$$

where

$$d_1 = \cos^{-1} \left(-\frac{A - 2 - \sqrt{(A - 2)^2 + 8A(A - 2)}}{4A} \right), \quad 0 < d_1 < \frac{\pi}{2},$$

$$d_2 = \cos^{-1} \left(\frac{A - 2 + \sqrt{(A - 2)^2 + 8A(A - 2)}}{4A} \right), \quad 0 < d_2 < \frac{\pi}{2}.$$

We note that $d_2 < d_1$ and $d < d_1$. Thus, taking into account (3.2), (3.3), and the inequalities (1.6), we find

$$\mu_{4n+1}^- = \left(\frac{4n\pi + 2d_2}{\delta}\right)^2, \quad \mu_{4n+1}^+ = \left(\frac{(4n+2)\pi - 2d_1}{\delta}\right)^2, \quad n = 0, 1, 2, \dots, \quad (3.4)$$

$$\mu_{4n+3}^- = \frac{[(4n+2)\pi + 2d_1]^2}{\delta^2}, \quad \mu_{4n+3}^+ = \frac{[(4n+4)\pi - 2d_2]^2}{\delta^2}, \quad n = 0, 1, 2, \dots \quad (3.5)$$

In view of (3.2), (3.3), (3.4), and (3.5), we have

$$I_{4n} = 0, \quad n = 1, 2, 3, \dots; \quad I_{4n+2} = \frac{16d}{\delta^2} \cdot (2n+1), \quad n = 0, 1, 2, \dots,$$

$$I_{4n+1} = \frac{4(\pi - d_1 - d_2)}{\delta^2} \cdot (4n\pi + \pi - d_1 + d_2),$$

$$I_{4n+3} = \frac{4(\pi - d_1 - d_2)}{\delta^2} \cdot (4n\pi + 3\pi - d_2 + d_1), \quad n = 0, 1, 2, \dots$$

We see that $I_{4n+1} \rightarrow \infty$, $I_{4n+2} \rightarrow \infty$, $I_{4n+3} \rightarrow \infty$ as $n \rightarrow \infty$.

(iii) Finally, let $\gamma = \frac{\delta}{3}$, that is $\alpha = \beta \cdot \frac{2(\omega-a)}{a}$ (and $a \neq \frac{2}{3}\omega$, since $\alpha \neq \beta$). In this case we have, by (2.1),

$$F(\lambda) = 4A \cos^3 \frac{s\delta}{3} + (B - 3A) \cos \frac{s\delta}{3}.$$

Therefore,

$$F(\lambda) - 2 = 0 \Leftrightarrow \cos \frac{s\delta}{3} = 1 \quad \text{or} \quad \cos \frac{s\delta}{3} = \frac{-A - \sqrt{A(A-2)}}{2A}$$

or

$$\cos \frac{s\delta}{3} = \frac{-A + \sqrt{A(A-2)}}{2A}.$$

Hence

$$\frac{s\delta}{3} = 2n\pi \quad \text{or} \quad \frac{s\delta}{3} = \pm d_3 + (2n+1)\pi \quad \text{or} \quad \frac{s\delta}{3} = \pm d_4 + (2n+1)\pi,$$

where $n = 0, \pm 1, \pm 2, \dots$ and

$$d_3 = \cos^{-1} \frac{A + \sqrt{A(A-2)}}{2A}, \quad d_4 = \cos^{-1} \frac{A - \sqrt{A(A-2)}}{2A},$$

$$0 < d_3 < \frac{\pi}{3} < d_4 < \frac{\pi}{2}.$$

Further, $F(\lambda) + 2 = 0 \Leftrightarrow$

$$\cos \frac{s\delta}{3} = -1 \quad \text{or} \quad \cos \frac{s\delta}{3} = \frac{-A + \sqrt{A(A-2)}}{2A} \quad \text{or} \quad \cos \frac{s\delta}{3} = \frac{A - \sqrt{A(A-2)}}{2A}.$$

Hence

$$\frac{s\delta}{3} = (2n + 1)\pi \quad \text{or} \quad \frac{s\delta}{3} = \pm d_3 + 2n\pi \quad \text{or} \quad \frac{s\delta}{3} = \pm d_4 + 2n\pi,$$

where $n = 0, \pm 1, \pm 2, \dots$.

