Tr. J. of Mathematics
21 (1997) , 207 – 212.

THE SPECTRA AND FINE SPECTRA FOR P-CESÀRO OPERATORS

Cafer Coşkun

Abstract

In [6], Rhaly computed the spctrum of p-Cesàro operator on the Hilbert space $\ell_2 = \{x = (x_k) : \sum_k |x_k|^2 < \infty\}$. In the present paper, we study the spectrum and fine spectrum for p-Cesàro operators acting on c_0, the space of null sequences.

1. Introduction

Let $x = (x_k)$ be a sequence of complex numbers. Following Rhaly [6], we define C_p, the Cesàro operator by

$$(C_p x)_n = \frac{1}{(n+1)^p} \sum_{k=0}^{n} x_k \quad n = 0, 1, 2, 3, \cdots, \tag{1}$$

where p is a real number. The case $p = 1$ is the Cesàro operator. We note that the spectrum of C_1 acting on ℓ and c_0 was studied respectively in [1] and [5]. Also the fine spectrum of C_1 on c, the space of convergent sequences, was computed in [7].

The matrix corresponding to (1) $C = (c_{nk})$ is given by

$$c_{nk} := \begin{cases} \frac{1}{(n+1)^p} & 0 \le k \le n \\ 0 & n < k \end{cases} \tag{2}$$

If $1 \le p$, then an application of Theorem 1 in [4, p.163] shows that $C_p \in B(c_0)$ and $\| C_p \| = 1$.
If $p < 1$, then C_p does not map c_0 into c_0. We assume throughout the paper that $p \ge 1$.

On the other hand, using the idea of [3, p.407–408] one can see that C_p is compact operator on c_0.

2. The Spectra

In this section we compute the spectrum and the point spectrum of C_p on c_0. By $\pi(C_p, c_0)$ and $\sigma(C_p, c_0)$ we respectively denote the point spectrum and spectrum of C_p on c_0. With this notation we have

Theorem 2.1. $\pi(C_p, c_0) = \left\{\frac{1}{m^p} : m = 1, 2, \cdots\right\}$.

Proof. Let $C_p x = \lambda x$ for $x \neq \theta = (0, 0, \cdots)$ in c_0. By (2) we get $x_0 = \lambda x_0$ and for $n \geq 1$

$$\frac{1}{(n+1)^p}(x_0 + x_1 + \cdots + x_n = \lambda x_n \tag{3}$$

If m is the smallest integer for which $x_m \neq 0$, then we have $\lambda = \frac{1}{(m+1)^p}$. It follows by (3) that

$$x_n = \prod_{j=m+1}^{n} \left(\frac{\lambda j^p}{\lambda(j+1)^p} - 1\right) x_m \tag{4}$$

where $n \geq m+1$. The sequence defined by (4) is in $\ell_2(see[6])$. Since $\ell_2 \subset c_0$ we get that $x = (x_n)$ is in c_0, this proves the theorem.

It should be noted that the adjoint oprerator C_p^* of C_p is an operator on the dual space of c_0 which is isometrically isomorphic to Banach space ℓ_1 of absolutely summable sequences normed by

$$\| x \| = \sum_k |x_k| < \infty.$$

The matrix of C_p^* is transpose of the matrix of C_p.

The next theorem shows that $\pi(C_p, c_0) = \pi(C_p^*, \ell_1)$. □

Theorem 2.2. $\pi(C_p^*, \ell_1) = \left\{\frac{1}{m^p} : m = 1, 2, \cdots\right\}$.

Proof. Let $C_p^* x = \lambda x$, for $x \neq \theta$ in ℓ_1. Hence we have

$$\sum_{k=n}^{\infty} \frac{x_k}{(k+1)^p} = \lambda x_n \tag{5}$$

If $\lambda = 0$, then it follows from (5) that $x = \theta$, so $\lambda = 0$ is not eigenvalue. Now, by (5), for $\lambda \neq 0$, we have

