ABOUT SOME NORMAL SUBGROUPS OF HECKE GROUPS

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Abstract

In this work we study the structure of some special normal subgroups of Hecke groups. Particularly those of genus 0 and 1 are studied by means of the regular map theory which has been a growing subject in recent years. As a nice application, we calculate the total number of genus 1 normal subgroups for a given index.

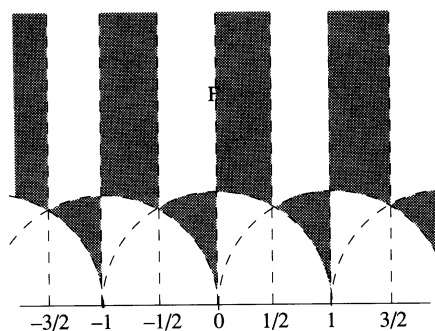**Key words: Hecke groups, level, parabolic class number.**

1. Introduction

Hecke groups are, in some sense, a generalisation of the well-known modular group

$$\Gamma = PSL(2,\mathbf{Z}) = \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbf{Z}, ad - bc = 1 \right\}.$$

Γ is generated by $R(z) = -1/z$ and $T(z) = z + 1$. A fundamental region for Γ is

$$F = \{z \in U : |z| > 1, |Rez| < 1/2\}.$$

Therefore Γ is a discrete group.

Hecke groups $H(\lambda)$ have been introduced by E. Hecke, [2]. They are subgroups of $PSL(2, \mathbf{R})$ and are generated by $R(z) = -1/z$ and $T(z) = z + \lambda$. Hecke asked the question that for what values of λ these groups are discrete. He found a fundamental region

$$F_\lambda = \{z \in U : |z| > 1, |Rez| < \lambda/2\}$$

when $\lambda = \lambda_q = 2\cos(\pi/q), q \geq 3$, integer if $\lambda < 2$; and for every real $\lambda \geq 2$. Therefore $H(\lambda)$ is discrete iff $\lambda = \lambda_q$ or $\lambda \geq 2$.

Here we deal with the former case where $\lambda = \lambda_q$ and denote the group by $H(\lambda_q)$. Then $S(z) = RT(z)$ is of order q and $H(\lambda_q)$ has signature $(0; 2, q, \infty)$. That is, it is a triangle group containing a parabolic element.
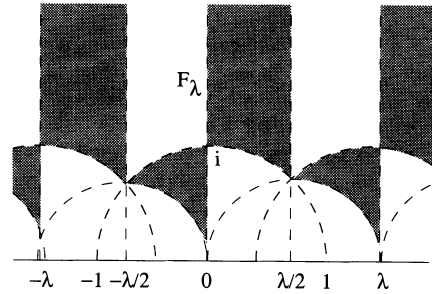
Let us now return to the fundamental region $F_{\lambda q}$. Each triangle has two vertices inside U and one on the line $\mathbf{R} \cup \{\infty\}$. Former two are the images of the fixed points $\mathbf{i}$ and $\zeta = \exp(i\pi/q)$ of R and S, respectively. The last one is an image of ∞ under a group element. It is called a *parabolic (cusp) point*. The problem of determining all parabolic points of $H(\lambda_q)$ is still open although there has been several attempts giving partial answers. To solve this problem, one needs to know the elements of $H(\lambda_q)$.

Most important and interesting Hecke group is the modular group $\Gamma = H(\lambda_3)$. As $\lambda_3 = 1$, the underlying field for this group is $\mathbf{Q}$, i.e. all coefficients are rational integers. Hence the parabolic number set of Γ is $\mathbf{Q}$.

Next two interesting Hecke groups are obtained for $q = 4$ and $q = 6$. As $\lambda_4 = \sqrt{2}$ and $\lambda_6 = \sqrt{3}$ we denote them by $H(\sqrt{2})$ and $H(\sqrt{3})$. The underlying fields are $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$-quadratic extensions of $\mathbf{Q}$ by $\sqrt{2}$ and $\sqrt{3}$, respectively.