From these results it follows that

$$\begin{aligned} \mu_0 &= 0, \quad \mu_{6n}^- = \mu_{6n}^+ = \left(\frac{6n\pi}{\delta}\right)^2, \quad n = 1, 2, 3, \dots; \\ \mu_{6n+1}^- &= \left(\frac{6n\pi+3d_3}{\delta}\right)^2, \quad \mu_{6n+1}^+ = \left(\frac{6n\pi+3d_4}{\delta}\right)^2, \quad n = 0, 1, 2, \dots; \\ \mu_{6n+2}^- &= \left(\frac{(6n+3)\pi-3d_4}{\delta}\right)^2, \quad \mu_{6n+2}^+ = \left(\frac{(6n+3)\pi-3d_3}{\delta}\right)^2, \quad n = 0, 1, 2, \dots; \\ \mu_{6n+3}^- &= \mu_{6n+3}^+ = \left(\frac{(6n+3)\pi}{\delta}\right)^2, \quad n = 0, 1, 2, \dots; \\ \mu_{6n+4}^- &= \left(\frac{(6n+3)\pi+3d_3}{\delta}\right)^2, \quad \mu_{6n+4}^+ = \left(\frac{(6n+3)\pi+3d_4}{\delta}\right)^2, \quad n = 0, 1, 2, \dots; \\ \mu_{6n+5}^- &= \left(\frac{(6n+6)\pi-3d_4}{\delta}\right)^2, \quad \mu_{6n+5}^+ = \left(\frac{(6n+6)\pi-3d_3}{\delta}\right)^2, \quad n = 0, 1, 2, \dots \end{aligned}$$

Therefore, $I_{6n} = 0$, $n = 1, 2, 3, \dots$; $I_{6n+3} = 0$, $n = 0, 1, 2, \dots$; and $I_{6n+j} \rightarrow \infty$ as $n \rightarrow \infty$, $j = 1, 2, 4, 5$.

4. General Case

Now we consider the case of arbitrary values of α, β, a and ω under the only condition $\alpha \neq \beta$. We investigate the roots of the functions $F(\lambda) - 2$ and $F(\lambda) + 2$.

From (2.1), we have

$$\Phi^+(\lambda) \stackrel{def}{=} F(\lambda) - 2 = -2A \sin^2 \frac{s\delta}{2} - 2B \sin^2 \frac{s\gamma}{2}, \tag{4.1}$$

where $\lambda = s^2$. Setting

$$\frac{s\delta}{2} = \pi z \Rightarrow s = \frac{2\pi}{\delta} z \quad \text{and} \quad \lambda = s^2 = \left(\frac{2\pi}{\delta}\right)^2 \cdot z^2, \tag{4.2}$$

we get

$$\Phi^+(\lambda) = \Phi^+ \left(\frac{4\pi^2}{\delta^2} z^2 \right) = \Phi_1^+(z), \tag{4.3}$$

where

$$\Phi_1^+(z) = -2A \sin^2 \pi z - 2B \sin^2 \frac{\gamma}{\delta} \pi z. \tag{4.4}$$

Further, for any natural number n define the square contour

$$\Gamma_n = \left\{ z \in \mathbb{C} : |\operatorname{Re} z| = n + \frac{1}{2}, \quad |\operatorname{Im} z| = n + \frac{1}{2} \right\}.$$

We will use the following well-known theorem (see, for example, [5, p.116]).

Rouché's Theorem. *If $f(z)$ and $g(z)$ are analytic functions inside and on a closed contour Γ , and $|g(z)| < |f(z)|$ on Γ , then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside Γ .*

We apply the Rouché theorem putting

$$\Gamma = \Gamma_n, \quad f(z) = -2A \sin^2 \pi z, \quad g(z) = -2B \sin^2 \frac{\gamma}{\delta} \pi z.$$

Thus $\Phi_1^+(z) = f(z) + g(z)$ and the inequality $|g(z)| < |f(z)|$ is equivalent to the inequality

$$\left| \frac{B}{A} \right| \cdot \left| \frac{\sin \frac{\gamma}{\delta} \pi z}{\sin \pi z} \right|^2 < 1. \tag{4.5}$$

Lemma 4.1. There exists a natural number n_0 such that

$$\left| \frac{\sin \frac{\gamma}{\delta} \pi z}{\sin \pi z} \right| \leq 1 \quad (\forall z \in \Gamma_n, \forall n \geq n_0). \tag{4.6}$$

Proof. On the vertical sides of Γ_n we have $z = \sigma + i\tau$, $\sigma = \pm(n + \frac{1}{2})$ and so

$$|\sin \pi z| = \cosh \pi \tau, \quad \left| \sin \frac{\gamma}{\delta} \pi z \right| \leq \cosh \left| \frac{\gamma}{\delta} \right| \pi \tau \leq \cosh \pi \tau,$$

since $|\frac{\gamma}{\delta}| < 1$ in view of (3.1). Therefore, the inequality (4.6) holds on the vertical sides of Γ_n for all n .