$$x_n = \lambda(n+1)^p [x_n - x_{n+1}],\ n \geq 0$$

or

$$x_{n+1} = \left(1 - \frac{1}{\lambda(n+1)^p}\right) x_n$$

thus, for $n \geq 1$

$$x_n = \prod_{j=1}^{n} \left(1 - \frac{1}{\lambda j^p}\right) x_0 \tag{6}$$

If $\lambda \in \{\frac{1}{m^p} : m = 1, 2, \cdots\}$ then (6) yields that, for $n \geq m$, $x_n = 0$ wihch implies that $x = (x_n) \in \ell_1$. If $\lambda \neq \frac{1}{m^p}$, then by (6)

$$\frac{|x_{n+1}|}{|x_n|} = |1 - \frac{1}{\lambda(n+1)^p}| = \frac{|\lambda(n+1)^p - 1|}{|\lambda|(n+1)^p} \to 1, n \to \infty$$

Hence the ratio test fails. We now turn to Raabe's test. Since

$$\begin{aligned} \lim_n \left(\frac{|x_n|}{|x_{n+1}|} - 1\right) n &= \lim_n \frac{\left\{\left|\frac{x_n}{x_{n+1}}\right|^2 - 1\right\} n}{\left|\frac{x_n}{x_{n+1}}\right| + 1} \\ &= \frac{1}{2} \lim_n n \left\{\frac{(\lambda + \bar{\lambda})(n+1)^p - 1}{|\lambda|^2(n+1)^{2p} - (\lambda + \bar{\lambda})(n+1)^p + 1}\right\} \\ &= 0 \end{aligned}$$

the series $\sum_n |x_n|$ is divergent, *i.e.*, $x \notin \ell_1$ □

Theorem 2.3. $\sigma(C_p, c_0) = \{\frac{1}{m^p} : m = 1, 2, \cdots\} \cup \{0\}$.
Proof. By the Theorem 2.1.

$$\{\frac{1}{m^p} : m = 1, 2, \cdots\} \subset \sigma(C_p, c_0).$$

Since c_0 is a Banach space $\sigma(C_p, c_0)$ is closed. Thus

$$\{\frac{1}{m^p} : m = 1, 2, \cdots\} \cup \{0\} \subset \sigma(C_p, c_0).$$

On the other hand, since C_p is a compact operator on c_0, every spectral value $\lambda \neq 0$ is an eigenvalue [1, p.420]. This completes the proof. □

3. The Fine Spectra

Let X be a Banach space and $T \in B(X)$. Then there are three possibilities for $R(T)$, the range of T:
(I) $R(T) = X$
(II) $\overline{R(T)} = X$, but $R(T) \neq X$
(III) $\overline{R(T)} \neq X$
and there are three possibilities for T^{-1}, the inverse of T:
(1) T^{-1}, exists and is continuous.
(2) T^{-1}, exists but is discontinuous.
(3) T^{-1}, does not exist.
$(See, e.g.$[2.p.66]$)$.

If these possibilities are combined in all possible ways, nine different states are created. These are written as: $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2$, and III_3. For example, if $T \in II_3$, then $\overline{R(T)} = X$, but $R(T) \neq X$ and T^{-1} does not exist. Similarly we write $T \in 2$ if T^{-1} exists but is discontinuonus. Also we write $T \in III$ if $\overline{R(T)} \neq X$.

If λ is a complex number such that $\lambda I - T \in I_1$ or II_1, then λ is in the resolvent set of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T on X. If $\lambda I - T \in II_3$, then we write $\lambda \in II_3\sigma(T, X)$. With this terminology we have

Theorem 3.1. *If $\lambda = 0$, then $\lambda \in II_2\sigma(C_p, c_0)$.*

Proof. By the Theorem 2.1, $\lambda = 0$ is not in $\pi(C_p, c_0)$. Hence C_p^{-1} exists, therefore $C_p \in 1 \cup 2$.

