Tr. J. of Mathematics
21 (1997) , 357 – 373.

ARITHMETIC OF A SEMIGROUP OF SERIES IN LEGENDRE FUNCTIONS OF THE SECOND KIND

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Abstract

In the framework of D. Kendall's theory of Delphic semigroups, a semigroup of series in Legendre functions of the second kind is studied. The basic factorization theorems are prowed, the classes of infinitely divisible elements and of elements without indecomposable factors are completely described, the density of the class of indecomposable elements is established.

Key words: D. Kendall's Delphic semigroup, infinitely divisible, indecomposable, Legendre functions.

1. Introduction and Statement of Results

The arithmetic of the convolution semigroup $\mathcal{P}$ of all probability distributions on the real line has been deeply investigated by several authors. The detailed information related to this field can be found in the book [4] and in the expository paper [6]. In the papers by D. Kendall and R. Davidson collected in the book [3], and the paper by K. Urbanik [9], it was shown that there are many semigroups essentially different from $\mathcal{P}$ whose arithmetical nature is similar to that of $\mathcal{P}$. Since then, the study of such semigroups was continued; the paper [6] contains a survey of results and references. In particular, the arithmetic of multiplicative semigroups of functions representable by series in Legendre, Gegenbauer, Jacobi polynomials was studied in [7], [8]. The aim of this paper is to study the arithmetic of a multiplicative semigroup essentially different from studied before. This semigroup consists of functions representable by series in Legendre functions of the second kind. We obtain solutions of main problems of the arithmetic of this semigroup.

The Legendre functions of the second kind are defined ([10, §15.3]) by the formula

$$Q_k(x) = 2^{-k-1} \int_{-1}^{1} (1-t^2)^k (x-t)^{-k-1} dt, \tag{1.1}$$

* AMS (1991) Subject Classification. Primary: 60E99, 33C45; Secondary: 20M14.

where $x \in (-\infty, -1) \cup (1, \infty), k = 0, 1, 2, ...$ Note that $Q_k(x) > 0$ for $x > 1$. Fix $r > 1$ and define the functions $q_k(x)$ for $|x| \geq r$ by the equality

$$q_k(x) = \frac{Q_k(x)}{Q_k(r)}, \quad |x| \geq r, \quad k = 0, 1, 2, \ldots \tag{1.2}$$

The fixed parameter r will not be explicitly shown in the notation of $q_k(x)$. Obviously, the function Q_k is a decreasing function of $x \in (1, \infty), k = 0, 1, 2, \ldots$. This and equalities (1.1), (1.2) yield that the function $q_k, k = 0, 1, 2, \ldots,$ possesses the field

(i) q_k is decreasing for $x \geq r$;
(ii) q_k is positive for $x \geq r$;
(iii) $q_k(\infty) = 0$;
(iv) $q_k(-x) = (-1)^{k+1} q_k(x)$ for $|x| > r$;
(v) $q_k(r) = 1, q_k(-r) = (-1)^{k+1}$;
(vi) for $|x| > r$ and $k = 0, 1, 2, \ldots$

$$|q_k(x)| < 1. \tag{1.3}$$

F. Neuman ([5, p. 90]) and W.N. Bailey ([1, formula (5.3) for $m = 0$]) obtained the linearization formula for the product of two Legendre functions. Using definition (1.2), we can write the formula in the following form:

$$q_p(x) q_s(x) = \sum_{m=0}^{\infty} A_{2m+1}(p, s) \frac{Q_{p+s+2m+1}(r)}{Q_p(r) Q_s(r)} q_{p+s+2m+1}(x), \; p, s = 0, 1, 2, \ldots, \tag{1.4}$$

where

$$\begin{aligned} A_{2m+1}(p, s) \; &= \; \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m + 1)} \frac{\Gamma(m + p + 1)}{\Gamma(m + p + \frac{3}{2})} \times \\ &\times \frac{\Gamma(m + s + 1)}{\Gamma(m + s + \frac{3}{2})} \frac{\Gamma(m + p + s + \frac{5}{2})}{\Gamma(m + p + s + 2)} \frac{(4m + 2p + 2s + 3)}{(2m + 2p + 2s + 3)}. \end{aligned} \tag{1.5}$$

It is clear, that

$$\forall m, p, s \geq 0 : \quad A_{2m+1}(p, s) > 0. \tag{1.6}$$

Define $q_{-1}(x) \equiv 1$ and denote by $\mathcal{L}_r$ the set of all functions f representable for $|x| \geq r$ in the form

$$f(x) = \sum_{k=-1}^{\infty} a_k q_k(x), \quad a_k \geq 0, \quad \sum_{k=-1}^{\infty} a_k = 1. \tag{1.7}$$

From the properties (v), (vi) of the functions q_k it follows that, for any $f \in \mathcal{L}_r$,

$$f(r) = 1, \text{ and } |f(x)| < 1, \quad |x| > r \text{ whenever } f \not\equiv 1. \tag{1.8}$$

If follows directly from (1.4), (1.6), (1.7) that $\mathcal{L}_r$ is a semigroup under multiplication. It is a (Hausdorff) topological semigroup under the topology of uniform convergence on $|x| \geq r$. The function $q_{-1}(x) \equiv 1 \in \mathcal{L}_r$ is the neutral element of the semigroup.