Now we consider the horizontal sides of Γ_n . Since, for each $z = \sigma + i\tau$,

$$|\sin \pi z|^2 = \frac{1}{4} (e^{-2\tau\pi} + e^{2\tau\pi} - 2 \cos 2\sigma\pi) \geq \frac{1}{4} (e^{-\pi\tau} - e^{\pi\tau})^2$$

and hence

$$|\sin \pi z| \geq \frac{1}{2} e^{\pi|\tau|} (1 - e^{-2\pi|\tau|}),$$

we have for $\tau = \pm(n + \frac{1}{2})$,

$$|\sin \pi z| > \frac{1}{4} e^{\pi|\tau|}.$$

Therefore for $\tau = \pm(n + \frac{1}{2})$,

$$\left| \frac{\sin \frac{\gamma}{\delta} \pi z}{\sin \pi z} \right| \leq \frac{8e^{\pi|\frac{\gamma}{\delta}\tau|}}{e^{\pi|\tau|}} = 8e^{-\pi(1-|\frac{\gamma}{\delta}|)|\tau|} \rightarrow 0 \tag{4.7}$$

as $n \rightarrow \infty$, since $|\frac{\gamma}{\delta}| < 1$. Consequently, there exists a natural number n_0 such that the inequality (4.6) holds for each $z = \sigma + i\tau$ if $\tau = \pm(n + \frac{1}{2})$ and $n \geq n_0$. The lemma is proved. \square

Remark 4.2. It follows from (4.7) that as n_0 in Lemma 4.1 we can take

$$n_0 = \left[\frac{3 \log 2}{\pi (1 - |\frac{\gamma}{\delta}|)} \right] + 1, \tag{4.8}$$

where $[\cdot]$ denotes the integral part.

Since $|\frac{B}{A}| < 1$ in view of (3.1), it follows from Lemma 4.1 that (4.5) holds for $z \in \Gamma_n$, $n \geq n_0$. Applying the Rouché theorem we conclude that for $n \geq n_0$ the number of roots of $\Phi_1^+(z)$ lying inside Γ_n is the same as that for $\sin^2 \pi z$. The latter function has roots inside Γ_n at the points $z = 0, \pm 1, \pm 2, \dots, \pm n$ and each of them is a double root. Therefore, $\Phi_1^+(z)$ must have $4n + 2$ roots inside Γ_n ($n \geq n_0$). Since, by (4.4), $\Phi_1^+(z)$ is an even function and $z = 0$ is a double root, we can denote the roots of $\Phi_1^+(z)$ lying inside Γ_n by

$$-z_{2n}^+, -z_{2n}^-, \dots, -z_2^+, -z_2^-, 0, 0, z_2^-, z_2^+, z_4^-, z_4^+, \dots, z_{2n}^-, z_{2n}^+.$$

We note that the roots of $\Phi_1^+(z)$ are real in virtue of (4.2) and (4.3), since the eigenvalues of the periodic boundary value problem (1.2) are non-negative.

Lemma 4.3. For $n \geq n_0 + 1$ the function $\Phi_1^+(z)$ has exactly two roots in the region

$$D_n = \left\{ z \in \mathbb{C} : n - \frac{1}{2} < \operatorname{Re} z < n + \frac{1}{2}, \quad |\operatorname{Im} z| < n + \frac{1}{2} \right\}.$$

Therefore, the roots z_{2n}^- and z_{2n}^+ of $\Phi_1^+(z)$ lie in the interval $(n - \frac{1}{2}, n + \frac{1}{2})$:

$$n - \frac{1}{2} < z_{2n}^\pm < n + \frac{1}{2} \quad (n \geq n_0 + 1). \tag{4.9}$$

Proof. It is evident from the proof of Lemma 4.1 that the inequality (4.5) holds also on the boundary of the region D_n if $n \geq n_0 + 1$. Consequently, by the Rouché theorem the function $\Phi_1^+(z)$ has in the D_n as many roots as the function $\sin^2 \pi z$, i.e. exactly two roots (the function $\sin^2 \pi z$ has a double root inside D_n at the point $z = n$). The lemma is proved. \square