One of the main reasons for $H(\sqrt{2})$ and $H(\sqrt{3})$ to be two of the most important Hecke groups is that apart from Γ, they are the only Hecke groups whose elements can be completely described. For $m = 2$ or 3, $H(\sqrt{m})$ consists of the set of all matrices of the following two types:

(i) $\begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}$; $a, b, c, d \in Z, ad - mbc = 1$

(ii) $\begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix}$; $a, b, c, d \in Z, mad - bc = 1$

Those of type (i) are called *even* while those of type (ii) called *odd*. Even elements from a subgroup of index 2 called the *even subgroup*. It is denoted by $H_e(\sqrt{m})$. In general for any even q, this subgroup is obtained in the following way:

These exists a homomorphism $\Theta : H(\lambda_q) \cong (2, q, \infty) \longrightarrow C_2 \cong (2, 2, 1)$ such that $R \longrightarrow (1\ 2)$ and $T \longrightarrow (1)(2)$. Then by the permutation method given by Singerman, $H_e(\lambda_q)$ is isomorphic to $\mathbf{Z} \,{}^*C_{q/2}$. Note that the even subgroup does not exist when q is odd.

By the above classification, the parabolic elements of $H(\sqrt{m})$ will have form $a\sqrt{m}/c$, $a, c \in \mathbf{Z}$ and therefore the parabolic point set is a subset of $\mathbf{Q}(\sqrt{m})$.

Another interesting Hecke group is $H(\lambda_5)$. Here $\lambda_5 = (1 + \sqrt{5})/2$ is known as the golden ratio, and has the property $\lambda_5 + 1 = \lambda_5^2$. Therefore all powers of λ_5 can be induced to a linear form $a\lambda_5 + b$. The underlying field is $\mathbf{Q}(\lambda_5)$. Hence the parabolic points are quotients of elements of $\mathbf{Q}(\lambda_5)$. A typical one is $(a_1 + a_2\lambda_5)/(b_1 + b_2\lambda_5)$.

There is no general method giving the parabolic point set for all Hecke groups.

2. Connection with Regular Maps

Normal subgroups of Γ have been studied by many people and classification theorems are given. Our aim here is to find normal subgroups of Hecke groups. One way of doing this is to use regular map theory.

A *map* M is an embedding (without crossings) of a finite connected graph G into a compact connected surface S without boundary such that $S - G$ is a union of 2-cells. Here, we also assume that S is orientable.
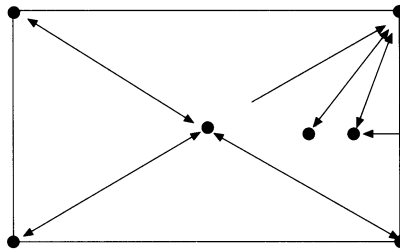
If m and n are the l.c.m.'s of the valencies of the faces and vertices of M, respectively, M is of type $\{m, n\}$.

The *dual map* of M has the same underlying surface S while the vertices and face centers interchanghed. It is of type $\{n, m\}$. e.g. cube and octahedron are dual maps.

A *dart* of M is a pair consisting of an edge and an incident vertex.
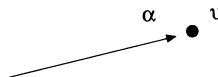
An *automorphism* of M is an orientation-preserving homeomorphism of S preserving the incidence of the darts of M. The set of all automorphisms of M forms the

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automorphism group Aut M of M. If Aut M is transitive on the set of darts, then M is called *regular*. If a map is regular then every vertex has the same valency and every face has the same valency.
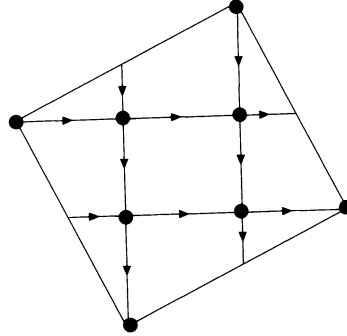
The study of maps is closely related to the study of subgroups of triangle groups $(2, m, n)$. Jones and Singerman showed the existence of a 1:1 correspondence between normal subgroups of $(2, m, n)$ and regular maps, [3]. Using this we are able to find many results about the normal subgroups of Hecke groups:

For m dividing q, we have a homomorphism

$$\Theta : H(\lambda_q) \cong (2, q, \infty) \longrightarrow (2, m, n)$$

Also associated with every regular map of type $\{m, n\}$ there is a normal subgroup N of $(2, m, n)$. Then $\Theta^{-1}(N)$ is a normal subgroup of $H(\lambda_q)$ corresponding to a regular map of type $\{m, n\}$.

