Tr. J. of Mathematics
21 (1997) , 213 – 225.

NEAR ULTRAFILTERS AND COMPACTIFICATION OF TOPOLOGICAL GROUPS

Mahmut Koçak & Zekeriya Arvasi

Abstract

In this work we will construct LUC-compactification of a topological group in terms of the new concept of near ultrafilters.

Introduction

The purpose of this paper is to describe the compactification of a topological group in terms of the concept of *"right (respectively left) near ultrafilters"*. If G is discrete topological space, the points of its Stone-Čech compactification βG can be regarded as ultrafilters on G, we were motivated by this in defining the analogous concept of *"right (respectively left) near ultrafilters"* to describe the points of an arbitrary compactification. A uniform space and therefore, a topological group has a compactivation $\widetilde{G}$ with the property that $C(\widetilde{G})$ is isomorphic to the algebra of bounded real-valued right uniformly continuous functions defined on G (cf. [5]). We believe that right near ultrafilters provide a natural and usefull method for describing $\widetilde{G}$.

1. Preliminaries

We firstly remind the reader of some basic definitions.

Compactifications: Let X be a topological space. By *a compactificaiton* of X, we shall mean a pair (C, e), where C is a compact Hausdorff space, $e : X \to C$ is an embedding and $e[X]$ is dence in C. In this case, we may simply refer to C as being a compactification of X. Two compactifications (C, e) and (C', e') are regarded as equivalent if there is a homeomorphism $h : C \to C'$ for which $e = e'h$.

Semigroups and Groups: Let $(G, \bullet)$ be a semigroup. For each $s \in G$, we shall use ρ_s and λ_s to denote the mapping from G to itself for which $\rho_s(x) = x \bullet s$ and $\lambda_s(x) = s \bullet x$. The maps ρ_s and λ_s are called *right and left translations* by s respectively. Suppose that G is also a topological space. G will be called a *topological semigroup* if the mapping $(s, t) \mapsto s \bullet t$ is continuous mapping from $G \times G$ to G. It will be called a *semitopological semigroup* if, for every $s \in G, \lambda_s$ and ρ_s are both continuous. It will be called a *right topological* if, for every $s \in G, \rho_s$ is continuous.

If G is right topological semigroup, $\{s \in G : \lambda_s : G \to G$ is continuous$\}$ called the *topological centre* of G.

A *topological group* G is a set which carries a group structure and a topology and satisfies the following two axioms:

i) The mapping $(x,y) \longrightarrow x \bullet y$ of $G \times G$ into G is continuous;

ii) The mapping $x \longrightarrow x^{-1}$ of G into G (the symmetry of the group) is continuous.

Let $(G, \bullet)$ be a topological group. If $X, Y \subseteq G$, we define X^{-1} and $X \bullet Y$ by stating that $\{x^{-1} \in G : x \in X\}$ and $X \bullet Y = \{x \bullet y \in G : x \in X, y \in Y\}$. We shall say that X *is symmetric* if $X = X^{-1}$. We use $\mathcal{B}$ to denote the fundamental system of neighborhood of the identity e of G.

Let $(G, \bullet)$ and $(G', *)$ be two topological groups. We say a function $f : G \longrightarrow G'$ is *right (respectively left) uniformly continuous* if f is uniformly continuous with respect to the right (respectively left) uniformities on G and G'. In other words, for each $V \in B_{G'}$ there exists $U \in B_G$ such that $x \bullet y^{-1} \in G$ *(respectively* $x^{-1} \bullet y \in U$) implies that $f(x) \bullet (f(y))^{-1} \in V$ *(respectively* $(f(x))^{-1} * f(y) \in V$*)*.

It is a well-known fact that the left and right translations are right (respectively left) uniformly continuous. Furthermore, they are isomorphisms of the right uniformity onto itself [2].

If G is abelian then left and right uniformities are coincides and hence, a left uniformy continuous function is also right uniformly continuous and vise-versa.

Suppose that $(G, \bullet)$ is a semitopological semigroup or a group and that (C, e) is a compactification of G. We shall say that (C, e) is a *semigroup compactification* of G if G is a right topological semigroup, e is a homomorphism and $e(G)$ is contained contained in the topological centre of C.