Putting $s_{2n}^\pm = \frac{2\pi}{\delta} \cdot z_{2n}^\pm$ we have, by (4.9),

$$s_{2n}^\pm = \frac{2n\pi + 2h_{2n}^\pm}{\delta}, \quad -\frac{\pi}{2} < h_{2n}^+ < \frac{\pi}{2} \quad (n \geq n_0 + 1).$$

To analyze the quantities h_{2n}^\pm we note that the numbers s_{2n}^\pm are the roots of the function (4.1). Therefore, h_{2n}^- and h_{2n}^+ are the roots of the equation

$$A \sin^2 t + B \sin^2 \frac{\gamma}{\delta} (n\pi + t) = 0,$$

lying in the interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Since $B = 2 - A$ and $A > 2$, the latter equation is equivalent to the union of equations

$$\sin t = \left| \sin \frac{\gamma}{\delta}(n\pi + t) \right| \cdot \sqrt{\frac{A-2}{A}} \tag{4.10}$$

and

$$\sin t = - \left| \sin \frac{\gamma}{\delta}(n\pi + t) \right| \cdot \sqrt{\frac{A-2}{A}}. \tag{4.11}$$

It is clear that each root of (4.10) lying in $(-\frac{\pi}{2}, \frac{\pi}{2})$ is non-negative and each root of (4.11) lying in $(-\frac{\pi}{2}, \frac{\pi}{2})$ is non-positive.

Lemma 4.4. For each $n \geq n_0 + 1$ the equation (4.10) has exactly one root in $[0, \frac{\pi}{2})$ and the equation (4.11) has exactly one root in $(-\frac{\pi}{2}, 0]$.

Proof. The function

$$f(t) = \sin t - \left| \sin \frac{\gamma}{\delta}(n\pi + t) \right| \cdot \sqrt{\frac{A-2}{A}}$$

is continuous on the segment $0 \leq t \leq \frac{\pi}{2}$ and besides $f(0) \leq 0$, $f(\frac{\pi}{2}) > 0$. Therefore, $f(t)$ vanishes at least at one point in $[0, \frac{\pi}{2})$.

Similarly it follows that the equation (4.11) has at least one root in $(-\frac{\pi}{2}, 0]$.

In order that one of the equations (4.10) and (4.11) have more than one root, it is necessary that the function $\Phi_1^+(z)$ have more than two roots in the region D_n . But the latter is impossible in view of Lemma 4.3. The lemma is proved. \square

Denote the root of the equation (4.10) in $[0, \frac{\pi}{2})$ by r_{2n}^+ and the root of the equation (4.11) in $(-\frac{\pi}{2}, 0]$ by $-r_{2n}^-$. Then $0 \leq r_{2n}^\pm < \frac{\pi}{2}$ and $h_{2n}^+ = r_{2n}^+$, $h_{2n}^- = -r_{2n}^-$. Thus, the eigenvalues of the periodic problem have the form

$$\mu_0 = 0, \quad \mu_{2n}^\pm = (s_{2n}^\pm)^2 = \left(\frac{2n\pi \pm 2r_{2n}^\pm}{\delta} \right)^2, \quad n = 1, 2, 3, \dots,$$

where $0 \leq r_{2n}^\pm < \frac{\pi}{2}$ ($n \geq n_0 + 1$) and

$$\sin r_{2n}^\pm = \left| \sin \frac{\gamma}{\delta}(n\pi \pm r_{2n}^\pm) \right| \cdot \sqrt{\frac{A-2}{A}}, \quad n = 1, 2, 3, \dots \tag{4.12}$$

We can analogously investigate the eigenvalues of the semi-periodic problem, that is the roots of the function

$$\Phi^-(\lambda) \stackrel{def}{=} F(\lambda) + 2 = 2A \cos^2 \frac{s\delta}{2} + 2B \cos^2 \frac{s\gamma}{2},$$

where $\lambda = s^2$. Setting $s\delta = 2\pi z$ we get

$$\Phi^-(\lambda) = \Phi^-\left(\frac{4\pi^2}{\delta^2} z^2\right) = \Phi_1^-(z),$$

where

$$\Phi_1^-(z) = 2A \cos^2 \pi z + 2B \cos^2 \frac{\gamma}{\delta} \pi z.$$

Applying the Rouché theorem to the function $\Phi_1^-(z) = f(z) + g(z)$ and the contour

$$\Gamma_n^- = \{z \in \mathbb{C} : |\operatorname{Re} z| = n + 1, |\operatorname{Im} z| = n + 1\},$$