The number n is the *level* of $\Theta^{-1}(N)$. Let us see this with an example.

We have the relations

$$R^2 = S^q = I \quad \text{in } H(\lambda_q).$$

In $H(\lambda_q)/H'(\lambda_q)$ we have

$$R^2 = S^q = I, \quad RS = SR.$$

Therefore $H(\lambda_q)/H'(\lambda_q) \cong C_2 \times C_q$. Hence $|H(\lambda_q)/H'(\lambda_q)| = 2q$. Also $(RS)^{2q} = I$ if q is odd, and $(RS)^q = I$ if q is even. Therefore there are homomorphisms

$$H(\lambda_q) \longrightarrow (2, q, 2q) \longrightarrow C_{2q} \cong H(\lambda_q)/H'(\lambda_q) \text{ if } q \text{ is odd,}$$

$$H(\lambda_q) \longrightarrow (2, q, q) \longrightarrow C_2 \times C_q \cong H(\lambda_q)/H'(\lambda_q) \text{ if } q \text{ is even.}$$

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3. Number of Genus 1 Normal Subgroups

As an application of regular map theory to Hecke groups, we want to calculate the number $N_4(\mu)$ of normal genus 1 subgroups of index μ in $H(\lambda_4)$. It is known that all regular maps on a torus are of type $\{3,6\}, \{4,4\}$ and $\{6,3\}$. As $H(\sqrt{2}) \cong (2,4,\infty)$, only the ones of type $\{4,4\}$ correspond to the normal subgroups of $H(\sqrt{2})$. They are classified as $\{4,4\}_{r,s}, r, s \in \mathbf{Z}$, such that the automorphism group has order $4(r^2 + s^2)$. Also as we are mapping onto a subgroup of (2,4,4), the level of such a subgroup is 4. Therefore

$$\mu = 4t = 4(r^2 + s^2)$$

and hence

$$t = r^2 + s^2.$$

Given $\mu = 4(r^2 + s^2), N_4(\mu)$ is equal to the number of "non-identical" pairs (r,s) such that $t = r^2 + s^2$. By identical pairs we understand the following:

1. if $r = s \neq 0$, then $(r,r), (r,-r)(-r,r), (-r,-r)$ are identical,

2. if $r \neq 0, s = 0$, then $(r,0)(0,r), (-r,0), (0,-r)$ are identical,

3. if $r \neq s$, and non-zero then $(r,s), (r,-s)(-r,s), (-r,-s)$ are identical, and (s,r), $(s,-r), (-s,r), (-s,-r)$ are identical.
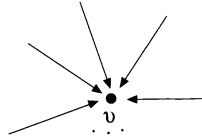
Therefore

$$N_4(\mu) = \frac{1}{4} \cdot \#\{(r,s) \in \mathbf{Z} \times \mathbf{Z} : r^2 + s^2 = t\}.$$

Using results from the number theory, $N_4(\mu)$ can easily be calculated.

4. Normal Genus 0 Subgroups

These can be found using regular map theory. Here we use a similar method to find them. They correspond to five platonic solids, dihedrons and star maps.

Let N be a normal subgroup of $H(\lambda_q)$ of genus 0. Then $H(\lambda_q)/N$ is a group of automorphisms of U_∞/N where $U_\infty = U \cup \{\infty\}$. Therefore $H(\lambda_q)/N \cong A_4, S_4, A_5, C_n$ or $D_n, n \in \mathbf{N}$. Now if $3|q$, then mapping onto A_4, S_4, A_5 respectively gives $(0; (q/3)^{(4)}$,

$\infty^{(4)}$), $(0; (q/3)^{(8)}, \infty^{(6)})$ and $(0; (q/3)^{(20)}, \infty^{(12)})$. Secondly, if $4|q$, then mapping onto S_4 we obtain $(0; (q/4)^{(6)}, \infty^{(8)})$. Thirdly, if $5|q$, then mapping onto A_5 gives $(0; (q/5)^{(12)}, \infty^{(20)})$. Also for any q and every $n|q$, mapping onto C_n and D_n gives $(0; 2^{(n)}, q/n, \infty)$ and $(0; q/n^{(2)}, \infty^{(n)})$.