The aim of this paper is to study the arithmetic of the semigroup $\mathcal{L}_r$. We shall use the following definitons similar to those generally accepted [4] in the arithmetic of the semigroup $\mathcal{P}$. A function $f_1 \in \mathcal{L}_r$ is called a *factor* of the function $f \in \mathcal{L}_r$ if the exists a function $f_2 \in \mathcal{L}_r$ such that $f = f_1 f_2, |x| \geq r$. A function $f \in \mathcal{L}_r, f \not\equiv 1$, is called *indecomposable* if it has only two factors: f and $q_{-1} \equiv 1$. A function $f \in \mathcal{L}_r$ is called *infinitely divisible* if for each $n = 2, 3 \dots$ there is a function $f_n \in \mathcal{L}_r$ such that $f = (f_n)^n$. Denote by $I(\mathcal{L}_r), I_0(\mathcal{L}_r), N(\mathcal{L}_r)$ the class of infinitely divisible elements of $\mathcal{L}_r$, the class of elements without indecomposable factors and the class of indecomposable elements, respectively.

Let us state the main results of the paper.

First we give the description of the class $I(\mathcal{L}_r)$.

Theorem 1. *The class $I(\mathcal{L}_r)$ consists of all functions f representable in the form*

$$f(x) = \exp\left(\sum_{k=0}^{\infty} b_k (q_k(x) - 1)\right), \; b_k \geq 0, \sum_{k=0}^{\infty} b_k < \infty. \tag{1.9}$$

The constants b_k are uniquely determined by the function f.

The general formula (1.9) $f \in I(\mathcal{L}_r)$ can be considered as an analogue of the well-known Levy-Khinchin formula ([4, p.9]) of characteristic function of an infinitely divisible probability distribution on the real line.

The following theorem can be considered as an analogue of the Khinchin factorization theorem ([4, p. 79]) related to the arithmetic of the semigroup $\mathcal{P}$.

Theorem 2. *Any function $f \in \mathcal{L}_r$ has at least one factorization of the form*

$$f = f_0 \prod_{i=1}^{k} f_i, \quad 0 \leq k \leq \infty, \tag{1.10}$$

where $f_0 \in I_0(\mathcal{L}_r), f_i \in N(\mathcal{L}_r), i \geq 1$. For $k = \infty$, the product in (1.10) is convergent in the topology of $\mathcal{L}_r$; for $k = 0$, the product is equal to 1.

The following theorem is an analogue of Khinchin's one ([4, p.88]) related to the semigroup $\mathcal{P}$ which claims the inclusion of the class $I_0(\mathcal{P})$ of all distributions without indecomposable components into the class $I(\mathcal{P})$ of all infinitely divisible distributions.

Theorem 3. *The following inclusion is valid*

$$I_0(\mathcal{L}_r) \subseteq I(\mathcal{L}_r).$$

Recall that the problem of the description of the class $I_0(\mathcal{P})$ has not been solved until now in spite of the fact that deep investigations of several mathematicians were devoted to the problem (the corresponding results are reflected in [4, Ch. IV, V, VI] and [6]). It seems to be interesting that the analogous problem for $I_0(\mathcal{L}_r)$ can be solved completely by the following theorem.

Theorem 4. *The class $I_0(\mathcal{L}_r)$ consists of a single element $q_{-1} \equiv 1$.*

To be sure, Theorem 3 is contained in Theorem 4, but the first is a base of the proof of the latter.

The following theorem is an analogue of the theorem of Parthasarathy, Rao and Varadhan ([4, p. 71]) related to the set $N(\mathcal{P})$ of all indecomposable elements of the semigroup $\mathcal{P}$.

Theorem 5. *The class $N(\mathcal{L}_r)$ is dense in $\mathcal{L}_r$ in the topology of $\mathcal{L}_r$.*

2. The Description of the Class $I(\mathcal{L}_r)$. Proof of Theorem 1.

To prove Theorem 1 we need several lemmas. The first of them is an analogue of the well-known compactness test for families of probability distributions (see, e.g. [4, p. 88]). Further, the symbol $\Rightarrow$ will denote the convergence in the topology of $\mathcal{L}_r$, i.e. the uniform convergence on the set $\{|x| \geq r\}$.

Lemma 1. *If a sequence of functions*

$$f_n(x) = \sum_{k=-1}^{\infty} a_k^{(n)} q_k(x) \in \mathcal{L}_r, \quad n = 1, 2, \dots, \tag{2.1}$$

satisfies the condition

$$\forall \varepsilon > 0, \quad \exists k_o = k_o(\varepsilon), \ \forall n : \quad \sum_{k > k_0} a_k^{(n)} < \varepsilon, \tag{2.2}$$

then there exists a subsequence $\{f_{n_j}(x)\}$ such that $f_{n_j} \Rightarrow f \in \mathcal{L}_r$.