Throughout this paper, we shall assume that all the topological groups referred to are Hausdorff.

Notation: We shall use $\mathbf{N}$ to denote the set of positive integers, $\mathbf{Z}$ to denote the set of all integers and $\mathbf{R}$ to denote the set of real numbers.

If X is a topological space, $C(X)$ will denote the set of continuous bounded real-valued functions defined on X, and βX will denote the *Stone-Čech compactification* of X.

2 The Space $\widetilde{G}$

Definition 2.1 *Suppose that $(G, \bullet)$ is a topological group and that $\mathcal{H} \subseteq \mathcal{P}(G)$. We shall say that $\mathcal{H}$ has the right (respectively left) near finite intersection property if $\mathcal{H}$ is non-empty and if, for every finite subset $\mathcal{F}$ of $\mathcal{H}$ and every $U \in \mathcal{B}, \cap_{Y \in \mathcal{F}}(U \bullet Y) \neq \emptyset$ (respectively $\cap_{Y \in \mathcal{F}}(Y \bullet U) \neq \emptyset$).*

Definition 2.2 *Let $\xi \subseteq \mathcal{P}(G)$. We shall say that ξ is a right (respectively left) near ultrafilter on G if ξ is maximal subject to being a subset of $\mathcal{P}(G)$ with the right (respectively left) near finite intersection property.*

If ξ is both a right near ultrafilter and left near ultrafilter then we say that ξ is a near ultrafilter. It is clear that a right near ultrafilter is also a left near ultrafilter if and only if G is abilian. We observe that the concept of right near ultrafilter and left near ultrafilter generalies the concept of an ultrafilter and so if G is discrete topological group then a right near ultrafilter and a left near ultrafilter are simply a ultrafilter.

It is immediate from Zorn's Lemma that every subset of $\mathcal{P}(G)$ with the right (respectively left) near finite intersection property is contained in a right (respectively left) near ultrafilter.

Notation: We shall use $\widetilde{G}$ (respectively $\widehat{G}$) to denote the set of all right (respectively left) near ultrafilters on G. We may simply denote this set by $\widetilde{G}$ (*respectively* $\widehat{G}$).

For each assertion for the right near ultrafilters there is a corresponding assertion for the left near ultrafilters whose proofs are entirely similar to those corresponding assertions for right near ultrafilters. Therefore, we shall give the proofs of assertions claimed for the right near ultrafilters.

Lemma 2.3 *Let $\xi \in \widetilde{G}$ and φ be a finite subset of ξ and $W \in \mathcal{B}$. Then $\bigcap_{Y \in \varphi}(W \bullet Y) \in \xi$.*

Proof. Suppose that $\bigcap_{Y \in \varphi}(W \bullet Y) \notin \xi$ for some finite subset φ of ξ and $W \in \mathcal{B}$. Then where will be a finite subset ψ of ξ and $U \in \mathcal{B}$ for which

$$[U \bullet (\bigcap_{Y \in \varphi}(W \bullet Y))] \cap [\bigcap_{Z \in \psi}(U \bullet Z]) = \emptyset.$$

We can choose $W' \in \mathcal{B}$ such that $W' \subseteq U \cap W$. This will imply that $\bigcap_{Y \in \varphi \cup \psi}(W' \bullet Y) = \emptyset-$ contradicting our assumption that ξ has the right near finite intersection property. $\square$

Lemma 2.4 *Let $\xi \in \widetilde{G}$ and let $Y \subseteq G$. The following statements are equivalent:*

i) *$Y \in \xi$;*

ii) *For every $U \in \mathcal{B}$ and every $Z \in \xi, (U \bullet Y) \cap Z \neq \emptyset$;*

iii) *For every $U \in \mathcal{B}$ and every $Z \in \xi, Y \cap (U \bullet Z) \neq \emptyset$;*

Proof. i) $\Leftrightarrow$ ii)

If $Y \notin \xi$ these will be a finite subset $\mathcal{F}$ of ξ and $U \in \mathcal{B}$ such that

$$(U \bullet Y) \bigcap) \bigcap_{Y' \in \mathcal{F}} (U \bullet Y')) = \emptyset.$$

If Z denotes $\bigcap_{Y' \in \mathcal{F}}(U \bullet Y')$, then $Z \in \xi$ by Lemma 2.3 and $(U \bullet Y) \cap Z = \emptyset$.

