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IMMERSIONS PRESERVED UNDER ROTATIONS WITH TOTALLY REDUCIBLE FOCAL SET

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Abstract

In [1] Carter and the author introduced the idea of an immersion $f : M^m \to R^n$ with totally reducible focal set (TRFS). Such an immersion has the property that, for all $p \in M$, the focal set with base p is a union of hyperplanes in the normal plane to $f(M)$ at $f(p)$. Here we show that if we take two immersions with TRFS then we can construct new immersions with TRFS. In particular, rotating an immersion with TRFS about an axis gives a new immersion with TRFS.

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1. Introduction

Let $f : M \to R^n$ be a smooth immersion of a connected smooth (C^∞) m-dimensional manifold without boundary into Euclidean n-space. For each $p \in M$, the focal set of f with base p is an algebraic variety. In this paper we consider immersions for which this variety is a union of hyperplanes. Trivially, this always holds if $n = m + 1$ so we only consider $n > m + 1$. In the section 2 we give a known definition and results and in the section 3 we shall give very interesting constructions of immersions with totally reducible focal set (TRFS).

2. Definition and Basic Properties of TRFS

For $p \in M$, let U be a neighbourhood of p in M such that $f|U : U \to R^n$ is an embedding and let $v_f(p)$ denote the $(n-m)$-plane which is normal to $f(U)$ at $f(p)$. Then the total space of the normal bundle is $N_f = \{(p,x) \in M \times R^n | x \in v_f(p)\}$. The projection map $\eta_f : N_f \to R^n$ is defined by $\eta_f(p,x) = x$ and the set of focal points with base p is $\Gamma_f(p) = \{x \in R^n | (p,x)$ is a singularity of $\eta_f\}$. For each $p \in M$, $\Gamma_f(p)$ is a real algebraic variety in $v_f(p)$ which can be defined as the zeros of polynomial on $v_f(p)$ of degree $\leq m$.

Definition 1. *The immersion $f : M \to R^n$ has totally reducible focal set (TRFS) if, for all $p \in M$, $\Gamma_f(p)$ can be defined as the zeros of a real polynomial which is a product of real linear factors.*

So each irreducible component of $\Gamma_f(p)$ is an affine plane in $v_f(p)$, and $\Gamma_f(p)$ is a union of $(n-m-1)$-planes (possible $\Gamma_f(p) = \phi$).

Examples of embeddings $f : M \to R^n$ with TRFS are embeddings whose image is an isoparametric submanifolds [2, 6]. For these examples the pattern of the planes of $\Gamma_f(p)$ in $v_f(p)$ is the same for all $p \in M$, i.e. for all $p, q \in M$, $\Gamma_f(p)$ is isometric to $\Gamma_f(q)$. In general, for an immersion with TRFS the pattern of planes of $\Gamma_f(p)$ will vary with p. It is shown in [5, 7] that f has TRFS if and only if f has flat normal bundle, where M is thought of as a Riemannian manifold with metric g induced from R^n. We will give explicit ways of constructing immersions and embeddings with TRFS.

In calculating focal sets it is often easiest to work with distance functions. For x/inR^n the distance function $L_x : M \to R$ is defined by $L_x(p) = \|f(p) - x\|^2$. Then $x \in R^n$ is a focal point of f with base p if and only if p is a degenerate critical point of L_X, where at p, $\frac{\partial L_x}{\partial p_i} = 0$ and $H = \left[\frac{\partial^2 L_x}{\partial p_i \partial p_j}\right]$ is singular for $i, j = 1, 2, \ldots, m$ [4].

Proposition 1. *[1] Every immersion $f : S^1 \to R^n$, $n > 2$, has TRFS.*

Proposition 2. *[1] Let $f : M_1 \to R^n$ and $g : M_2 \to R^n$ be immersions with TRFS. Then $f \times g : M_1 \times M_2 \to R^{n_1+n_2}$ defined by $(f \times g)(p, q) = (f(p), g(q))$ has TRFS.*

From Proposition 1 and 2 we can construct more examples of immersions with TRFS by taking products of immersions of circles.