Proof. Since $0 \leq a_k^{(n)} \leq 1$, there exists a subsequence $\{n_j\}$ such that

$$\exists \lim_{j \to \infty} a_k^{(n_j)} =: b_k, \quad k = -1, 0, 1, \dots \tag{2.3}$$

Evidently, for any $m = 1, 2, \ldots,$ we have

$$\sum_{k=-1}^{m} b_k = \lim_{j\to\infty} \sum_{k=-1}^{m} a_k^{(n_j)} \le 1.$$

Therefore the series $\sum_{k=-1}^{\infty} b_k$ converges. Using (1.3), we conclude that the function $f := \sum_{k=-1}^{\infty} b_k q_k$ is well-defined and

$$|f_{n_j}(x) - f(x)| \le \sum_{k=-1}^{k_1} |a_k^{(n_j)} - b_k| + \sum_{k>k_1} a_k^{(n_j)} + \sum_{k>k_1} b_k < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \tag{2.4}$$

for k_1 and n_j being sufficiently large. Therefore $f_{n_j}(x) \rightrightarrows f(x)$. Substituting $x = r$ into (2.4) and using the equality $q_k(r) = 1$ we conclude that $f(r) = \sum_{k=-1}^{\infty} b_k = 1$. Hence $f \in \mathcal{L}_r$. □

Lemma 2. *Assume that the sequence of functions*

$$f_n(x) = \sum_{k=0}^{\infty} a_k^{(n)}(q_k(x) - 1), a_k^{(n)} \ge 0, \quad \sum_{k=0}^{\infty} a_k^{(n)} < \infty,$$

satisfies the condition (2.2) and, for all $k = 0, 1, 2, \ldots,$ *there exists* $\lim_{n\to\infty} a_k^{(n)} =: b_k$. *Then* $\sum_{k=0}^{\infty} b_k < \infty$ *and*

$$f_n(x) \rightrightarrows \sum_{k=0}^{\infty} b_k(q_k(x) - 1), \quad \textit{as} \quad n \to \infty.$$

The proof of Lemma 2 is quite similar to that of Lemma 1 and therefore can be omitted.

Lemma 3. *If the sequence of functions (2.1) satisfies the condition*

$$\forall \varepsilon > 0, \exists \lambda = \lambda(\varepsilon) > r, \forall n : \quad 1 - f_n(\lambda) < \varepsilon,$$

then the condition (2.2) is fulfilled.

Proof. We need the following property of functions $q_k(x)$:

$$\forall x, |x| > r : \quad \lim_{k\to\infty} q_k(x) = 0. \tag{2.5}$$

This is a simple corallary of (1.1). Indeed, (1.1) yields that, for $|x| > r$, we have

$$|Q_k(x)| \leq d^{k+1} Q_k(r), \text{ where } d := \max_{-1 \leq t \leq 1} \left| \frac{r-t}{x-t} \right| < 1.$$

Choose λ from the condition of the lemma. It follows from (2.5) that $\exists k_o = k_o(\varepsilon), \forall k > k_0 : q_k(\lambda) \leq 1/2$. Hence

$$\varepsilon > 1 - f_n(\lambda) = \sum_{k=0}^{\infty} a_k^{(n)} (1 - q_k(\lambda)) \geq \frac{1}{2} \sum_{k > k_0} a_k^{(n)}.$$

□

Lemma 4. *If the sequence of functions (2.1) converges pointwise on $\{|x| \geq r\}$ to a function f right-continuous at $x = r$, then $f \in \mathcal{L}_r$.*

Proof. Since $f_n(r) = 1$, we have $f(r) = 1$. The conditions of the lemma yield

$$\forall \varepsilon > 0, \exists \lambda = \lambda(\varepsilon) > r, \exists n_0 = n_0(\varepsilon), \forall n > n_0 : \; f_n(\lambda) > f(\lambda) - \varepsilon > 1 - 2\varepsilon.$$

Decrease $\lambda > r$ in such a way that the inequality $f_n(\lambda) > 1 - 2\varepsilon$ will be valid for any n. Using Lemma 3, we obtain (2.2). Then, $f \in \mathcal{L}_r$ by Lemma 1. □

Lemma 5. *If the sequence (2.1) satisfies the condition*

$$\exists \lambda > r : \quad f_n(\lambda) \to 1, n \to \infty,$$

then $f_n(x) \Rightarrow 1$.

Proof. By the condition of the lemma, we have $\forall \varepsilon > 0, \exists n_0, \forall n > n_0 : 1 - f_n(\lambda) < \varepsilon$. Decrease $\lambda > r$ in such a way that the last inequality will be valid for any n. By Lemma 3 the condition (2.2) holds for the sequence $\{f_n\}$. Choose k_0 according to (2.2). Then we have

$$\varepsilon > 1 - f_n(\lambda) \geq \sum_{k=0}^{k_0} a_k^{(n)} (1 - q_k(\lambda)) \geq c \sum_{k=0}^{k_0} a_k^{(n)}, \tag{2.6}$$

where $c = \min_{0 \leq k \leq k_0} [1 - q_k(\lambda)]$. From (1.3) it follows that $c > 0$. Using (2.2) and (2.6), we get $\sum_{k=0}^{\infty} a_k^{(n)} \to 0,\, n \to \infty$. Therefore

$$1 - f_n(x) = \sum_{k=0}^{\infty} a_k^{(n)}(1 - q_k(x)) \leq 2 \sum_{k=0}^{\infty} a_k^{(n)} \to 0.$$

