THE RANK AND THE CRANK MODULO 5

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Abstract

Let $p(n)$ denote the number of partitions of n. Ramanujan's partition congruences are $p(5n+4)$, $p(7n+5)$ and $p(11n+6) \equiv \bmod 5$, 7, and 11, respectively. These have been proved in number of ways. Atkin and Swinnerton-Dyer proved the congruences and some more relations about partition in the case of mod 5 and 7 in terms of *rank*, Garvan proved them in three cases in terms of *crank*.

In this study, we give an another proof of their results in the case of mod 5 by using the theory of modular forms. Although our method is more tedious and complicated, it shows us how Modular forms of integral weight on a certain subgroups of $SL_2(\mathbb{Z})$ play role in partition theory. Our method could be applied to the case mod 7, but not mod 11 since the components of $\prod_{n=1}^{\infty}(1-q^n)^{-1}$ are not known explicitly.

1 Introduction

A *partition* of a positive integer n is a representation of n as the sum of any number of positive integers. The number of partitions of n is denoted by $p(n)$. Let $p(0) = 1$. The *generating function* of $p(n)$ is the power series

$$\mathbf{P} = \sum_{n \geq 0} p(n)q^n = \prod_{r=1}^{\infty}(1-q^r)^{-1} \tag{1.1}$$

where $|q| < 1$, (see [9]). The well-known Ramanujan's congruences are

$$p(5n+4) \equiv 0 \pmod{5} \tag{1.2}$$

$$p(7n+5) \equiv 0 \pmod{7} \tag{1.3}$$

$$p(11n+6) \equiv 0 \pmod{11} \tag{1.4}$$

Dyson[4] conjectured and Atkin Swinnerton-Dyer proved combinatorial results which imply the congruences (1.2) and (1.3). Let

$$\pi = \pi_0 + \pi_1 + \cdots + \pi_{s-1} \tag{1.5}$$

be a partition with $\pi_0 \geq \pi_1 \geq \cdots \geq \pi_{s-1}$. Dyson[4] defined the *rank* of π to be the number $\pi_0 - s$. He denoted by $N(r, m, n)$ the number of partitions of n with rank congruent to r modulo m and remarked that several relations appeared to hold between the numbers $N(r, m, mn + c)$ when $m = 5$ and $m = 7$. Of particular interest are the relations

$$N(0, 5, 5n + 4) = N(1, 5, 5n + 4) = \cdots = N(4, 5, 5n + 4)$$
$$N(0, 7, 7n + 5) = N(1, 7, 7n + 5) = \cdots = N(6, 7, 7n + 5)$$

since they provide a combinatorial interpretation to the congruences (1.2) and (1.3). Dyson also conjectured the existence of a "*crank*" which would likewise imply the congruence (1.4). In 1987, Andrews and Garvan[2] discovered the crank as follows

$$crank(\pi) := \begin{cases} \pi_0, & \text{if } w(\pi) = 0 \\ \mu(\pi) - w(\pi), & \text{if } w(\pi) > 0 \end{cases}$$

where $w(\pi)$ is the number of ones in π and $\mu(\pi)$ is the number of parts of π bigger than $w(\pi)$. Let $M(r, m, n)$ denote the number of partitions of n with crank r modulo m. Several relations are known to hold between the numbers $M(r, m, an + c)$ when $(m, a) = (5, 5)$, $(7, 7)$ and $(11, 11)$ [7] and when $(m, a) = (8, 4)$, $(9, 3)$ and $(10, 15)$ [8].

Dyson[5] (following Andrews and Garvan) defined the crank of π in the following way, set $t = \pi_0 - \pi_1$,

$$crank(\pi) := \begin{cases} -s, & \text{if } t = 0 \\ t - \pi_t, & \text{if } t > 0 \end{cases}$$

(supposing $\pi_r = 0$ when $r \geq s$). Since the Andrews-Garvan crank of a partition, π, is the negative of the Dyson crank of the conjugate of π, both definitions give the same value for $M(r, m, n)$ unless $n = 1$ Let $N(m, n)$ and $M(m, n)$ denote the number of partitions of n with rank and crank respectively congruent to r modulo m. We change this definition of $M(m, n)$ just a little, setting $M(0, 1) = -1$ and $M(-1, 1) = 1 = M(1, 1)$, and modify $M(r, m, n)$ accordingly. We shall also suppose that the empty partition of zero has rank zero.