3. Immersions Preserved Under Rotations

Theorem 1. *Let $f : M^m \to R^n$ and $g : N^p \to S^q \subset R^{q+1}$ be immersions with TRFS and assume that $f_n(\theta) \neq 0$ for all $\theta \in M^m$. Then $h : M^m \times N^p \to R^{n+q}$ defined by $h(\theta, \varphi) = (f_1(\theta), \ldots, f_{n-1}(\theta), f_n(\theta)g(\varphi))$ is an immersion with TRFS.*

Proof. Let $x \in R^n$ and $L_x(\theta) = \sum_{k=1}^n (x_k - f_k(\theta))^2$. Then $x \in \Gamma_f(\theta)$ if and only if, at, θ,

$$\frac{\partial L_x}{\partial \theta_i} = -2\sum_{k=1}^n (x_k - f_k)\frac{\partial f_k}{\partial \theta_i} = 0, \quad i = 1, 2, \ldots, m \tag{1}$$

and if

$$A_{ij} = \frac{\partial^2 L_x}{\partial \theta_i \partial \theta_j} = -2\left(\sum_{k=1}^n (x_k - f_k)\frac{\partial^2 f_k}{\partial \theta_i \partial \theta_j} - \left(\frac{\partial f_k}{\partial \theta_i}\frac{\partial f_k}{\partial \theta_j}\right)\right) \tag{2}$$

at θ, then $\det(A_{ij}) = 0$. □

Thus f has TRFS if and only if using (1), (2) factorizes into linear factors. Let $y \in R^{q+1}$ and

$$L_y(\varphi) = \sum_{s=1}^{q+1} (y_s - g_s(\varphi))^2.$$

Then $y \in \Gamma_g(\varphi)$ if and only if, at φ,

$$\sum_{s=1}^{q+1} g_s^2 = 1 \Rightarrow \sum_{s=1}^{q+1} g_s \frac{\partial g_s}{\partial \varphi_r} = 0, \quad r = 1, 2, \ldots, p \tag{3}$$

$$\frac{\partial L_y}{\partial \varphi_r} = -2 \sum_{s=1}^{q+1} y_s \frac{\partial g_s}{\partial \varphi_r} = 0, \quad r = 1, 2, \ldots, p \tag{4}$$

and if

$$B_{rt} = \frac{\partial^2 L_y}{\partial \varphi_r \partial \varphi_t} = -2 \left(\sum_{s=1}^{q+1} y_s \frac{\partial^2 g_s}{\partial \varphi_r \partial \varphi_t} \right) \tag{5}$$

at φ then $\det(B_{rt}) = 0$.

Thus g has TRFS if and only if using (3), (4), (5) factorizes into linear factors. For

$$z = (x_1, \ldots, x_{n-1},\ y_1, \ldots, y_{q+1}) \in R^{n+q} \text{ and } L_z(\theta, \varphi) = \sum_{k=1}^{n-1} (x_k - f_k)^2 + \sum_{s=1}^{q+1} (y_s - f_n g_s)^2.$$

Then $z = (x_1, \ldots, x_{n-1}, y_1, \ldots, y_{q+1}) \in \Gamma_h(\theta, \varphi)$ if and only if at (θ, φ), and we get

$$\frac{\partial L_z}{\partial \theta_i} = -2 \sum_{k=1}^{n-1} (x_k - f_k) \frac{\partial f_k}{\partial \theta_i} - 2 \left(\sum_{s=1}^{q+1} y_s g_s - f_n \right) \frac{\partial f_n}{\partial \theta_i} = 0 \tag{6}$$

and using (3)

$$\frac{\partial L_z}{\partial \varphi_r} = -2 f_n \sum_{s=1}^{q+1} y_s \frac{\partial g_s}{\partial \varphi_r} = 0, \tag{7}$$

and if

$$\begin{aligned} C_{ij} &= \frac{\partial^2 L_z}{\partial \theta_i \partial \theta_j} = -2 \left(\sum_{t=1}^{n-1} (x_t - f_t) \frac{\partial^2 f_t}{\partial \theta_i \partial \theta_j} - \left(\frac{\partial f_t}{\partial \theta_i} \frac{\theta f_t}{\partial \theta_j} \right) \right. \\ &\quad \left. + \left(\left(\sum_{s=1}^{q+1} y_s g_s \right) - f_n \right) \frac{\partial^2 f_n}{\partial \theta_i \partial \theta_j} - \left(\frac{\partial f_n}{\partial \theta_i} \frac{\partial f_n}{\partial \theta_j} \right) \right), \\ \textit{then} \quad \det(C_{ij}) &= 0; \end{aligned} \tag{8}$$

and if

$$\begin{aligned} D_{rt} &= \frac{\partial^2 L_z}{\partial\varphi_r \partial\varphi_t} = -2f_n \left(\sum_{s=1}^{q+1} y_s \frac{\partial^2 g_s}{\partial\varphi_r \partial\varphi_t}\right) \\ \textit{then} \quad \det(D_{rt}) &= 0, \end{aligned} \tag{9}$$

since if $\frac{\partial L_z}{\partial \varphi_r} = 0$ then

$$\frac{\partial^2 L_z}{\partial\theta_i \partial\varphi_r} = \frac{\partial^2 L_z}{\partial\varphi_r \partial\theta_i} = -2\left(\sum_{s=1}^{q+1} y_s \frac{\partial g_s}{\partial\varphi_r}\right)\frac{\partial f_n}{\partial\theta_i} = 0 \text{ as } f_n \neq 0.$$