Conversely, suppose that $(U \bullet Y) \cap Z = \emptyset$ for some $Z \in \xi$ and some $U \in \mathcal{B}$. We can choose a symmetric neighborhood $W \in \mathcal{B}$ of e satisfying $W^2 \subseteq U$. We claim that $(W \bullet Y) \cap (W \bullet Z) = \emptyset$. To see this, assume that there is a point $x \in (W \bullet Y) \cap (W \bullet Z)$. Therefore, $x = w_1 \bullet y = w_2 \bullet z$ for some $y \in Y, z \in Z$ and $w_w, w_2 \in W$. Hence, we have $z \in (W \bullet W \bullet Y) \cap Z \subseteq (U \bullet Y) \cap Z$-contradiction. This shows that $(W \bullet Y) \cap (W \bullet Z) = \emptyset$ and hence that $Y \notin \xi$.

ii) $\Leftrightarrow$ iii)

For every symmetric neighborhood $U \in \mathcal{B}$ of e and every $Y, Z \subseteq G$,

$$(U \bullet Y) \cap Z \neq \emptyset \Leftrightarrow Y \cap (U \bullet Z) \neq \emptyset.$$

□

Lemma 2.5 *Let $\xi \in \widetilde{G}$ and let $Y \subseteq G$. Then $Y \in \xi$ if and only if $(U \bullet Y) \in \xi$ for every $U \in \mathcal{B}$. Furthermore, this is the case if and only if $\bar{Y} \in \xi$.*

Proof. Clearly, if $Y \in \xi$, then $(U \bullet Y) \in \xi$ for every $U \in \mathcal{B}$, because $Y \subseteq (U \bullet Y)$.

Conversely, if $Y \notin \xi$, then $(U \bullet Y) \cap Z = \emptyset$ for some $U \in \mathcal{B}$ and some $Z \in \xi$ (by Lemma 2.4). Let $V \in \mathcal{B}$ be a symmetric neighborhood of e satisfying $V^2 \subseteq U$. Then $(V \bullet (V \bullet Y)) \subseteq (U \bullet Y)$ and so $(V \bullet (V \bullet Y)) \cap Z = \emptyset$ and so $(V \bullet Y) \notin \xi$-contradiction. Now, for every $U \in \mathcal{B}, Y \subseteq \bar{Y} \subseteq (U \bullet Y)$. It follows that $Y \in \xi$ if and only if $\bar{Y} \in \xi$. □

Lemma 2.6 *Let $\xi \in \widetilde{G}$. For any $Y_1, Y_2 \subseteq G, Y_1 \cup Y_2 \in \xi$ implies that $Y_1 \in \xi$ or $Y_2 \in \xi$.*

Proof. If $Y_1, Y_2 \notin \xi$, there will be sets $Z_1, Z_2 \in \xi$ and $U_1, U_2 \in \mathcal{B}$ for which $Y_1 \cap (U_1 \bullet Z_1) = \emptyset$ and $Y_2 \cap (U_2 \bullet Z_2) = \emptyset$ (by Lemma 2.4). We choose a symmetric neighborhood $U \in \mathcal{B}$ of the identity e satisfying $U^2 \subseteq U_1 \cap U_2$, and claim that $(U \bullet Y_1) \cap (U \bullet Z_1) = (U \bullet Y_2) \cap (U \bullet Z_2) = \emptyset$. To see this, suppose that $x \in (U \bullet Y_i) \cap (U \bullet Z_i)$, where $i \in \{1, 2\}$. Then there will be points $y \in Y_i, z \in Z_i$ and $u_1, u_2 \in U$ for which $x = u_1 \bullet y_i = u_2 \bullet z_i$. Hence,

$$y_i = u_1^{-1} \bullet u_2 \bullet z_i \in (U \bullet U \bullet Z_i) \subseteq (U_i \bullet Z_i) \cap Y_i.$$