For convenience, we write N_1 for M and N_3 for N. By (12) in [11] and (1.11) in [2] when $k = 1$, and (2.12) in [3] when $k = 3$, we have

$$\sum_{m \in \mathbb{Z}} \sum_{n \geq 0} N_k(m, n) z^m q^n = \mathbf{P}(1 - z) \sum_{m \in \mathbb{Z}} (-1)^n \frac{q^{n(kn+1)/2}}{1 - zq^n} \tag{1.6}$$

where $\mathbf{P} = \prod_{n=1}^{\infty} (1 - q^n)^{-1}$, $|q| < 1$ and $|q| < |z| < |q^{-1}|$

For a power series $X = X(q) = \sum_{n \geq 0} a_n q^n$, we define

$$X^{(r)} := q^r \sum_{n \geq 0} a_{5n+r} y^n \quad (r = 0, \ 1, ..., \ m - 1)$$

and say that $X^{(r)}$ is r*th* component of X. Here, and below, $y = q^5$.

We introduce some notations

$$(z;q)_\infty := \prod_{r=1}^{\infty}(1 - zq^{r-1}),$$

$$J(z;q) := (z;q)_\infty(z^{-1}q;q)_\infty = \prod_{r=1}^{\infty}(1 - zq^{r-1})(1 - z^{-1}q^r).$$

Note that

$$-zJ(zq;q) = -zJ(z^{-1};q) = J(z;q) = J(z^{-1}q;q). \tag{1.7}$$

We also write

$$J(a) := J(y^a;y^5) = \prod_{r=1}^{\infty}(1 - y^{a+5(r-1)})(1 - y^{5r-a})$$

$$J(0) := \prod_{r=1}^{\infty}(1 - y^{5r})$$

It should be noted that $J(0)$ is not the expression that would be obtained by writing 0 instead of a in the definition of $J(a)$. From (1.7), we have

$$J(5 - a) = J(a), \qquad J(-a) = J(5 + a) = -y^{-a}J(a). \tag{1.8}$$

we also define

$$\mathbf{R}_{ij}(k) := N(i)^{(k)} - N(j)^{(k)} \quad \text{and} \quad \mathbf{C}_{ij}(k) := M(i)^{(k)} - M(j)^{(k)} \tag{1.9}$$

so that

$$\mathbf{R}_{ij}(k) = q^k \sum_{n \geq 0} \left(N(i, 5, 5n + k) - N(j, 5, 5n + k)\right) y^n$$

and

$$\mathbf{C}_{ij}(k) = q^k \sum_{n \geq 0} \left(M(i, 5, 5n + k) - M(j, 5, 5n + k)\right) y^n.$$

Theorem 1 (Thm.4 of [3])

$$\mathbf{R}_{12}(0) = \frac{y}{J(0)}T(y, 1, y^5) \tag{1.10}$$

$$\mathbf{R}_{02}(3) = -q^3\frac{y}{J(0)}T(y^2, 1, y^5) \tag{1.11}$$

$$\mathbf{R}_{12}(1) = \mathbf{R}_{02}(2) = \mathbf{R}_{02}(4) = \mathbf{R}_{12}(4) = 0 \tag{1.12}$$

$$\mathbf{R}_{02}(0) + 2\mathbf{R}_{12}(0) = \frac{J(2)J(0)}{J(1)^2} \tag{1.13}$$

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$$\mathbf{R}_{02}(1) \;=\; q\frac{J(0)}{J(1)} \tag{1.14}$$

$$\mathbf{R}_{12}(2) \;=\; q^2\frac{J(0)}{J(2)} \tag{1.15}$$

$$\mathbf{R}_{01}(3) + \mathbf{R}_{02}(3) \;=\; q^3\frac{J(1)J(0)}{J(2)^2} \tag{1.16}$$

Theorem 2 (Thm.4.7 of [7])

$$\mathbf{C}_{01}(0) \;=\; \frac{J(2)J(0)}{J(1)^2} \tag{1.17}$$

$$\mathbf{C}_{01}(1) \;=\; -2q\frac{J(0)}{J(1)} \tag{1.18}$$

$$\mathbf{C}_{12}(1) \;=\; q\frac{J(0)}{J(1)} \tag{1.19}$$

$$\mathbf{C}_{12}(2) \;=\; -q^2\frac{J(0)}{J(2)} \tag{1.20}$$

$$\mathbf{C}_{01}(3) \;=\; q^3\frac{J(1)J(0)}{J(2)^2} \tag{1.21}$$

$$\mathbf{C}_{12}(3) \;=\; -q^3\frac{J(1)J(0)}{J(2)^2}. \tag{1.22}$$

and all other functions $\mathbf{C}_{i,i+1}(d) \;=\; 0, \quad where \quad i = 0 \; or \; 1, \quad are \; zero.$ (1.23)

