Tr. J. of Mathematics
21 (1997) , 317 – 324.

ON THE SOLUTION OF THE E.P.D. EQUATION USING FINITE INTEGRAL TRANSFORMATIONS

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Abstract

In this paper, a solution is given for the following initial boundary value problem:

$$\Delta u = u_{tt} + \frac{k}{t} u_t + g(x,t) \quad (t > 0)$$

$$u(0,t) = u(a,t) = 0$$

$$u(x,0) = f(x), u_t(x,0) = 0$$

where $x, a \in R^n$, t is the time variable, $k < 1, k \neq -1, -2, -3, \ldots$ is a real parameter, Δ is the n dimensional Laplace operator, f and g real analytic functions. The equation in this problem is known as the nonhomogeneous Euler-Poisson-Darboux (E.P.D.) Equation. The solution is obtained using finite integral transformation technique and is the sum of two uniformly and absolutely convergent power series.

Key words: Hyperbolic equations, initial boundary value problems

1. Introduction

Let us consider the following nonhomogeneous initial boundary value problem

$$\Delta u = u_{tt} + \frac{k}{t} u_t + g(x,t) \ (t > 0) \tag{1}$$

$$u(0,t) = u(a,t) = 0 \tag{2}$$

$$u(x,0) = f(x), u_t(x,0) = 0 \tag{3}$$

*AMS Classification 35L25

where $x = (x_1, x_2, \ldots, x_n), a = (a_1, a_2, \ldots, a_n) \in R^n$, t is the time variable, $k < 1$ is a real parameter, Δ is the n dimensional Laplace operator, f and g are real analytic functions.

(1) is known as the nonhomogeneous Euler-Poisson-Darboux (E.P.D.) Equation. The initial value problem (1), (3) was solved by Germain and Bader [3] for $f(x) = 0$ and $n = 2, k = \frac{1}{3}$ using Riemann's method. The case $n = 3, k > 0$ was considered by Diaz and Lunford [2] using Hadamard's method. In this case however it didnot lead to divergent integrals. The solution of the n dimensional initial value problem (1), (3) was given by Young [5] for $f(x) = 0$. In Young's paper extensive use of the techniques of Davis [1] and Diaz and Lunford [2] was made. The solution of the nonhomogeneous problems can result in quite complicated formulas. In the present paper we shall consider the initial value problem (1), (2), (3) and apply a new solution method which consists of using finite integral transformations (see [4]). Finite integral transformation method was used to solve some initial boundary value problems (For example, Daniel Bernoulli's problem of the vibrations of a heavy thread, the problems of small vibrations of a rectangular membrane fastened at the edges etc. (see [4], pp. 542-562). In the present paper finite integral transformation method is used for the first time for the nonhomogeneous E.P.D. equation. The solution is expressed in terms of uniformly convergent power series for $k < 1$ and $k \neq, 1, -2, \ldots$. The solution has been obtained a much simpler manner than by other methods mentioned above.

Let us seek a solution to initial boundary value problem (1), (2), (3) in the form of a sum

$$u(x,t) = v(x,t) + w(x,t) \tag{4}$$

where $v(x,t)$ is the solution to the nonhomogeneous equation

$$\Delta v = v_{tt} + \frac{k}{t} v_t + g(x,t) \quad (t > 0) \tag{5}$$

satisfying the initial-boundary conditions

$$v(x,0) = v_t(x,0) = 0 \tag{6}$$

$$v(0,t) = v(a,t) = 0. \tag{7}$$

and $w(x,t)$ is the solution to the homogeneous equation

$$\Delta w = w_{tt} + \frac{k}{t} w_t \quad (t > 0) \tag{8}$$

satisfying the initial-boundary conditions

$$w(x,0) = f(x), w_t(x,0) = 0 \tag{9}$$

$$w(x,0) = w(a,t) = 0 \tag{10}$$

We first consider the problem (5), (6), (7) and apply a finite integral transformation to each variable $x_1, x_2, \ldots, x_n$ respectively on the intervals $(0, a_i)$ $(i = 1, 2, \ldots, n)$. According to the terminology of finite integral transformations given in [4], the kernel of this transformation in the i-th step is