□

Proof of Theorem 1. Sufficiency. Obviously, the function $f(x) = \exp\left(\sum_{k=0}^{\infty} b_k(q_k(x) - 1)\right), b_k \geq 0, \sum_{k=0}^{\infty} b_k < \infty$, can be written in the form $f(x) = \exp(c(g(x) - 1))$, where $c \geq 0, g \in \mathcal{L}_r$. Note that a convex linear combination of functions of $\mathcal{L}_r$ belongs to $\mathcal{L}_r$. Therefore, using lemma 4, we conclude that the function

$$f(x) = \exp(c(g(x) - 1)) = \lim_{N \to \infty} \left(\sum_{k=0}^{N} \frac{c^k}{k!}\right)^{-1} \left(\sum_{k=0}^{N} \frac{c^k}{k!}(g(x))^k\right)$$

belongs to $\mathcal{L}_r$ for any $c \geq 0$ and any $g(x) \in \mathcal{L}_r$. For this reason $(f(x))^{1/n} = \exp((c/n)(g(x) - 1)) \in \mathcal{L}_r, n = 1, 2, \ldots$ Thus, $f \in I(\mathcal{L}_r)$.

Necessity. Let $f \in I(\mathcal{L}_r)$. Then, for any $n = 1, 2, \ldots$, we have $f = (f_n)^n$, where f_n has the form (2.1). Since $q_k(x) > 0$ for $x \geq r$, we have $f(x) > 0$ for $x \geq r$. Let us show that $f(x) > 0$ for $x \leq -r$. Take any $\lambda > r$. Then $f_n(\lambda) = (f(\lambda))^{1/n} \to 1$, as $n \to \infty$ and, by Lemma 5, $f_n \Rightarrow 1$. Therefore $f_n(x) > 0$ for all sufficiently large n and $f(x) > 0$.

Further, we have

$$\begin{aligned} \log f(x) &= \lim_{n \to \infty} n((f(x))^{1/n} - 1) = \lim_{n \to \infty} n(f_n(x) - 1) \\ &= \lim_{n \to \infty} \sum_{k=0}^{\infty} n a_k^{(n)}(q_k(x) - 1). \end{aligned} \tag{2.7}$$

Since the series in (2.7) converges and all its members are stricktly negative, the sequence $\{n a_k^{(n)}\}_{n=1}^{\infty}$ is bounded for each $k \geq 0$. Therefore, there exists a subsequence $\{n_j\}$ such that $n_j a_k^{(n_j)} \to b_k \geq 0$, as $j \to \infty, k = 0, 1, 2, \ldots$. Using (2.7) and the equality $f(r) = 1$, we see that

$$\forall \varepsilon > 0, \exists \lambda > r, \exists n_0, \forall n > n_0 :$$

$$\sum_{k=0}^{\infty} n a_k^{(n)}(1 - q_k(\lambda)) \leq \log f(\lambda) + \varepsilon < 2\varepsilon. \tag{2.8}$$

Using (2.5), we choose k_0 so that $q_k(\lambda) \leq 1/2$ for $k > k_0$. Then, from (2.8), we obtain

$$2\varepsilon > \sum_{k>k_0} na_k^{(n)}(1 - q_k(\lambda)) \geq \frac{1}{2} \sum_{k>k_0} na_k^{(n)}.$$

Hence, (2.2) is satisfied for $na_k^{(n)}$ instead of $a_k^{(n)}$. Thus, the conditions of Lemma 2 are satisfied for $n_j a_k^{(n_j)}$ instead of $a_k^{(n)}$. Applying Lemma 2, we conclude, that $\sum_{k=0}^{\infty} b_k < \infty$ and

$$\log f(x) = \sum_{k=0}^{\infty} b_k(q_k(x) - 1).$$

It remains to verify the uniqueness of the representation. Assume,

$$\exp\left(\sum_{k=0}^{\infty} c_k(q_k(x) - 1)\right) = \exp\left(\sum_{k=0}^{\infty} b_k(q_k(x) - 1)\right), \tag{2.9}$$

where $c_k, b_k \geq 0, \sum_{k=0}^{\infty} c_k < \infty, \sum_{k=0}^{\infty} b_k < \infty$. Letting $x \to \infty$ and using the equality $q_k(\infty) = 0, k = 0, 1, 2, \ldots$, we get $\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} b_k$. Therefore (2.9) yields

$$\sum_{k=0}^{\infty} c_k q_k(x) = \sum_{k=0}^{\infty} b_k q_k(x). \tag{2.10}$$

From (1.1) and (1.2) follows that $q_k(x) \sim d_k x^{-k-1}$, as $x \to \infty$, where $d_k, k = 0, 1, 2, \ldots$ are positive constants. Multiplying (2.10) successively by $x^m, m = 0, 1, 2, \ldots$, and letting $x \to \infty$, we obtain $c_k = b_k, k = 0, 1, 2, \ldots$

□

Note that we did not use non-negativity of the constants c_k, b_k in (2.9) for the proof of $c_k = b_k, k = 0, 1, 2, \ldots$. Therefore the following remark which we shall use in the proof of Theorem 4 is valid.