2 Preparation

We define

$$N_k(r) := \sum_{n\geq 0} N_k(r,5,n)q^n,$$

and

$$S_k(b) := {\sum_n}'(-1)^n\frac{q^{n(kn+1)/2+bn}}{1-q^{5n}}$$

where the sum is taken over all non-zero integers n, and

$$T_k(z,\zeta,q) = \sum_{n=-\infty}^{\infty} (-1)^n\frac{\zeta^n\, q^{3n(n+1)/2}}{1-z\,q^n}$$

which is an analytic function of z in every region $0 < r_1 \leq z \leq r_2$, except for simple poles at $z = q^{-n}$. From (3.6) in [6], we have

$$
\begin{array}{llll}
N(0) &=& \mathbf{P}\left(1 - 2S_3(4)\right), & \qquad M(0) &=& \mathbf{P}\left(2S_1(0) + 1\right) \\
N(1) &=& \mathbf{P}\left(S_3(1) + S_3(4)\right), & \qquad M(1) &=& \mathbf{P}\left(S_1(1) - S_1(0)\right) \qquad (2.1) \\
N(2) &=& -\mathbf{P}S_3(1), & \qquad M(2) &=& -\mathbf{P}S_1(1)
\end{array}
$$

From [10],

$$\mathbf{P}^{(0)} = \frac{J(0)^5}{(y;y)_\infty^6} \left\{ \frac{J(2)^4}{J(1)^4} - 3y\frac{J(1)}{J(2)} \right\} \tag{2.2}$$

$$\mathbf{P}^{(1)} = q\frac{J(0)^5}{(y;y)_\infty^6} \left\{ \frac{J(2)^3}{J(1)^3} + 2y\frac{J(1)^2}{J(2)^2} \right\} \tag{2.3}$$

$$\mathbf{P}^{(2)} = q^2\frac{J(0)^5}{(y;y)_\infty^6} \left\{ 2\frac{J(2)^2}{J(1)^2} - y\frac{J(1)^3}{J(2)^3} \right\} \tag{2.4}$$

$$\mathbf{P}^{(3)} = q^3\frac{J(0)^5}{(y;y)_\infty^6} \left\{ 3\frac{J(2)}{J(1)} + y\frac{J(1)^4}{J(2)^4} \right\} \tag{2.5}$$

$$\mathbf{P}^{(4)} = 5q^4\frac{J(0)^5}{(y;y)_\infty^6} \tag{2.6}$$

We have the components of $S_3(1)$ and $S_3(4)$ from (6.21) and (6.22) of [3] and the components of $S_1(0)$ and $S_1(1)$ from (3.7) of [6].

Finally, we can state the components of $N(r)$ and $M(r)$:
for $k = 0, 1, 2, 3, 4$

$$N(0)^{(k)} = -2\delta_{0k}\frac{y}{J(0)}T_3(y,1,y^5) + \frac{\mathbf{P}^{(k)}}{5}$$
$$+ \left\{ \frac{\mathbf{P}^{(k)}}{5}\left(4\frac{J(2)^2}{J(1)^3} - 2y\frac{J(1)^2}{J(2)^3}\right) - \mathbf{P}^{(k+3)}\frac{2q^2}{J(1)} \right\} J(0)^2 \ ,$$

$$N(1)^{(k)} = \delta_{0k}\frac{y}{J(0)}T_3(y,1,y^5) - \delta_{3k}q^3\frac{y}{J(0)}T_3(y^2,1,y^5) + \frac{\mathbf{P}^{(k)}}{5}$$
$$+ \left\{ \frac{\mathbf{P}^{(k)}}{5}\left(-\frac{J(2)^2}{J(1)^3} + 3y\frac{J(1)^2}{J(2)^3}\right) + \mathbf{P}^{(k+3)}\frac{q^2}{J(1)} - \mathbf{P}^{(k+2)}\frac{q^3}{J(2)} \right\} J(0)^2 \ ,$$