$$K(x_i, y_i) = \frac{2}{a_i} \sin \frac{\pi y_i}{a_i} x_i.$$

The transformed equation will not contain the derivatives with respect to $x_i (i = 1, 2, \ldots, n)$. There is the following relation between the kernels of direct transformations and inverse transformaitons:

$$K(x_i, y_i) = \frac{1}{c_{y_i}} \tilde{K}_{y_i}(x_i),$$

where c_{y_i}'s are normalizing divisors and are given by

$$c_{y_i} = \int_0^{a_i} \sin^2 \frac{\pi y_i}{a_i} x_i dx_i = \frac{a_i}{2}.$$

The kernel of the inverse transformation is the solution of the following boundary value problem:

$$\frac{\partial^2 \tilde{K}}{\partial x_i^2} + \lambda_i^2 \tilde{K} = 0, \; \lambda_i = \frac{\pi y_i}{a_i}$$

$$\tilde{K}|_{x_i=0} = \tilde{K}|_{x_i=a_i} = 0.$$

The direct transformation is applied to equation (5) for the variable x_i and

$$\frac{\partial^2 v}{\partial x_i^2} = -\lambda_i^2 \tilde{v}$$

is obtained. This transformation is applied n times to equation (5) leading to

$$\Delta v = -\lambda^2 \tilde{v}, \; \lambda^2 = \sum_{i=1}^{n} \lambda_i^2$$

is arrived at. Under this transformation, the problem (5), (6), (7) is transformed to the following:

$$\tilde{v}_{tt} + \frac{k}{t}\tilde{v}_t + \lambda^2 \tilde{v} + \tilde{g}(y,t) = 0 \; (t > 0) \tag{11}$$

$$\tilde{v}(y,0) = \tilde{v}_t(y,0) = 0 \tag{12}$$

(11) is an ordinary differential equation and (12) gives the initial conditions, where $\tilde{v}(y,t), \tilde{g}(y,t)$ are transformed functions obtained respectively, from $v(x,t), g(x,t)$ and

$$\begin{aligned}\tilde{g}(y,t) &= \frac{2^n}{a^n}\int_0^{a_1}\int_0^{a_2}\cdots\int_0^{a_n} g(x_1,x_2,\ldots,x_n;t)\prod_{i=1}^{n}\sin\frac{\pi y_i}{a_i}x_i d\sigma_x \\ &= \sum_{n=0}^{\infty}\tilde{p}_n(y)t^n\end{aligned}$$

is a real analytic function $(d\sigma_x = dx_1, dx_2 \ldots dx_n, y = (y_1, y_2, \ldots, y_n))$.

Let us seek a solution $\tilde{v}$ in the form of the series $\tilde{v}(y,t) = \sum_{n=0}^{\infty} d_n(y)t^n$ to the problem (11), (12) and apply inverse transformations to the solution. We obtain

$$v(x_1,x_2,\ldots,x_n;t) = \sum_{y_1,y_2,\ldots,y_n=1}^{\infty} \tilde{v}(y,t)\tilde{K}_{y_1}\cdots\tilde{K}_{y_n}$$

where the kernels of inverse transformations are $\tilde{K}(x_i,y_i) = \tilde{K}_{y_i}(x_i) = \sin\frac{\pi y_i}{a_i}x_i$.

2. The Solution

We define $(k)_{(2n-1)}$ and $[k]_{(2n)}$ as following for $n \in N$

$$(k)_{(-1)} = 1,\ (k)_{(2n-1)} = (1+k)(3+k)(5+k)\cdots(2n-1+k)$$

and

$$[k]_{(0)} = 1, [k]_{(2n)} = (2+k)(4+k)\cdots(2n+k)$$

where $k < 1, k \notin Z^-$.

Lemma 1. *The power series* $\sum_{n=1}^{\infty} B_n \frac{t^{2n}}{(k)_{(2n-1)}}, B_n = \frac{(-1)^{n+1}\lambda^{2n}}{2^n n!}$, *is absolutely and uniformly convergent for* $t > 0$. *Furthermore, the recurrence relations*

$$2nB_n + \lambda^2 B_{n-1} = 0 \ \ (n = 1,2,\ldots) \tag{13}$$

are satisfied for the coefficients B_n, *where* $k < 1$ *and* $k \neq -1, -3, \ldots$.