This is a contradiction. Since $(U \bullet (Y_1 \cup Y_2)) = (U \bullet Y_1) \cup (U \bullet Y_2)$, we have shown that $(U \bullet (Y_1 \cup Y_2)) \cap (U \bullet Z_1) \cap (U \bullet Z_2) = \emptyset$ and hence that $Y_1 \cup Y_2 \notin \xi$. □

Definition 2.7 *For each $Y \subseteq G$, we put $\mathcal{C}_Y = \{\xi \in \widetilde{G} : Y \in \xi\}$.*

Lemma 2.8 *For every* $Y_1, Y_2 \subseteq G, \mathcal{C}_{Y_1 \cup Y_2} = \mathcal{C}_{Y_1} \cup \mathcal{C}_{Y_2}$. *Furthermore,* $\mathcal{C}_\emptyset = \emptyset$ *and* $\mathcal{C}_G = \widetilde{G}$.

Proof. The first statement follows from Lemma 2.6, and the second is immediate from the definition. □

3. The Topological Space $\widetilde{G}$

Definition 3.1 *We define the topology of* $\widetilde{G}$ *by choosing the sets of the form* $\mathcal{C}_Y$, *where* $Y \in \mathcal{P}(G)$, *as a base for the closed sets.*

Theorem 3.2 $\widetilde{G}$ *is a compact Hausdorff space.*

Proof. Let $(\mathcal{C}_{Y_\alpha})_{\alpha \in A}$ be a family of basic closed subsets of $\widetilde{G}$ with the finite intersection property. We shall show that $\bigcap_{\alpha \in A} \mathcal{C}_{Y_\alpha} \neq \emptyset$. It will follow that $\widetilde{G}$ is compact.

For any finite $F \subseteq A$ and any $U \in \mathcal{B}$, there will be a right near ultrafilter $\xi_F \in \bigcap_{\alpha \in F} \mathcal{C}_{Y_\alpha}$ and so, since $Y_\alpha \in \xi_F$ for every $\alpha \in F, \bigcap_{\alpha \in F}(U \bullet Y_\alpha) \neq \emptyset$. This shows that the family $(Y_\alpha)_{\alpha \in A}$ has the right near finite intersection property and hence that it is contained in a right near ultrafilter ξ. Since $\xi \in \bigcap_{\alpha \in A} \mathcal{C}_{Y_\alpha}$, it follows that $\bigcap_{\alpha \in A} \mathcal{C}_{Y_\alpha} \neq \emptyset$.

The see that $\widetilde{G}$ is Hausdorff, suppose that ξ_1, ξ_2 are distinct elements of $\widetilde{G}$. Choose any $Y_1 \in \xi_1 \setminus \xi_2$. There will be a set $Y_2 \in \xi_2$ and $U \in \mathcal{B}$ for which $Y_1 \cap (U \bullet Y_2) = \emptyset$ (by Lemma 2.4). We choose a neighborhood $V \in \mathcal{B}$ of the identity e satisfying $V^2 \subseteq U$ and put $Z = G \setminus (V \bullet Y_2)$. It is easy to check that $Y_1 \cap (V \bullet (V \bullet Y_2)) = \emptyset$ and hence that $\xi_1 \in \widetilde{G} \setminus \mathcal{C}_{V \bullet Y_2}$ (by Lemma 2.4). Also, since $Z \cap (V \bullet Y_2) = \emptyset$, $\xi_2 \in \widetilde{G} \setminus \mathcal{C}_Z$. Now $\mathcal{C}_{V \bullet Y_2} \cup \mathcal{C}_Z = \widetilde{G}$ (by Lemma 2.8), and so $(\widetilde{G} \setminus \mathcal{C}_{V \bullet Y_2}) \cap (\widetilde{G} \setminus \mathcal{C}_Z) = \emptyset$. Thus $\widetilde{G}$ is indeed Hausdorff. □

Definition 3.3 *We define a mapping* e *on* G *by stating that, for each* $x \in G, e(x) = \{Y \in \mathcal{P}(G) : x \in \bar{Y}\}$.