Note that all these give finitely many normal subgroups of genus 0 of $H(\lambda_q)$. There is also an infinite class of such subgroups when q is even: For even q, mapping onto $D_n \cong (2, 2, n)$ gives $(0; 2^{(n)}, \infty^{(2)})$.

Theorem 1. $H(\lambda_q)$ has finitely many normal genus 0 subgroups if q is odd, and infinitely many if q is given.

Note that most genus 0 normal subgroups of $H(\lambda_q)$ have torsion. In fact there is only a few torsion-free genus 0 normal subgroups:

Theorem 2. All normal torsion-free subgroups of genus 0 of $H(\lambda_q)$ are

N	**$H(\lambda_q)$**	$\mu = \text{index}$
$(0; \infty^{(4)})$	Γ	12
$(0; \infty^{(6)})$	Γ	24
$(0; \infty^{(12)})$	Γ	60
$(0; \infty^{(8)})$	$H(\sqrt{2})$	24
$(0; \infty^{(20)})$	$H(\lambda_5)$	60
$(0; \infty^{(q)})$	$H(\lambda_q)$	$2q$

Corollary 1. The number of normal torsion-free genus 0 subgroups of $H(\lambda_q)$ is

$$\begin{cases} 4 & \text{if} \quad q = 3 \\ 2 & \text{if} \quad q = 4, 5 \\ 1 & \text{if} \quad q \geq 6 \end{cases}$$

The number of parabolic classes of each normal subgroup is actually equal to the number of vertices of the corresponding regular solid. In fact every vertex can be thought of as a cusp on the sphere, e.g. there are four classes of parabolic points for the subgroup $(0; \infty^{(4)})$ and the corresponding regular solid is a hyperbolic tetrahedron with four ideal vertices.
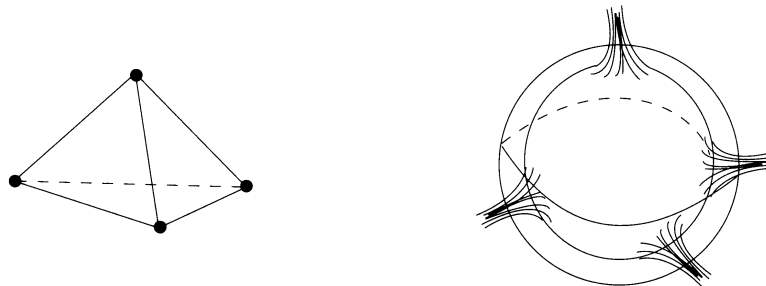
5. Principal Congruence Subgroups of $\mathbf{H}(\lambda_q)$

These are the most important and interesting normal subgroups of the modular group Γ. There are many results about them. Newman [6] and McQuillan [5] gave a complete classification of them. The *principal congruence subgroup of level n of* Γ is defined by

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mp \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, mod\, n \right\}.$$

$\Gamma(n)$ is a normal subgroup of Γ. Here we say a principal congruence subgroup is of level n if it contains T^n and n is minimal with this property.

Another way of obtaining $\Gamma(n)$ is to consider the *"reduction homomorphism"* which reduces everything modulo n. Then $\Gamma(n)$ is the kernel of this homomorphism.