Case (i) If (6), (7) and (8) hold then since (7) is linear in $y_1, \ldots, y_{q+1}$ and $(x_1, \ldots, x_{n-1}, w_n)$ satisfies (1) and (2) where $w_n = \sum_{s=1}^{q+1} y_s g_s$ and using (6), (8) factorizes into linear factors in $x_1, \ldots, x_{n-1}, w_n$ since f has TRFS. But w_n is linear in $y_1, \ldots, y_{q+1}$, so (8) factorizes into linear factors in $x_1, \ldots, x_{n-1}, \; y_1, \ldots, y_{q+1}$.

Case (ii) If (6), (7) and (9) hold then (7) is linear in $x_1, \ldots, x_{n-1}, \; y_1, \ldots,_{q+1}$ and $(y_1, \ldots, y_{q+1}$ satisfies (4) and (5) and using (7), (9) factorizes into linear factors in $y_1, \ldots, y_{q+1}$, since g has TRFS.

Hence the equations defining $\Gamma_h(\theta, \varphi)$ factorize into linear factors and therefore h has TRFS.
□

Corollary 1. *If $f : M^m \to R^n$ has TRFS and the immersion $h : M^m \times S^q \to R^{n+q}$ is defined by $h(p, \theta) = (f_1(p), \cdots, f_{n-1}(p), \; f_n(p)g(\theta))$, where $g : S^q \to S^q \subset R^{q+1}$ is defined by*

$$g(\theta) = \left(\prod_{i=1}^{q} \cos\theta_i, \; \prod_{j=2}^{q} \cos\theta_j \sin\theta_1, \; \prod_{k=3}^{q} \cos\theta_k \sin\theta_2, \ldots, \cos\theta_q \sin\theta_{q-1}, \sin\theta_q\right),$$

Then h has TRFS.

Proof. Since g has TRFS as the embedding g has codimension one, the results follows immediately from Theorem 2.1. □

As a particular case consider an immersion $f : M^m \to R^{m+1}$, with $f_{m+1}(p) \neq 0$ for all $p \in M^m$. Then the immersion $h : M^m \times S^1 \to R^{m+2}$ is defined by $h(p, \theta) = (f_1(p), \ldots, f_m(p), f_{m+1}(p)\cos\theta, f_{m+1}(p)\sin\theta)$ has TRFS. This immersion is obtained by rotating the image of f about the hyperplane $R^m \times \{(0,0)\}$ in R^{m+2}.

Theorem 2. *Let $f : M^m \to R^n$ $(f(M^m) \subset S^{n-1})$ and $g : N^k \to R^d$ $(g(N^k) \subset S^{d-1})$ be immersions with TRFS. Then the embedding $h : M^m \times N^k \times (0, \pi/2) \to R^{n+d}$ defined by $h(p, q, \theta) = (\cos\theta \cdot f(p), \sin\theta \cdot g(q))$ has TRFS.*

Proof. Let $x = (x_1, \ldots, x_n) \in R^n$, $y = (y_1, \ldots, y_d) \in R^n$ and $z = (x, y) \in R^{n+d}$. We know that

$$\sum_{i=1}^{n} f_i^2 = 1 \Rightarrow \sum_{i=1}^{n} f_i \frac{\partial f_i}{\partial p_s} = 0, \quad s = 1, 2, \ldots, m \tag{10}$$

$$\sum_{j=1}^{d} g_j^2 = 1 \Rightarrow \sum_{j=1}^{d} g_j \frac{\partial g_j}{\partial q_r} = 0, \quad r = 1, 2, \ldots, k. \tag{11}$$

So from (10),

$$\frac{\partial L_z}{\partial p_s} = 2\cos\theta \left(\sum_{i=1}^{n} x_i \frac{\partial f_i}{\partial p_s} \right) \tag{12}$$

since

$$\cos\theta \neq 0, \quad \frac{\partial L_z}{\partial p_s} = 0 \quad \text{if and only if} \quad \left(\sum_{i=1}^{n} x_i \frac{\partial f_i}{\partial p_s} \right) = 0. \tag{13}$$