Remark. Let $c_k, b_k (k = 0, 1, 2, \ldots)$ be real constants such that $\sum_{k=0}^{\infty}(|c_k| + |b_k|) < \infty$. Assume, the equality (2.9) holds for $|x| \geq r$. Then $c_k = b_k$ for $k = 0, 1, 2, \ldots$.

3. Semigroup $\mathcal{L}_r$ is Delphic. Proof of Theorems 2 and 3.

We shall use D. Kendall's theory of Delphic semigroups ([2], [3]) to prove analogues for the semigroup $\mathcal{L}_r$ of the Khintchin factorization theorems. Recall that a (Hausdorff) topological semigroup $\mathcal{D}$with the neutral element e is called *Delphic* if there exists a continuous homomorphism Δ of $\mathcal{D}$ into the additive semigroup of non-negative numbers and the following conditions are satisfied:

(i) The implication holds : $\Delta(f) = 0 \Longleftrightarrow f = e$.

(ii) For any $f \in \mathcal{D}$, the set of all factors of f is compact.

(iii) Let $\{f_{ij} : j = 1, 2, \dots i; i = 1, 2 \dots\}$ be a triangular array of elements of $\mathcal{D}$ such that

$$\max_{1 \le j \le i} \Delta(f_{ij} \to 0, \text{ as } i \to \infty, \tag{3.1}$$

$$\exists \lim_{i \to \infty} \prod_{j=1}^{i} f_{ij} =: f \in \mathcal{D}, \tag{3.2}$$

then $f \in I(\mathcal{D})$.

The definitions of the classes $I(\mathcal{D}), I_0(\mathcal{D})$ and $N(\mathcal{D})$ are quite similar to those of $I(\mathcal{L}_r), I_0(\mathcal{L}_r), N(\mathcal{L}_r)$ given in the Introduction.

D. Kendall ([2], [3]) proved that, for any Delphic semigroup $\mathcal{D}$, the analogues of both of the Khinchin theorems mentioned in the Introduction are valid in the following form.

Theorem A. *Any element $f \in \mathcal{D}$ has at least one representation of the form*

$$f = f_0 \prod_{i=1}^{k} f_i, \quad 0 \le k \le \infty,$$

where $f_0 \in I_0(\mathcal{D}), f_i \in N(\mathcal{D})$. For $k = \infty$, the product is convergent in the topology of $\mathcal{D}$; for $k = 0$, the product is equal to e.

Theorem B. *The following inclusion is valid*

$$I_0(\mathcal{D}) \subseteq I(\mathcal{D}).$$

In order to prove Theorems 2 and 3, it suffices to prove the following theorem which is of an interest by itself.

Theorem 6. *The semigroup $\mathcal{L}_r$ is Delphic.*
Proof. . Fix $a\mu > r$ and consider the following functional

$$\Delta(f) := -\log f(\mu), \quad f \in \mathcal{L}_r.$$

It is clear that Δ is a continuous homomorphism of $\mathcal{L}_r$ into the additive semigroup of non-negative numbers. Let us verify that the abovementioned conditions (i)-(iii) from the definition of a Delphic semigroup are satisfied.

Evidently, (i) is satisfied in virtue of (1.8). In order to verify (ii), we show that for any sequence of factors of a function $f \in \mathcal{L}_r$ there exists a subsequence uniformly convergent to a factor of f. Assume, $f = g_n h_n$ where $g_n, h_n \in \mathcal{L}_r$, $n = 1, 2, , ,$. For any $\varepsilon > 0$ there exists $\lambda = \lambda(\varepsilon) > r$ such that $f(\lambda) > 1-\varepsilon$ and, hence, $g_n(\lambda) \geq g_n(\lambda)h_n(\lambda) = f(\lambda) > 1 - \varepsilon$. The condition of Lemma 3 is satisfied for the function $g_n = \sum_{k=-1}^{\infty} a_k^{(n)} q_k$. Therefore (2.2) is valid and, using Lemma 1, we see that there exists a subsequence $\{n_j\}$ such that $g_{n_j}(x) \Rightarrow g(x) \in \mathcal{L}_r$. Similarly, there exists a subsequence $\{n'_j\}$ of the sequence $\{n_j\}$ such that $h_{n'_j}(x) \Rightarrow h(x) \in \mathcal{L}_r$. Evidently, $f = gh$ i.e. g is a factor of f. Thus, (ii) is satisfied for $\mathcal{L}_r$.