It is easy to verify that $e(x) \in \widetilde{G}$.

Theorem 3.4 *The mapping* e *embeds* G *as a dense subspace in* $\widetilde{G}$.

Proof. We first remark that e is injective. To see this, suppose that x_1, x_2 are distinct points of G. Then $\{x_1\} \in e(x_1) \setminus e(x_2)$ and so $e(x_1) \neq e(x_2)$.

Now, for any $Y \subseteq G$ and any $x \in G$,

$$x \in \bar{Y} \Leftrightarrow Y \in e(x) \Leftrightarrow e(x) \in \mathcal{C}_Y.$$

This shows that $e^{-1}(\mathcal{C}_Y) = \bar{Y}$ and hence that e is continuous.

It also shows that, for any closed subset Y of $G, e[Y] = \mathcal{C}_Y \cap e[G]$. Since this is a closed subset of $e[G], e$ is a closed mapping from G to $e[G]$ and therefore defines a homeomorphism from G to $e[G]$.

Finally, suppose that $\mathcal{C}_Y \neq \widetilde{G}$. If $\xi \in \widetilde{G} \setminus \mathcal{C}_Y$, then $Y \cap (U \bullet Z) = \emptyset$ for some $Z \in \xi$ and some symmetric neighborhood $U \in \mathcal{B}$ of e. This implies that $(U \bullet Y) \cap Z = \emptyset$ and hence that $\bar{Y} \neq G$, because $\bar{Y} \subseteq (U \bullet Y)$. Thus we can choose $x \in G \setminus \bar{Y}$. This implies that $e(x) \in \widetilde{G} \setminus \mathcal{C}_Y$ and shows that $e[G]$ is dense in $\widetilde{G}$, because every non-empty open subset of $\widetilde{G}$ will contain a non-empty set of the form $\widetilde{G} \setminus \mathcal{C}_Y$. □

Let $(X, \mathcal{U})$ be a Hausdorff uniform space. Then for each $Y \subseteq X$ and each $U \in \mathcal{U}$, the set $\{z \in X : (y,z) \in U$ for some $y \in Y\}$ is denoted by $\widehat{U}(Y)$. If $\xi \subset \mathcal{P}(X)$ and ξ is maximal subject to being a subset of $\mathcal{P}(X)$ with the near finite intersection property, that is ξ is non-empty and for every finite subset φ of ξ and every $U \in \mathcal{U}, \bigcap_{Y \in \varphi} \widehat{U}(Y) \neq \emptyset$, then ξ is called a near ultrafilter. The set of near ultrafilters on X is denoted by $\widetilde{X}$ and the topology of $\widetilde{X}$ is defined by choosing the sets of the form $\mathcal{C}_Y$, where $Y \in \mathcal{P}(X)$, as a base for the closed sets. This space is a compactification of X (cf. [6]).

Theorem 3.5 *Suppose that $(G, \bullet)$ is a topological group, $(G', \mathcal{U})$ is Hausdorff uniform space and that $f : G \to G'$ is uniformly continuous function with respect to the right uniformity on G and $\mathcal{U}$ on G'. Then there is a continuous function $\tilde{f} : \widetilde{G} \to \widetilde{G}'$ which is an extension of f in the sense that $\tilde{f} e_G = e_{G'} f$, where $e_G, e_{G'}$ denote the natural embeddings of G, G', in $\widetilde{G}, \widetilde{G}'$, respectively.*

Proof. Given $\xi \in \widetilde{G}$, we define

$$\eta = \{Y \in \mathcal{P}(G') : f^{-1}(\tilde{V}(Y)) \in \xi \text{ for every } V \in \mathcal{U}\}$$

We shall show that $\eta \in \widetilde{G}'$.