Lemma 2. *The coefficients b_n which are defined by*

$$b_{2n} = \frac{(-1)^n \lambda^{2n}}{2^n n!(k)_{(2n-1)}}(1+\sum_{i=1}^{n}(-1)^{i-1}2^{i-1}(i-1)!(k)_{(2i-3)}\tilde{p}_{2i-2}\lambda^{-2i}) \tag{14a}$$

$$b_{2n+1} = \frac{(-1)^n \lambda^{2n}}{1.3.5\ldots(2n+1)[k]_{(2n)}}\sum_{i=1}^{n}\frac{(-1)^{i-1}(2i-1)![k]_{(2i-2)}\tilde{p}_{2i-1}\lambda^{-2i}}{2^{i-1}(i-1)!} \tag{14b}$$

satisfy the following relations

$$(n+2)(n+1+k)b_{n+2} + \lambda^2 b_n = -\tilde{p}_n. \ (n = 1, 2, \ldots) \tag{15}$$

Lemma 3. *The solution of (11), (12) is*

$$\tilde{v}(y,t) = \sum_{n=1}^{\infty} B_n \frac{t^{2n}}{(k)_{(2n-1)}} - D\sum_{n=0}^{\infty} B_n(1-k)\frac{t^{2n+1-k}}{(-k)_{(2n+1)}} + \sum_{n=2}^{\infty} b_n t^n \tag{16}$$

where the coefficient B_n and b_n are given, respectively, by Lemma 1 and Lemma 2 for $k < 1$, $k \notin Z^-$.

Proof. Let us consider (11). The solution of the adjoint homogeneous equation is

$$\tilde{v}_h(y,t) = C[1 + \sum_{n=1}^{\infty} B_n \frac{t^{2n}}{(k)_{(2n-1)}}] - D\sum_{n=0}^{\infty} B_n(1-k)\frac{t^{2n+1-k}}{(-k)_{(2n+1)}} \tag{17a}$$

where C and D are arbitrary constants. The special solution of (11) has the form

$$\tilde{v}_s(y,t) = \sum_{n=0}^{\infty} b_n t^n, b_1 = 0 \tag{17b}$$

If we choose $b_0 = 1$, we obtain the coefficients b_n given by (14.a) and (14.b) for $n = 1, 2, 3, \ldots$. From the initial conditions (12), we have $C = -b_0 = -1$ and D is arbitrary, $k < 1$ and $k \neq -1, -2, \ldots$. The general solution of the problem (11), (12) is

$$\tilde{v}(y,t) = \tilde{v}_h(y,t) + \tilde{v}_s(y,t) \tag{16}$$

where $\tilde{v}_h$ and $\tilde{v}_s$ are given respectively by (17.a) and (17.b). □

Theorem 1. *The solution of problem (5), (6), (7) is*

$$v(x_1, x_2, \dots, x_n; t) = \sum_{y_1, y_2, \dots, y_n = 1}^{\infty} \tilde{v}(y,t) \prod_{i=1}^{n} \sin \frac{\pi y_i}{a_i} x_i \tag{18}$$

where $\tilde{v}(y,t)$ is given by (16).

Proof. It can be easily seen from (13) and (15) that

$$\begin{aligned} \Delta v - v_{tt} - \frac{k}{t} v_t - g(x,t) &= -\lambda^2 v - (v_{tt} + \frac{k}{t} v_t) - g(x,t) \\ &= - \sum_{y_1, y_2, \dots y_n = 1}^{\infty} (\tilde{v}_{tt} + \frac{k}{t} \tilde{v}_t + \lambda^2 \tilde{v}) \prod_{i=1}^{n} \sin \frac{\pi y_i}{a_i} x_i - g(x,t) \\ &= - \sum_{y_1, y_2, \dots, y_n = 1}^{\infty} (-\tilde{g}(y,t)) \prod_{i=1}^{n} \sin \frac{\pi y_i}{a_i} x_i - g(x,t) \\ &= 0. \end{aligned}$$