Also, from (11),

$$\frac{\partial L_z}{\partial q_r} = 2\sin\theta \left(\sum_{j=1}^{d} y_j \frac{\partial g_j}{\partial q_r} \right) \tag{14}$$

since

$$\sin\theta \neq 0, \quad \frac{\partial L_z}{\partial q_r} = 0 \quad \text{if and only if} \quad \left(\sum_{j=1}^{d} y_j \sum^{\partial g_j} \partial q_r \right) = 0. \tag{15}$$

From (10), (11),

$$\frac{\partial L_z}{\partial \theta} = 2\sin\theta \left(\sum_{i=1}^{n} x_i f_i \right) - 2\cos\theta \left(\sum_{j=1}^{d} y_j g_j \right). \tag{16}$$

So

$$\frac{\partial L_z}{\partial \theta} = 0 \quad \text{if and only if } 2\sin\theta \left(\sum_{i=1}^{n} x_i f_i \right) - 2\cos\theta \left(\sum_{j=1}^{d} y_j g_j \right) = 0. \tag{17}$$

From (12), (14), $\dfrac{\partial^2 L_z}{\partial p_s \partial q_r} = \dfrac{\partial^2 L_z}{\partial q_r \partial p_s} = 0$. From (12), (16),

$$\frac{\partial^2 L_z}{\partial p_s \partial \theta} = \frac{\partial^2 L_z}{\partial \theta \partial p_s} = 2\sin\theta \left(\sum_{i=1}^{n} x_i \frac{\partial f_i}{\partial p_s} \right).$$

So $\dfrac{\partial^2 L_z}{\partial p_s \partial \theta} = 0$ if and only if $\dfrac{\partial L_z}{\partial p_s} = 0$ from (13). From (14), (16),

$$\frac{\partial^2 L_z}{\partial q_r \partial \theta} = \frac{\partial^2 L_z}{\partial \theta \partial q_r} = -2\cos\theta \left(\sum_{j=1}^{d} y_j \frac{\partial g_j}{\partial q_r} \right).$$

So $\dfrac{\partial^2 L_z}{\partial q_r \partial \theta} = 0$ if and only if $\dfrac{\partial L_z}{\partial q_r} = 0$ from (15), and

$$\begin{aligned} A_{st} &= \frac{\partial^2 L_z}{\partial p_s \partial p_t} = -2\cos\theta \left(\sum_{i=1}^{n} x_i \frac{\partial^2 f_i}{\partial p_s \partial p_t} \right), \\ B_{rv} &= \partial^2 L_z \partial q_r \partial q_v = 2\sin\theta \left(\sum_{j=1}^{d} y_j \frac{\partial^2 g_j}{\partial q_r \partial q_v} \right) \\ C &= \frac{\partial^2 L_z}{\partial \theta^2} = -2\cos\theta \left(\sum_{i=1}^{n} x_i f_i \right) - 2\sin\theta \left(\sum_{j=1}^{d} y_j g_j \right). \end{aligned}$$

Thus $z \in \Gamma_h(p, q, \theta)$ if and only if $\dfrac{\partial L_z}{\partial p_s} = 0$, $\dfrac{\partial L_z}{\partial q_r} = 0$, $\dfrac{\partial L_z}{\partial \theta} = 0$ and either

$$\det(A_{st}) = 0, \tag{18}$$

or

$$\det(B_{rv}) = 0, \tag{19}$$

or

$$C = 0. \tag{20}$$

Case (i) (13), (15), (17) and (18) hold. (15) is linear in $y_1, \ldots, y_d$, (17) is linear in $x_1, \ldots, x_n$, $y_1, \ldots, y_d$ and using (13) which is linear in $x_1, \ldots, x_n$, (18) factorizes into linear factors in $x_1, \ldots, x_n$ since f has TRFS and $x_1, \ldots, x_n \in \Gamma_f(p)$ if and only if

$$\left(\sum_{i=1}^{n} x_i \frac{\partial f_i}{\partial p_s} \right) = 0 \quad \text{and} \quad \det \left(\sum_{i=1}^{n} x_i \frac{\partial^2 f_i}{\partial p_s \partial p_t} \right) = 0,$$

as in the proof of Theorem 1.

Case (ii) (13), (15), (17) and (19) hold. This is similar to case (i) but uses the fact that g has TRFS.

Case (i) (13), (15), (17) and (20) hold. In this case, all equations are linear in $x_1, \ldots, x_n, y_1, \ldots, y_d$.

□

It follows that h has TRFS.

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