We first show that η has the near finite intersection property. To see this, suppose that $\mathcal{F}$ is a finite subset of η and that $W \in \mathcal{U}$. We choose $V \in \mathcal{U}$ satisfying $V^2 = VV = \{(x,y) : (x,z) \in V \text{ and } (x,y) \in V \text{ for some } z \in G'\} \subseteq W$. Then, since f is uniformly continuous with respect to right uniformity on G and $\mathcal{U}$ on G', there exists $U \in \mathcal{B}_G$ such that $(f(x), f(y)) \in V$ whenever $x \bullet y^{-1} \in U$. It follows that $\bigcap_{T \in \mathcal{F}} (U \bullet (f^{-1}(\widetilde{V}(T)))) \neq \emptyset$. If x is in this set, then, for each $T \in \mathcal{F}$, there will be a point $x_T \in f^{-1}(\widetilde{V}(T))$ for which $x \bullet x_T^{-1} \in U$. This implies that $(f(x), f(x_T)) \in V$ and hence, since $f(x_T) \in \widetilde{V}(T)$, that $f(x) \in (\widetilde{V}(\widetilde{V}(T))) \subseteq \widetilde{W}(T)$. Thus $\bigcap_{T \in \mathcal{F}} \widetilde{W}(T) = \emptyset$ and η does have the near finite intersection property.

We now show that η is a near ultrafilter. If $T \notin \eta, f^{-1}(\widetilde{V}(T)) \notin \xi$ for some $V \in \mathcal{U}$. This implies that $^{-1}(\widetilde{V}(T)) \cap S = \emptyset$ for some $S \in \xi$, and hence that $\widetilde{V}(T) \cap f[S] = \emptyset$. Now $f[S] \in \eta$, because, for every $W \in \mathcal{U}, f^{-1}(\widetilde{W}(f[S])) \supseteq f^{-1}(f[S]) \supseteq S$. It follows that η is maximal subject to heaving the near finite intersection property.

We can thus define a mapping $\tilde{f} : \widetilde{G} \to \widetilde{G'}$ by stating that $\tilde{f}(\xi) = \eta$. It is immediate that $\tilde{f}$ is continuous, because, if $T \subseteq G', (\tilde{f})^{-1}(\mathcal{C}_T) = \bigcap_{V \in \mathcal{U}} \mathcal{C}_{f^{-1}(\widetilde{V}(T))}$.

Finally, let $x \in G$. It is obvious that $\{f(x)\} \in \tilde{f}(e_G(x))$ and hence that $\tilde{f}(e_G(x)) = e_{G'}(f(x))$. □

Lemma 3.6 *Let $\xi \in \widetilde{G}$ and let $Y \subseteq G$. Then $\xi \in \mathbf{cl}_{\widetilde{G}} e[Y]$ if and only if $Y \in \xi$.*

Proof. Clearly, $\mathrm{cl}_{\widetilde{G}} e[Y] = \bigcap\{\mathcal{C}_Z : \mathcal{C}_Z \supseteq e[Y]\}$. Now $y \in Y \Rightarrow Y \in e(y) \Rightarrow e(y) \in \mathcal{C}_Y \supseteq e[Y]$. On the other hand, suppose that $Z \in \mathcal{P}(G)$ satisfies $\mathcal{C}_Z \supseteq e[Y]$. Then $y \in Y \Rightarrow e(y) \in \mathcal{C}_Z \Rightarrow Z \in e(y) \Rightarrow y \in \mathrm{cl}_{\widetilde{G}} Z$. So $Y \subseteq \bar{Z}$ and hence $\mathcal{C}_Y \subseteq \mathcal{C}_{\bar{Z}} = \mathcal{C}_Z$ (by Lemma 2.5). Thus $\mathrm{cl}_{\widetilde{G}} e[Y] = \mathcal{C}_Y$. □

Corollary 3.7 *For any $Y_1, Y_2 \in \mathcal{P}(G)$, $cl_{\widetilde{G}}(Y_1) \cap cl_{\widetilde{G}}(Y_2) \neq \emptyset$ if and only if $(U \bullet Y_1) \cap (U \bullet Y_2) = \emptyset$ for every $U \in \mathcal{B}$.*

Proof. The condition that $(U \bullet Y_1) \cap (U \bullet Y_2) \neq \emptyset$ for every $U \in \mathcal{B}$ is equivalent to the condition that $\mathcal{C}_{Y_1} \cap \mathcal{C}_{Y_2} \neq \emptyset$. □

Remark 3.8 *We shall henceforward regard G as being a subspace $\widetilde{G}$ by identifying the point $x \in G$ with the point $e(x) \in \widetilde{G}$.*

The following Lemma is elementary and obviously well-known. We include it for the sake of completeness.