□

Theorem 2. *The solution of the problem (8), (9), (10) is given by*

$$w(x_1, x_2, \dots, x_n; t) = \sum_{y_1, y_2, \dots, y_n = 1}^{\infty} \tilde{w}(y,t) \prod_{i=1}^{n} \sin \frac{\pi y_i}{a_i} x_i \tag{19}$$

where $k \neq -1, -3, \dots, c$ is an arbitrary constant and $\tilde{w}$ is given by

$$\tilde{w}(y,t) = \tilde{f}(y)[1 - \sum_{n=1}^{\infty} B_n \frac{t^{2n}}{(k)_{(2n-1)}}] - c \sum_{n=0}^{\infty} B_n (1-k) \frac{t^{2n+1-k}}{(-k)_{(2n+1)}}. \tag{20}$$

Proof. If we use integral transformations on the intervals $(0, a_i), n$ times to the problem (8), (9), (10) we have the following ordinary differential equation and its initial values;

$$\tilde{w}_{tt} + \frac{k}{t} \tilde{w}_t + \lambda^2 \tilde{w} = 0 \ (t > 0) \tag{21}$$

$$\tilde{w}(y,0) = \tilde{f}(y), \tilde{w}_t(y,0) = 0. \tag{22}$$

The kernel of this transformation is $\tilde{K}(x_i, y_i) = \sin \frac{\pi y_i}{a_i} x_i, i = 1, 2, \dots, n$. The functions $\tilde{w}(y,t), \tilde{f}(y)$ are transformed functions respectively of $w(x,t), f(x)$. The solution of (21), (22) is given by (20), where $k < 1, k \neq -1, -3, \dots, -(2n+1), \dots$ and c is an arbitrary constant. When inverse transformation is applied to this solution, is obtained (19). The solution of the problem (8), (9), (10) is obtained as (19). Indeed, as in Theorem 1 we see that from (13)

$$\begin{aligned}\Delta w - (w_{tt} + \frac{k}{t} w_t) &= -\lambda^2 w - (w_{tt} + \frac{k}{t} w_t) \\ &= - \sum_{y_1, y_2, \ldots, y_n = 1}^{\infty} (\tilde{w}_{tt} + \frac{k}{t} \tilde{w}_t + \lambda^2 \tilde{w}) \prod_{i=1}^{n} \sin \frac{\pi y_i}{a_i} x_i \\ &= 0.\end{aligned}$$

□

From the above analysis we arrive at the following theorem:

Theorem 3. *The solution of the initial boundary value problem (1), (2), (3) is expressed in the form of the sum of two uniformly and absolutely convergent power series*

$$u(x, t) = v(x, t) + w(x, t)$$

where $v(x, t)$ is given by (18) and $w(x, t)$ is given by (19).

Corollary. *When the arbitrary constant D is chosen as $D = 0$ in (16) Walter's [6] solution for singular homogeneous E.P.D. equation is obtained.*

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Sonlu İntegral Dönüşümü ile E.P.D. Denkleminin Çözümü

Özet

Δ, n-boyutlu Laplace operatörü, $k < 1, k \neq -1, -2, \ldots$ bir reel parametre, $x, a \in R^n, f$ ve g reel analitik fonksiyonları göstermek üzere

$$\Delta u = u_{tt} + \frac{k}{t} u_t + g(x,t) \quad (t > 0)$$

$$u(0,t) = u(a,t) = 0$$

$$u(x,0) = f(x), u_t(x,0) = 0$$

başlangıç ve sınır değer probleminin çözümü, sonlu integral dönüşümleri kullanılarak düzgün ve mutlak yakınsak iki kuvvet serisinin toplamı olarak elde edilmiştir. $\Delta u = u_{tt} + \frac{k}{t} u_t + g(x,t)$ $(t > 0)$, homogen olmayan Euler - Poisson - Darboux (E.P.D.) denklemidir. E.P.D. denkleminin çözümünde sonlu integral dönüşüm yöntemi ilk kez bu çalışmada kullanılmıştır. Çözüm, daha önce kullanılan yöntemlere göre daha yalın bir biçimde elde edilmiştir.

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Received 9.2.1996
Revised 4.12.1996