setting $f(z) = 2A \cos^2 \pi z$, $g(z) = 2B \cos^2 \frac{\gamma}{\delta} \pi z$, we conclude that $\Phi_1^-(z)$ and $\cos^2 \pi z$ have the same number of zeros inside Γ_n^- if $n \geq n_0$, where n_0 is a sufficiently large number (as n_0 we can take the integer defined by (4.8)). Since the function $\cos^2 \pi z$ has double roots inside Γ_n^- at the points $z = \frac{1}{2} + m$, where $m = 0, \pm 1, \pm 2, \dots, \pm n, -n - 1$, the function $\Phi_1^-(z)$ will have $4n + 4$ zeros inside Γ_n^- . Denote them by

$$-z_{2n+1}^+, -z_{2n+1}^-, \dots, -z_1^+, -z_1^-, z_1^-, z_1^+, z_3^-, z_3^+, \dots, z_{2n+1}^-, z_{2n+1}^+.$$

Further, setting $s_{2n+1}^\pm = \frac{2\pi}{\delta} \cdot z_{2n+1}^\pm$ we can for the eigenvalues $\mu_{2n+1}^\pm = (s_{2n+1}^\pm)^2$, $n = 0, 1, 2, \dots$ of the semi-periodic problem obtain the formula

$$\mu_{2n+1}^\pm = \left(\frac{(2n+1)\pi \pm 2r_{2n+1}^\pm}{\delta} \right)^2, \quad n = 0, 1, 2, \dots,$$

where $0 \leq r_{2n+1}^\pm < \frac{\pi}{2}$ ($n \geq n_0 + 1$) and

$$\sin r_{2n+1}^\pm = \left| \cos \frac{\gamma}{\delta} \left(n\pi + \frac{\pi}{2} \pm r_{2n+1}^\pm \right) \right| \cdot \sqrt{\frac{A-2}{A}}, \quad n = 0, 1, 2, \dots \quad (4.13)$$

Thus, we have proved the following theorem:

Theorem 4.5. *The eigenvalues $\{\mu_0, \mu_{2n}^\pm\}$ of the periodic and the eigenvalues $\{\mu_{2n+1}^\pm\}$ of the semi-periodic boundary value problems have the form*

$$\mu_0 = 0, \quad \mu_n^\pm = \left(\frac{n\pi \pm 2r_n^\pm}{\delta} \right)^2, \quad n = 1, 2, 3, \dots, \quad (4.14)$$

where $0 \leq r_n^\pm < \frac{\pi}{2}$ ($n \geq n_0 + 1$) and the equations (4.12) and (4.13) hold.

Corollary 4.6. *For the length $I_n = \mu_n^+ - \mu_n^-$ of the instability interval (μ_n^-, μ_n^+) the formula*

$$I_n = \frac{4n\pi}{\delta} (r_n^+ + r_n^-) + \frac{4}{\delta^2} (r_n^{+2} - r_n^{-2}) \quad (4.15)$$

holds, where r_n^\pm is the same as in Theorem 4.5.

Theorem 4.7. *If $\alpha \neq \beta$, then the length I_n (if in addition $\gamma \neq 0$, then, what is more, the lengths I_{2n} and the lengths I_{2n+1}) are unbounded as $n \rightarrow \infty$.*

Proof. In Section 3 had been shown the unboundedness of I_{2n+1} in the case of $\gamma = 0$. Let now $\gamma \neq 0$. In view of (4.15) it will be sufficient to show that the sequences $\{r_{2n}^+\}$ and $\{r_{2n+1}^+\}$ do not tend to zero as $n \rightarrow \infty$. Let us assume the contrary: let $r_{2n}^+ \rightarrow 0$ as $n \rightarrow \infty$. Then the equation

$$\sin r_{2n}^+ = \left| \sin \frac{\gamma}{\delta} n\pi \cos \frac{\gamma}{\delta} r_{2n}^+ + \cos \frac{\gamma}{\delta} n\pi \sin \frac{\gamma}{\delta} r_{2n}^+ \right| \cdot \sqrt{\frac{A-2}{A}},$$

arising from (4.12), gives $\lim_{n \rightarrow \infty} \sin \frac{\gamma}{\delta} n\pi = 0$. Using this we obtain from the identity

$$\sin \frac{\gamma}{\delta} (n+1)\pi = \sin \frac{\gamma}{\delta} n\pi \cos \frac{\gamma}{\delta} \pi + \cos \frac{\gamma}{\delta} n\pi \sin \frac{\gamma}{\delta} \pi$$

also $\lim_{n \rightarrow \infty} \cos \frac{\gamma}{\delta} n\pi = 0$. Thus, we arrive at a contradiction, since $\sin^2 \frac{\gamma}{\delta} n\pi + \cos^2 \frac{\gamma}{\delta} n\pi = 1, \forall n$.