In order to verify (iii), consider a triangular array $\{f_{ij} : j = 1, 2, \ldots, i, i = 1, 2, \ldots\}$ of elements of $\mathcal{L}_r$ satisfying the conditions (3.1), (3.2). From (3.1) and Lemma 5, it follows that

$$\max_{1 \leq j \leq i} |f_{ij} - 1| \Rightarrow 0 \text{ as } i \to \infty. \tag{3.3}$$

Hence $f_{ij}(x) > 0$ for $|x| \geq r$ and large i. In the case of $\mathcal{L}_r$, (3.2) means that

$$\prod_{j=1}^{i} f_{ij}(x) \Rightarrow f(x), \text{ as } i \to \infty, \tag{3.4}$$

therefore f in (3.4) is non-negative. The relation (3.4) can be rewritten in the form

$$\exp\left(\sum_{j=1}^{i} \log f_{ij}(x)\right) \Rightarrow f(x). \tag{3.5}$$

Using (3.3), we have for large i

$$|\log f_{ij} - (f_{ij} - 1)| \leq |f_{ij} - 1|^2$$

whence, using (3.3) once more, we have

$$|\sum_{j=1}^{i} \log f_{ij} - \sum_{j=1}^{i} (f_{ij} - 1)| \leq$$

$$\sum_{j=1}^{i}|f_{ij}-1|^2 \le \max_{1\le j\le i}|f_{ij}-1|\cdot\sum_{j=1}^{i}|f_{ij}-1| = \circ\left(\sum_{j=1}^{i}|f_{ij}-1|\right), \text{ as } i\to\infty,$$

and

$$\sum_{j=1}^{i}\log f_{ij} = (1+\circ(1))\sum_{j=1}^{i}(f_{ij}-1),$$

as $i\to\infty$ uniformly with respect to $x\in\{|x|\ge r\}$. If follows that

$$\prod_{j=1}^{i}\exp(f_{ij}-1)\Rightarrow f. \tag{3.6}$$

By Theorem 1, $\exp(f_{ij}-1)\in I(\mathcal{L}_r)$, therefore, $\exp\left(\frac{1}{n}(f_{ij}-1)\right)\in\mathcal{L}_r$, for any $n=1,2,\ldots$. From (3.6) it follows

$$\prod_{j=1}^{i}\exp\left(\frac{1}{n}(f_{ij}-1)\right)\Rightarrow(f)^{1/n}.$$

But $(f)^{1/n}$ is right-continuous at $x=r$ since f is. Using Lemma 4, we conclude that $(f)^{1/n}\in\mathcal{L}_r, n=1,2,\ldots$. Thus, $f\in I(\mathcal{L}_r)$. □

4. The description of $I_0(\mathcal{L}_r)$. Proof of Theorem 4.

We need some lemmas.

Lemma 6. *For any $k=0,1,2,\ldots$, there exists $\varepsilon_0=\varepsilon_0(k)>0$ such that the following representations are valid for $0<\varepsilon<\varepsilon_0$:*

$$q_k^2-\varepsilon q_k q_{3k+2}=\sum_j\delta_j q_j, \tag{4.1}$$

$$q_k^2-\varepsilon q_{3k+2}^2=\sum_j\nu_j q_j, \tag{4.2}$$

$$q_k^3-\varepsilon q_{3k+2}=\sum_j\theta_j q_j, \tag{4.3}$$

where δ_j,ν_j,θ_j are non-negative constants.

Lemma 7. *For any* $b > 0$ *and* $k = 0, 1, 2, \ldots,$ *there exists* $\varepsilon_0 = \varepsilon_0(k, b) > 0$ *such that the following representations are valid for* $0 < \varepsilon < \varepsilon_0$:

$$(q_k - \varepsilon q_{3k+2})^2 = \sum_j \alpha_i q_j, \tag{4.4}$$

$$(q_k - \varepsilon q_{3k+2})^3 = \sum_j \beta_j q_j, \tag{4.5}$$

$$-\varepsilon b q_{3k+2} + \frac{1}{6} b^3 (q_k - \varepsilon q_{3k+2})^3 = \sum_j \gamma_j q_j, \tag{4.6}$$

where $\alpha_j, \beta_j, \gamma_j$ *are non-negative constants.*

Lemma 8. *For any* $b > 0$ *and* $k = 0, 1, 2, \ldots,$ *we have*

$$w_k(x) := \exp(b(q_k(x) - 1)) \notin I_0(\mathcal{L}_r).$$

Theorem 4 is an immediate corollary of Lemma 8 and Theorems 1 and 3. Therefore, it suffices to prove the lemmas.

Proof of Lemma 6. Using (1.4), we have

$$q_k^2 = \sum_{m=0}^{\infty} A_{2m+1}(k, k) \frac{Q_{2k+2m+1}(r)}{Q_k^2(r)} q_{2k+2m+1}, \tag{4.7}$$

$$q_k q_{3k+2} = \sum_{j=0}^{\infty} A_{2j+1}(k, 3k+2) \frac{Q_{4k+2j+3}(r)}{Q_k(r) Q_{3k+2}(r)} q_{4k+2j+3} =$$

$$= \sum_{m=k+1}^{\infty} A_{2(m-k-1)+1}(k, 3k+2) \frac{Q_{2k+2m+1}(r)}{Q_k(r) Q_{3k+2}(r)} q_{2k+2m+1}. \tag{4.8}$$