$$N(2)^{(k)} = \delta_{3k}q^3\frac{y}{J(0)}T_3(y^2,1,y^5) + \frac{\mathbf{P}^{(k)}}{5}$$
$$+ \left\{ \frac{\mathbf{P}^{(k)}}{5}\left(-\frac{J(2)^2}{J(1)^3} - 2y\frac{J(1)^2}{J(2)^3}\right) + \mathbf{P}^{(k+2)}\frac{q^3}{J(2)} \right\} J(0)^2 \tag{2.7}$$

where $\delta_{ij} = 1$ if $i = j$, $= 0$ otherwise ,

and

$$M(0)^{(k)} = \frac{\mathbf{P}^{(k)}}{5} + \left\{ \frac{\mathbf{P}^{(k)}}{5} \left(4\frac{J(2)^2}{J(1)^3} - 2y\frac{J(1)^2}{J(2)^3} \right) - \mathbf{P}^{(k+4)} 2q\frac{J(2)}{J(1)^2} + \mathbf{P}^{(k+2)} 2\frac{q^3}{J(2)} \right\} J(0)^2 \ ,$$

$$M(1)^{(k)} = \frac{\mathbf{P}^{(k)}}{5} + \left\{ \frac{\mathbf{P}^{(k)}}{5} \left(-\frac{J(2)^2}{J(1)^3} + 3y\frac{J(1)^2}{J(2)^3} \right) + \mathbf{P}^{(k+4)} q\frac{J(2)}{J(1)^2} \right.$$
$$\left. -\mathbf{P}^{(k+3)} \frac{q^2}{J(1)} - \mathbf{P}^{(k+2)} \frac{q^3}{J(2)} + \mathbf{P}^{(k+1)} q^4 \frac{J(1)}{J(2)^2} \right\} J(0)^2$$

$$M(2)^{(k)} = \frac{\mathbf{P}^{(k)}}{5} + \left\{ \frac{\mathbf{P}^{(k)}}{5} \left(-\frac{J(2)^2}{J(1)^3} - 2y\frac{J(1)^2}{J(2)^3} \right) + \mathbf{P}^{(k+3)} \frac{q^2}{J(1)} \right.$$
$$\left. -\mathbf{P}^{(k+1)} q^4 \frac{J(1)}{J(2)^2} \right\} J(0)^2 \qquad (2.8)$$

(we take $\mathbf{P}^{(i)} = \mathbf{P}^{(j)}$ when $i \equiv j(\mathrm{mod}5)$)

3 Modular Forms

By (2.7) and (2.8), (1.10)-(1.12) and (1.23) are straightforward. For the rest, let, for $i = 1,\ 2,\ 3,\ 4$, H_i be the right hand-sides minus the left hand-sides of (1.13)-(1.16), respectively and let, for $j = 1,\ 2,\ 3,\ 4,\ 5,\ 6$, G_j be the r.h.s minus the l.h.s of (1.17)-(1.22) respectively. But, (2.7) and (2.8) also give that

$$\begin{array}{lll} G_1 = H_1 & G_2 = -2G_3 & G_3 = H_2 \\ G_4 = -H_3 & G_5 = H_4 & G_6 = -H_4 \end{array}$$

Therefore, we must only show that $H_i \equiv 0$ for $i = 1,\ 2,\ 3,\ 4$. In what follows, τ denotes a variable ranging over $\mathbb{H} := \left\{ z \in \mathbb{C} \ : \ \mathrm{im}z > 0 \right\}$ and $q := \exp(2\pi i \tau)$.

We now define, for $r = 1$ and 5, the functions on $\mathbb{H}$

$$\eta(r) = \eta(r, \tau) := \exp(\pi i \tau / 12)(q^r; q^r)_\infty$$

and, for k an integer,

$$\begin{aligned} s(k) = s(k, \tau) &:= q^{(5/2-k)^2/10} J(q^r; q^5)(q^5; q^5)_\infty \\ &= -\exp(k^2 \pi \tau / 5) \Theta_{1,1}(k\tau | 5\tau). \end{aligned}$$

where $\Theta_{1,1}$ is the Mock theta function (see §76 of [12]). (1.8) gives that

$$s(-k) = -s(k) = s(k+5) \qquad (3.1)$$

We set, $i = 1,\ 2,\ 3,\ 4$

$$\mathcal{H}_i := \exp(-\pi i \tau / 60) H_i(q^{1/5})$$

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so that

$$\mathcal{H}_1 = \frac{\eta(5)^8}{\eta(1)^6}\left(\frac{s(2)^6}{s(1)^7} - 11\frac{s(2)}{s(1)^2} - \frac{s(1)^3}{s(2)^4}\right) - \frac{s(2)}{s(1)^2}\eta(5)^2 \tag{3.2}$$