Similarly, with the help of equation (4.13), it can be shown that the sequence $\{r_{2n+1}^+\}$ does not tend to zero as $n \rightarrow \infty$. The theorem is proved. \square

5. Coexistence

Let $\frac{\gamma}{\delta}$ be a rational number: $\frac{\gamma}{\delta} = \frac{p}{q}$, where p and q are relatively prime integers, and $q > 0$. Since $|\frac{\gamma}{\delta}| < 1$ we can assume that $0 \leq |p| < q$.

Theorem 5.1. (i) *If $p = 0$ (i.e., if $\gamma = 0$), then*

$$\mu_{2n}^- = \mu_{2n}^+, \quad n = 1, 2, 3, \dots$$

(ii) *If $p \neq 0$, then*

$$\mu_{2kq}^- = \mu_{2kq}^+, \quad k = 1, 2, 3, \dots \tag{5.1}$$

(iii) *If both p and q are odd, then, in addition to (5.1),*

$$\mu_{(2k+1)q}^- = \mu_{(2k+1)q}^+, \quad k = 0, 1, 2, \dots$$

Proof. Statement (i) has been proved in Section 3. Notice that it follows also from (4.12): if $\gamma = 0$, we have $\sin r_{2n}^\pm = 0$, and hence $r_{2n}^- = r_{2n}^+ = 0 \quad (n = 1, 2, 3, \dots)$.

To prove (ii) we put in (4.12) $n = kq \quad (k = 1, 2, \dots)$. Then we get

$$\sin r_{2kq}^\pm = \left| \sin \frac{p}{q} r_{2kq}^\pm \right| \cdot \sqrt{\frac{A-2}{A}}.$$

Hence $r_{2kq}^- = r_{2kq}^+ = 0$ and (5.1) follows from (4.14).

Let us now to prove (iii). Setting $q = 2m + 1$ and $n = kq + m$ ($k = 0, 1, 2, \dots$) we get, from (4.13),

$$\sin r_{(2k+1)q}^\pm = \left| \cos \left[p(2k+1) \frac{\pi}{2} \pm \frac{p}{q} r_{(2k+1)q}^\pm \right] \right| \cdot \sqrt{\frac{A-2}{A}}.$$

Hence (since p is odd)

$$\sin r_{(2k+1)q}^\pm = \left| \sin \frac{p}{q} r_{(2k+1)q}^\pm \right| \cdot \sqrt{\frac{A-2}{A}}$$

and therefore, $r_{(2k+1)q}^- = r_{(2k+1)q}^+ = 0$. The theorem is proved. \square

Theorem 5.2. *If the periodic problem has at least one double eigenvalue, then $\frac{\gamma}{\delta}$ is rational. If the semi-periodic problem has at least one double eigenvalue, then $\frac{\gamma}{\delta}$ is a ratio of two odd integers.*

Proof. Let the periodic problem have at least one double eigenvalue. Then for some $n \in \{1, 2, \dots\}$ we will have $r_{2n}^- = r_{2n}^+ = 0$ and from (4.12) we will then get $\sin \frac{\gamma}{\delta} n\pi = 0$. Hence, $\frac{\gamma}{\delta} n\pi = m\pi$ for some integer m . Therefore, $\frac{\gamma}{\delta} = \frac{m}{n}$ is rational.

Let us now assume the semi-periodic problem has at least one double eigenvalue. Then for some $n \in \{1, 2, \dots\}$ we will have $r_{2n+1}^- = r_{2n+1}^+ = 0$ and from (4.13) we will then get $\cos \frac{\gamma}{\delta} (2n+1) \cdot \frac{\pi}{2} = 0$. Hence $\frac{\gamma}{\delta} (2n+1) \cdot \frac{\pi}{2} = (2m+1) \cdot \frac{\pi}{2}$ for some integer m . Therefore $\frac{\gamma}{\delta} = \frac{2m+1}{2n+1}$ is a ratio of the two odd integers. The theorem is proved. \square

Comparing of Theorem 5.1 and Theorem 5.2 gives the following corollary.

Corollary 5.3. *In the case of $\rho(x)$ of the form (1.8), if the periodic or the semi-periodic problem has at least one double eigenvalue, then this problem has infinitely many double eigenvalues.*

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