Lemma 3.9 *Let (f_n) be a sequence of right uniformly continuous real-valued functions defined on a topological group $(G, \bullet)$. If (f_n) converges uniformly on G to a function f, then f is right uniformly continuous.*

Proof. Let $\epsilon > 0$. We can choose $n \in \mathbf{N}$ so that $|f(x) - f_n(x)| < \frac{\epsilon}{3}$ for every $x \in G$. We can then choose $U \in \mathcal{B}$ so that $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ whenever $x \bullet y^{-1} \in U$. It follows that $|f(x) - f(y)| < \epsilon$ whenever $x \bullet y^{-1} \in U$. □

Theorem 3.10 *A bounded continuous function $f : G \to \mathbb{R}$ has a continuous extension $\tilde{f} : \widetilde{G} \to \mathbb{R}$ if and only if it is right uniformly continuous.*

Proof. Let $C(\widetilde{G})$ denote the set of all continuous real-valued functions defined on $\widetilde{G}$. We know from Theorem 3.5 that a bounded uniformy continuous function $f : G \to \mathbf{R}$ does have a continuous extension $\tilde{f} : \widetilde{G} \to \mathbf{R}$. The set of all functions $\tilde{f}$ which arise in this way will be a uniformly closed subalgebra of $C(G)$ by Lemma 3.9 and will contain the constant functions. By the Stone-Weierstrass Theorem, it will be the whole of $C(\widetilde{G})$ if it separates the points of $\widetilde{G}$.

To see that it does, let ξ_1, ξ_2 be distinct points of $\widetilde{G}$. By Lemma 2.4, we can choose $Y_1 \in \xi_1, Y_2 \in \xi_2$ and $U \in \mathcal{B}$ for which $(U \bullet Y_1) \cap Y_2 = \emptyset$. There will be a right uniformly continuous function $f : G \to [0,1]$ for which $Y[Y_1] = \{0\}$ and $f[Y_2] = \{1\}$ (Cf.[4]). Since $\xi_1 \in \mathrm{cl}_{\widetilde{G}} Y_1$ and $\xi_2 \in \mathrm{cl}_{\widetilde{G}} Y_2$ by Lemma 3.6, it follows that $\tilde{f}(\xi_1) = 0$ and $\tilde{f}(\xi_2) = 1$. Thus the functions of the form $\tilde{f}$ do separate the points of $\widetilde{G}$. □

Corallary 3.11 *$C(\widetilde{G})$ can be identified with the algebra of right uniformly continuous bounded real-valued functions defined on G.*

Theorem 3.12 *Suppose that the topological group $(G, \bullet)$ is not totally bounded. Then $\widetilde{G}$ contains a topological copy of $\beta \mathbb{N}$.*

Proof. We can choose a symmetric neighborhood $U \in \mathcal{B}$ of the identity e for which the covering $\{U \bullet x : x \in G\}$ of G has no finite subcovering. We can then choose a sequence $(x_n) \subseteq G$ with the property that, for each $n \in \mathbb{N}$, $x_n \notin \bigcup_1^{n-1}(U \bullet x_m)$.

We then choose $V \in \mathcal{B}$ to be a symmetric neighborhood of the identity e satisfying $V^2 \subseteq U$. This implies that the sets $(V \bullet x_n)$ will be pairwise disjoint.

Let D denote the discrete subspace $\{x_n : n \in \mathbb{N}\}$ of G. We shall show that $\mathrm{cl}_{\widetilde{G}} D \simeq \beta\mathbb{N}$.