$$\mathcal{H}_2 = \frac{\eta(5)^8}{\eta(1)^6}\left(\frac{s(2)^5}{s(1)^6} - 11\frac{1}{s(1)} - \frac{s(1)^4}{s(2)^5}\right) - \frac{1}{s(1)}\eta(5)^2 \tag{3.3}$$

$$\mathcal{H}_3 = \frac{\eta(5)^8}{\eta(1)^6}\left(\frac{s(2)^4}{s(1)^5} - 11\frac{1}{s(2)} - \frac{s(1)^5}{s(2)^6}\right) - \frac{1}{s(2)}\eta(5)^2 \tag{3.4}$$

$$\mathcal{H}_4 = \frac{\eta(5)^8}{\eta(1)^6}\left(\frac{s(2)^3}{s(1)^4} - 11\frac{s(1)}{s(2)^2} - \frac{s(1)^6}{s(2)^7}\right) - \frac{s(1)}{s(2)^2}\eta(5)^2 \tag{3.5}$$

From these, we see that

$$\mathcal{H}_1 = \frac{s(2)}{s(1)}\mathcal{H}_2 = \frac{s(2)^2}{s(1)^2}\mathcal{H}_3 = \frac{s(2)^3}{s(1)^3}\mathcal{H}_4 \tag{3.6}$$

Thus, we only show that

$$\mathcal{H}_1 \equiv 0. \tag{3.7}$$

We define the subgroups

$$\Gamma_0(10) := \{A \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{10}\}$$
$$\Gamma_1(10) := \{A \in \Gamma_0(10) : a \equiv d \equiv \mp 1 \pmod{10}\}$$

of $SL_2(\mathbb{Z})$. Here, and below, A denotes a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The subgroup $\Gamma_0(10)$ has index 18 in $SL_2(\mathbb{Z})$ (see (1.4.28) in [13]) and $\Gamma_1(10)$ has index 36 in $SL_2(\mathbb{Z})$. We establish (3.7) by showing that $\mathcal{H}_1$ is a modular form on $\Gamma_1(10)$ of weight $1/2$ and that

$$\sum_{\zeta \in \mathcal{F}^*} \mathrm{ord}\,(\mathcal{H}_1, \zeta, \Gamma_1(10)) > 3/2 \tag{3.8}$$

where $\mathcal{F}^*$ is a fundamental region of $\Gamma_1(10) \cup \{\text{cups}\}$. (See [13] for the definitions of these terms.) We then appeal to the following well-known result.

Lemma 1 (Thm.4.1.4 in [13]) *Suppose that Γ is a subgroup of $SL_2(\mathbb{Z})$ (containing $-I$) of finite index μ. If f is a modular form of weight k on Γ, then either f is identically zero, or*

$$\sum_{\zeta \in \mathcal{F}^*} ord(f, \zeta, \Gamma) = \mu k/12.$$

The group $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ in the usual way. Suppose $A \in \Gamma_0(10)$. The transformation rules for the η and Θ functions ((74.11) and (81.2) in [12]) give

$$\eta(A\tau) = \varepsilon(A)\sqrt{c\tau + d}\,\eta(\tau) \tag{3.9}$$

where $\varepsilon(A)$ is a root of unity.

$$s(k, A\tau) = (-1)^{kb}\varepsilon(A_{10})^3 \exp(k^2\pi iab/5)\sqrt{c\tau + d}\, s(ak, \tau) \tag{3.10}$$

where $A_{10} = \begin{pmatrix} a & 10b \\ c/10 & d \end{pmatrix}$. Now, with the help of (3.1), (3.9) and (3.10), we see that $\mathcal{H}_i$'s are modular forms of weight 1/2 on $\Gamma_1(10)$

The set of cusps of $\Gamma_1(10)$ may be identified with the set

$$C = C_1 \cup C_2 \cup C_5 \cup C_0$$

where

$$C_1 = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right\}, \quad C_2 = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}, \quad C_3 = \left\{\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}\right\},$$

$C_0 = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right\}$. Here $\begin{bmatrix} x \\ y \end{bmatrix}$ denotes the equivalence class of $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}_1^2$ under the equivalence relation

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} x' \\ y' \end{pmatrix} \Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} \equiv \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \pmod{18}$$

for some $A \in \Gamma_1(10)$.