To verify that $C_p \in 2$, from [2, p.60] it is enough to slow that C_p^* is not onto. Now consider C_p^*. If $C_p^* x = y$, then we have for $n \geq 0$,

$$y_n = \sum_{k=n}^{\infty} \frac{x_k}{(k+1)^p}.$$

A few calculation yield that □

$$\begin{aligned} x_0 &= y_0 - y_1 \\ x_1 &= 2^p(y_1 - y_2) \\ x_2 &= 3^p(y_2 - y_3) \\ &\vdots \\ x_n &= (n+1)^p(y_n - y_{n+1}) \qquad (7) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If we defined the sequence $y = (y_n)$ by

$$y_n = \frac{(-1)^n}{(n+1)^p} \tag{8}$$

then, $y \in \ell$ for $(p > 1)$. On the other hand, it follows from (7) and (8) that

$$\begin{aligned} x_n &= (n+1)^p \frac{(-1)^n}{(n+1)^p} - (n+1)^p \frac{(-1)^{n+1}}{(n+2)^p} \\ &= (-1)^n \left(1 + \frac{(n+1)^p}{(n+2)^p}\right) \end{aligned}$$

Hence we have

$$\lim_n |x_n| = 2 \neq 0.$$

this yields that $x \notin \ell_1, i.e., C_p^*$ is not onto. So, C_p^{-1} is not bounded.

We can get from (5) that $C_p^* x = \theta$ if and only if $x = \theta$. Hence C_p^* is one to one. From [2, p.59], we have $\overline{R(C_p)} = c_0$. So, $C_p \in I \cup II$.

We now complete the proof by showing that C_p is not onto. If $C_p x = y$, then, by (1) $x_0 = y_0$ and for $n \geq 1$

$$x_n = (n+1)^p y_n - n^p y_{n-1}$$

The sequence (y_n) defined by (8) is in c_0. But the sequence

$$x_n = (n+1)^p \frac{(-1)^n}{(n+1)^p} - n^p \frac{(-1)^{n-1}}{n^p} = (-1)^n 2$$

is not in c_0 which implies that $R(C_p) \neq c_0, i.e., C_p \in II$. Hence the proof is completed.

Theorem 3.2. *If* $\lambda \in \{\frac{1}{m^p} : m = 1, 2, \cdots\}$, *then* $\lambda \in III_3\sigma(C_p, c_0)$.

Proof. If $\lambda \in \{\frac{1}{m^p} : m = 1, 2, \cdots\}$ by consulting Theorem 2.1, we deduce that $\lambda I - C_p \in 3$. From [2, p.59] it is sufficient to show that $(\lambda I - C_p)^*$ is not one to one.

Now consider the operator $(\lambda I - C_p)^* = \lambda I - C_p^*$ where $\lambda = \frac{1}{m^p}, (m = 1, 2, \cdots)$ If we define

$$x_k^{(m)} := \begin{cases} 1 & k = 0 \\ \prod_{j=1}^{k} \left(1 - \frac{m^p}{j^p}\right) & 0 < k < m \\ 0 & m \leq k \end{cases}$$

then for each $m, x^{(m)} = (x_k^{(m)}) \in c_0$. Observe that

$$\left(\frac{1}{m^p} I - C_p^*\right) x^m = \theta,$$

thus $\frac{1}{m^p} I - C_p^*$ is not one to one. This proves that $\frac{1}{m^p} I - C_p \in III$, whence the result. □

References

[1] Brown A. Halmos P.R. and Shields A.L., Cesàro Operators. Acta Sci. Math., **26** (1963), 125-137.

[2] Goldberg S., Unbounded Linear Operators, Mc Graw-Hill Book Comp., (New York, 1966).

[3] Kreysizg E., Introductory Functional Analysis with Applications, John Wiley and Sons., (New York, 1978).

[4] Maddox I.J., Elements of Functional Analysis with Applications, Cambridge University Press., (1970).

[5] Reade J.B., On the Spectrum of the Cesàro Operator, Bull. London Math. Soc. ,**17** (1985), 263-267.

[6] Rhaly JR H.C., p-Cesàro Matrices, Houston J.Math., **15** (1989), 137-146.

[7] Wenger R.B., The fine Spectra of the Hödler Summability Operators, Indian J. Pure and Appl. Math. **6** (1975), 695-712.

P-Cesàro Operatörünün Spektrumu ve Fine Spektrumu

Özet

Bu çalışmada p-Cesàro operatörünün, c_0 üzerindeki spektrumu ve fine spektrumu incelenmiştir.

Cafer COŞKUN
Department of Mathematics
Faculty of Science
Ankara Universty
06100 Tandoğan
Ankara,-TURKEY

Received 3.10.1995