From (4.7), (4.8) we find

$$q_k^2 - \varepsilon q_k q_{3k+2} = \sum_{m=0}^{k} A_{2m+1}(k, k) \frac{Q_{2k+2m+1}(r)}{Q_k^2(r)} q_{2k+2m+1}$$

$$+ \sum_{m=k+1}^{\infty} A_{2m+1}(k, k) \frac{Q_{2k+2m+1}(r)}{Q_k^2(r)} [1 - \varepsilon B(m, k)] q_{2k+2m+1}, \tag{4.9}$$

$$B(m, k) := \frac{A_{2(m-k-1)+1}(k, 3k+2)}{A_{2m+1}(k, k)} \frac{Q_k(r)}{Q_{3k+2}(r)}.$$

By the Stirling formula $\Gamma(n+1/2)/\Gamma(n+1) \sim n^{-1/2}$, as $n \to \infty$. Hence, from (1.5) we find

$$\forall p, s \geq 0: \ A_{2m+1}(p,s) \sim 2m^{-1}, \text{ as } m \to \infty \tag{4.10}$$

This implies that $B(m,k)$ is bounded with respect to m. Therefore, there exists $\varepsilon_0 = \varepsilon_0(k) > 0$ such that the expression in the brackets in (4.9) is non-negative for all m and $0 < \varepsilon < \varepsilon_0$. This yields (4.1).

To obtain (4.2), we use the equality

$$q_{3k+2}^2 = \sum_{m=2k+2}^{\infty} A_{2(m-2k-2)+1}(3k+2, 3k+2) \frac{Q_{2k+2m+1}(r)}{Q_{3k+2}^2(r)} q_{2k+2m+1}. \tag{4.11}$$

It follows from (4.7) by replacing k by $3k+2$ and m by $m-2k-2$. From (4.7), (4.11) we obtain

$$q_k^2 - \varepsilon q_{3k+2}^2 = \sum_{m=0}^{2k+1} A_{2m+1}(k,k) \frac{Q_{2k+2m+1(r)}}{Q_k^2(r)} q_{2k+2m+1} +$$

$$\sum_{m=2k+2}^{\infty} A_{2m+1}(k,k) \frac{Q_{2k+2m+1}}{Q_k^2(r)} [1 - \varepsilon C(m,k)] q_{2k+2m+1},$$

$$C(m,k) := \frac{A_{2(m-2k-2)+1}(3k+2, 3k+2)}{A_{2m+1}(k,k)} \left(\frac{Q_k(r)}{Q_{3k+2}(r)} \right)^2.$$

Using (4.10), we see that $C(m,k)$ is bounded with respect to m. This yields (4.2).

To obtain (4.3), we multiply (4.7) by q_k. We get

$$q_k^3 = \sum_{m=0}^{\infty} A_{2m+1}(k,k) \frac{Q_{2k+2m+1}(r)}{Q_k^2(r)} q_{2k+2m+1} q_k.$$

Using, for the product $q_{2k+2m+1} q_k$, the representation (1.4) with $p = 2k+2m+1, s = k$, we get a representation of q_k^3 by a series in q_j with non-negative coefficients where the coefficient of q_{3k+2} is stricktly positive. This yields (4.3).

Proof of Lemma 7. The representation (4.4) follows immediately from (4.1) since $(q_k - \varepsilon q_{3k+2})^2 = (q_k^2 - 2\varepsilon q_k q_{3k+2}) + \varepsilon^2 q_{3k+2}^2$.

To prove (4.6) we represent its left side as

$$-\varepsilon b q_{3k+2} + \frac{1}{6} b^3 q_k^3 - \frac{1}{2} \varepsilon b^3 q_k^2 q_{3k+2} + \frac{1}{2} \varepsilon^2 b^3 q_k q_{3k+2}^2 - \frac{1}{6} \varepsilon^3 q_{3k+2}^3 =$$

$$= \frac{1}{18}b^3\left[q_k^3 - \frac{18\varepsilon}{b^2}q_{3k+2}\right] + \frac{1}{18}b^3 q_k[q_k^2 - 9\varepsilon q_k q_{3k+2}] +$$

$$+\frac{1}{18}b^3[q_k^3 - 3\varepsilon^3 q_{3k+2}^3] + \frac{1}{2}\varepsilon^2 b^3 q_k q_{3k+2}^2.$$

By (4.3), for sufficiently small ε, the expression in the first brackets can be written as a linear combination of functions q_j with non-negative coefficients. By (4.1) the analogous statement holds for the expression in the second brackets. Let us write the expression in the third brackets in the form

$$q_k^3 - 3\varepsilon^3 q_{3k+2}^3 = q_k(q_k^2 - \varepsilon_1 q_k q_{3k+2}) + \varepsilon_1 q_{3k+2}\left(q_k^2 - \frac{3\varepsilon^3}{\varepsilon_1}q_{3k+2}^2\right).$$

Using (4.1), we choose $\varepsilon_1 = \varepsilon_1(k) > 0$ such that the expression in the first parentheses can be written as a linear combination of q_j with non-negative coefficients. Then, using (4.2), we find $\varepsilon_0 = \varepsilon_0(k)$ such that the expression in the second parentheses can be written in the same way for $0 < \varepsilon < \varepsilon_0$. Thus, (4.6) is proved.