For $r \neq 0$, the cusps in C_r have width $10/r$ and those in C_0 have width 1. We now give the orders of $\eta(1)$, $\eta(5)$, $s(1)$ and $s(2)$ at the cusps in C_r, $(r \neq 0)$,

	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 5 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 5 \end{bmatrix}$
$\eta(1)$	5/24	5/24	1/12	1/2
$\eta(5)$	1/24	1/24	5/12	5/12
$s(1)$	1/8	1/8	9/20	1/20
$s(2)$	1/8	1/8	1/20	9/20

cusp	0/1	0/3	1/2	1/4	1/5	2/5
ord $\mathcal{H}_1$	$\geq -\frac{25}{12}$	$\geq -\frac{25}{12}$	$\geq -\frac{25}{24}$	$\geq -\frac{25}{24}$	$\geq -\frac{1}{60}$	$\geq \frac{71}{60}$

Therefore

$$\sum_{\zeta \in C\backslash C_0} \mathrm{ord}(\mathcal{H}_1, \zeta, \Gamma_1(10)) \geq -\frac{61}{12} \tag{3.11}$$

where $\mathcal{H}_i = q^{-1/120}H_i(q^{1/5})$ for $i = 1, 2, 3, 4$. By (3.8) and (3.11), we need to show that $\mathcal{H}_1$ has order bigger than

$$\frac{1}{2}\left(\frac{3}{2} + \frac{61}{12}\right) = \frac{89}{24} < 4 - \frac{1}{120}$$

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at each of 1/0 and 3/0,

$$H_1(q^{1/5}) = \sum_{n \geq 0} \left(N(0,5,5n) + 2N(1,5,5n) - 3N(2,5,5n)\right) q^n - \frac{J(q^2;q^5)(q^5;q^5)_\infty}{j(q;q^5)^2} \quad (3.12)$$

$$H_4(q^{1/5}) = q^{3/5} \left(\sum_{n \geq 0} \left(2N(0,5,5n+3) - N(1,5,5n+3) - N(2,5,5n+3)\right) q^n \right.$$
$$\left. - \frac{J(q;q^5)(q^5;q^5)_\infty}{J(q^2;q^5)^2} \right) \quad (3.13)$$

If we calculate the first four coefficients of $H_1(q^{1/5})$ and $H_4(q^{1/5})$, we see that they are zero.

From (3.10) we have

$$\text{ord}\left(\mathcal{H}_1, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \Gamma_1(10)\right) = \text{ord}\left(\mathcal{H}_4, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \Gamma_1(10)\right).$$

Consequently,

$$\sum_{\zeta \in C_0} \text{ord}(\mathcal{H}_1, \zeta, \Gamma_1(10)) \geq (4 - \frac{1}{120}) + (4 + \frac{3}{5} - \frac{1}{120}) = \frac{89}{24} \quad (3.14)$$

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Rank ve Crank Modülo 5

Özet

$p(n)$, n'nin kısıtlanmamış ayrışımları sayısını göstersin. Ramanujan'nın ayrışım kongrüansları $p(5n + 4)$, $p(7n + 5)$ ve $p(11n + 6)$ $\equiv$ mod 5, 7, ve 11, sırasıyla. Bunlar değişik yollarla ispat edildi. Atkin ve Swinnerton-Dyer bu kongrüanslarla birlikte bazı ayrışım ilişkilerini mod 5 ve 7 durumlarında *rank* yöntemiyle ispatladılar, Garvan da benzeri ilişkileri her üç durum için *crank* yöntemini kullanarak ispatladı.

Bu çalışmada, modüler formlar teorisini kullanarak, mod 5 durumda bunların sonuclarına başka bir ispat veriyoruz. Bizim yöntemimiz her nekadar daha karmaşık olsa da, bu yöntem $SL_2(\mathbb{Z})$'in bir kongrüans altgrubu üzerinde yarım tamsayı ağırlıklı Modüler formların Ayrışım Teoride nasıl bir rol oynadığını gösterir. Yöntemimiz mod 7 durumuna da uygulanabilir fakat $\prod_{n=1}^{\infty}(1 - q^n)^{-1}$'nın komponentleri açıkca bilinmediğinden mod 11 durumuna uygulanamaz.

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