The *principal congruence subgroup of level p of* $H(\lambda_q)$ is defined by

$$\Gamma_p(\lambda_q) = \{T \in H(\lambda_q) : T \equiv \mp I (mod\, p)\}.$$

The kernel of reduction homomorphism for $q > 3$ is not always $\Gamma_p(\lambda_q)$. Let us denote it by $K_p(\lambda_q)$. By the definition

$$\Gamma_p(\lambda_q) \lhd K_p(\lambda_q).$$

e.g. let $q = 4, p = 7$. Here $3^2 \equiv 2$ mod 7 and therefore $\sqrt{2}$ can be considered as an element, 3, of GF(7). Therefore there exists a homomorphism

$$\Theta : H(\sqrt{2}) \longrightarrow PSL(2,7)$$

induced by

$$\begin{pmatrix} a\sqrt{2} & b \\ c & d\sqrt{2} \end{pmatrix} \rightarrow \begin{pmatrix} 3a & b \\ c & 3d \end{pmatrix} \text{ and } \begin{pmatrix} a & b\sqrt{2} \\ c\sqrt{2} & d \end{pmatrix} \rightarrow \begin{pmatrix} a & 3b \\ 3c & d \end{pmatrix}.$$

By Macbeath's results, [4], Θ is onto and (R_7, S_7, T_7) generates whole PSL (2,7), *i.e.*

$$H(\sqrt{2})/K_7(\sqrt{2}) \cong PSL(2,7).$$

We want to determine whether $K_7(\sqrt{2}) = \Gamma_7(\sqrt{2})$. Therefore we are looking for an element in $K_7(\sqrt{2}) - \Gamma_7(\sqrt{2})$. Indeed there is such an odd element

$$A = \begin{pmatrix} 5\sqrt{2} & 7 \\ 7 & 5\sqrt{2} \end{pmatrix}.$$

A is of order 2 mod 7 and hence

$$K_7(\sqrt{2}) \cong \Gamma_7(\sqrt{2}) \cup A\Gamma_7(\sqrt{2}).$$

Therefore

$$H(\sqrt{2})/\Gamma_7(\sqrt{2}) \cong K_7(\sqrt{2})/\Gamma_7(\sqrt{2}) \times H(\sqrt{2})/K_7(\sqrt{2})$$
$$\cong C_2 \times PSL(2,7)$$

Theorem 3. $H(\sqrt{2})/K_p(\sqrt{2}) \cong \begin{cases} PSL(2,p) & if \quad p \equiv \pm 1 \mod 8 \\ PSL(2,p) & if \quad p \equiv \pm 3 \mod 8 \\ C_2 & if \quad p = 2 \end{cases}$

$$H(\sqrt{2})/\Gamma_p(\sqrt{2}) \cong \begin{cases} C_2 \times PGL(2,p) & if \quad p \equiv \pm 1 \mod 8 \\ PGL(2,p) & if \quad p \equiv \pm 3 \mod 8 \\ C_4 & if \quad p = 2. \end{cases}$$

Theorem 4. $H(\lambda_5)/\Gamma_p(\lambda_5) \cong \begin{cases} PSL(2,p) & if \quad p \equiv \pm 1 \quad \mod 10 \\ PSL(2,p^2) & if \quad p \equiv \pm 3 \quad \mod 10, \, p \neq 3 \\ D5 & if \quad p = 2 \\ A_5 & if \quad p = 3,5 \end{cases}$

Theorem 5. *Let* $q = r^n, r$ *prime,* $n \in \mathbf{N}.$ *Then*

$$H(\lambda_q)/\Gamma_p(\lambda_q) \cong \begin{cases} PSL(2,p) & if \quad p \equiv \pm 1 \mod q \quad or \quad if \quad p = r \\ PSL(2,p^d) & if \quad p \neq \pm 1 \mod q \quad p \neq 2, r; \quad d \quad odd \\ PGL(2,p^{d/2}) & if \quad p \neq \pm 1 \mod q \quad p \neq 2, r; \quad d \quad even \\ D_q & if \quad p = 2, \end{cases}$$

where $d = \varphi(2q)/2$ *is the degree of the minimal polynomial of* $\lambda_q.$

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Hecke Gruplarının Bazı Normal Altgrupları Üzerine

Özet

Bu çalışmada Hecke gruplarının bazı özel normal altgruplarının yapısı incelenmiştir. Özellikle cinsi 0 ve 1 olanlarla temel denklik altgrupları yeni gelişmekte olan düzgün figürler teorisinin yardımıyla incelenmiştir. Bu teorinin bir uygulaması olarak cinsi 1 olan normal altgrupların verilen indeks için toplam sayısı hesaplanmıştır.

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