Evidently, we get the left hand side of (4.5) from that of (4.6) by adding to the latter the term $\varepsilon b q_{3k+2}$. Therefore, (4.5) is an immediate corollary of (4.6).

Proof of Lemma 8. We write w_k in the form

$$w_k = c\varphi_k\psi_k,$$

where

$$c := \exp(-b + \varepsilon b), \varphi_k := \exp(b(q_k - \varepsilon q_{3k+2})), \psi_k := \exp(\varepsilon b(q_{3k+2} - 1)), \varepsilon > 0.$$

By Theorem 1, we have $\psi_k \in \mathcal{L}_r$. Show that $c\varphi_k \in \mathcal{L}_r$ for some $\varepsilon > 0$. For this we rewrite φ_k in the form

$$\varphi_k = \sum_{m=0}^{\infty}\frac{b^m}{m!}z_k^m = 1 + bq_k + \frac{b^2 z_k^2}{2} + \left[-\varepsilon b q_{3k+2} + \frac{b^3}{6}z_k^3\right] + \sum_{m=4}^{\infty}\frac{b^m}{m!}z_k^m,$$

where $z_k := q_k - \varepsilon q_{3k+2}$. By (4.4) and (4.6) of Lemma 7, the term z_k^2 and the expression in the brackets are linear combinations of q_i's with non-negative coefficients for $\varepsilon > 0$ being small enough. Since any integer $m > 3$ can be represented in the form $m = 2p+3q$, we have $z_k^m = (z_k^2)^p(z_k^3)^q$ for $m > 3$. Using (4.4), (4.5) of Lemma 7, we conclude that $z_k^m, m > 3$, are linear combinations of q_j's with non-negative coefficients. Hence, $c\varphi_k \in \mathcal{L}_r$.

Assume, Lemma 8 is not valid, i.e. $w_k \in I_0(\mathcal{L}_r)$. Then, evidently, each factor of w_k belongs to $I_0(\mathcal{L}_r)$ as well. Since $c\varphi_k \in I(\mathcal{L}_r)$. Then, by Theorem 1, $c\varphi_k$ can be represented in the form (1.9). But, on the other hand, we have the representation $c\varphi_k = \exp(b(q_k - 1) - b\varepsilon(q_{3k+2} - 1))$. Using the Remark at the end of no. 2, we obtain a contradiction.

□

5. Indecomposable elements of $\mathcal{L}_r$. Proof of Theorem 5.

We need the following sufficient conditions of indecomposability of an element of $\mathcal{L}_r$.

Theorem 7. *Assume,* $f = \sum_{k=-1}^{\infty} a_k q_k \in \mathcal{L}_r, f(x) \not\equiv 1$. *If either*

$$\text{(i) } \exists m \geq 0, \forall k > m: \ a_k = 0,$$

or

$$\text{(ii) } \forall t \geq 0, \ \exists l \geq t: \ a_l = a_{l+1} = 0,$$

then $f(x) \in N(\mathcal{L}_r)$.

Proof. Asssume, f is not indecomposable. Then we have a factorization $f = f_1 f_2$ where $f_j \in \mathcal{L}_r$,

$$f_j = \sum_{k=-1}^{\infty} a_k^{(j)} q_k \not\equiv 1, \ \ j = 1, 2.$$

Hence

$$f = \sum_{p,s=-1}^{\infty} a_p^{(1)} a_s^{(2)} q_p q_s$$

where $a_p^{(1)} a_s^{(2)} > 0$ at least for one pair (p, s) such that $p \geq 0, s \geq 0$. Using (1.4) and (1.6), we see that neither (i) nor (ii) can be satisfied for f. □

Proof of Theorem 5. Let $f = \sum_{k=-1}^{\infty} a_k q_k \in \mathcal{L}_r$. In the case $f \not\equiv 1$ we set $A_n := \sum_{k=-1}^{n} a_k, f_n = A_n^{-1} \sum_{k=-1}^{n} a_k q_k$ where n is so large that A_n is stricktly positive. In the cases $f(x) \equiv 1$ we set $f_n = \left(1 - \frac{1}{n}\right) q_{-1} + \frac{1}{n} q_0$. In both cases $f_n \in N(\mathcal{L}_r)$ by Theorem 7. It is obvious that $f_n \Rightarrow f$.

□

Acknowledgements

The author is grateful to Professor I.V. Ostrovskii for the statement of the problem and the attention to the work, to A.M. Vishnyakova for the remark which simplified the proof of (4.6), and to T.V. Misyura for help in the preparation of the paper.

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İkinci Tip Legendre Fonksiyonları Serilerinin Yarı-Grubunun Aritmetiği

Özet

Delphic yarı-gruplarının D. Kendall teorisi üzerine düzenlenmiş bu çalışmada, ikinci tip Legendre fonksiyonları serilerinin yarı-grupları irdelendi. Temel faktorizasyon teoremleri ispatlandı, sonsuz bölünebilir elemanların ve ayrıştırılmayan çarpanları olmayan elemanların sınıfları tamamen belirlendi, ayrıştırılamayan elemanların sınıfının yoğunluğu saptandı.

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Received 13.5.1996