The mapping $f : \mathbb{N} \to \widetilde{G}$, defined by stating that $f(n) = x_n$, has a continuous extension $f^\beta : \beta\mathbb{N} \to \widetilde{G}$. It will be sufficient to show that f^β is injective. Suppose then that μ_1 and μ_2 are distinct elements of $\beta\mathbb{N}$, and that G_1 and G_2 are disjoint open subsets of $\beta\mathbb{N}$ containing μ_1 and μ_2 respectively. Let $M_i = \mathbb{N} \cap G_i (i = 1,2)$. Since $(V \bullet f[M_1]) \cap (V \bullet f[M_2] = \emptyset$, $\mathrm{cl}_{\widetilde{G}}(f[M_1]) \cap \mathrm{cl}_{\widetilde{G}}(f[M_2]) = \emptyset$, by the Corallary to Lemma 3.6 Now $f^\beta(\mu_i) \in \mathrm{cl}_{\widetilde{G}}(f[M_i])$ for $i = 1,2$, and so $f^\beta(\mu_1) \neq f^\beta(\mu_2)$. □

Remark 3.13 *It follows from Theorem 3.12 that $\widetilde{G}$ has a least 2^C points if $(G, \bullet)$ is not totally bounded, because it is well known that $|\beta \mathbb{N}| = 2^C$ (cf. [3]).*

Definition 3.14 *Suppose that S is a subgroup of a topological group $(G, \bullet)$. Then S is also a topological group with the group structure induced by that of G. Hence, $\mathcal{B}_S = \{U \cap S : U \in \mathcal{B}\}$ is the neighborhood system of the identity in S.*

Theorem 3.15 *Suppose that S is a subgroup of a topological group $(G, \bullet)$ and that S has the induced neighborhood system $\mathcal{B}_S = \{U \cap S : U \in \mathcal{B}\}$. Then $\widetilde{S} \simeq \mathrm{cl}_{\widetilde{G}} S$.*

Proof. The inclusion map $i : S \to G$ is right uniformly continuous and therefore has a continuous extension $\tilde{i} : \widetilde{S} \to \widetilde{G}$ by Theorem 3.5. We shall show that $\tilde{i}$ is injective.

Suppose that μ_1, μ_2 are distinct points in $\widetilde{S}$. There will then be sets $Z_1, Z_2 \subseteq S$ and a neighborhood $U \in \mathcal{B}$ of the identity for which $(U_S \bullet Z_1) \cap Z_2 = \emptyset$, where U_S denotes $U \cap S$. Now $(U_S \bullet Z_1) \cap Z_2 = \emptyset$ implies that $(U \bullet Z_1) \cap Z_2 = \emptyset$ and hence that $\mathrm{cl}_{\widetilde{G}}(Z_1) \cap \mathrm{cl}_{\widetilde{G}}(Z_2) = \emptyset$, by the Corollary to Lemma 3.6. Since $\tilde{i}(\mu_i) \in \mathrm{cl}_{\widetilde{G}}(Z_i)$ for $i = 1, 2$, it follows that $\tilde{i}(\mu_1) \neq \tilde{i}(\mu_2)$. □

Remark 3.16 *For any assertion we made in this section for the space $\widetilde{G}$, there are similar assertions for the space $\widehat{G}$. Hence, $\widehat{G}$ is a topological compactificaiton of G.*

The Semigroup Compactification of a Topological Group

We shall now show that the group operation G can extended to a semigroup operation on $\widetilde{G}$ giving $\widetilde{G}$ the structure of a compact right topological semigroup.

Theorem 4.1 *The group operation on G extends to a semigroup operation on $\widetilde{G}$ in a such a way that $\widetilde{G}$ becomes a compact right topological semigroup.*

Proof. Clearly for each $s \in G$, the mapping $\lambda_s : G \to G$ is rightuniformly continuous and hence, it extends to a continuous mapping $\tilde{\lambda}_s$ from $\widetilde{G}$ into itself (by Theorem 3.5). If $\eta \in \widetilde{G}$, we shall denote $\tilde{\lambda}(\eta)$ by $s \bullet \eta$.

We shall show that for each $\eta \in \widetilde{G}$, the mapping $s \mapsto s \bullet \eta$ from G to $\widetilde{G}$ is uniformly continuous with respect to the right uniformity on G and the unique uniformity $\mathcal{U}$ on $\widetilde{G}$.

Let $\phi : \widetilde{G} \to \mathbf{R}$ be continuous. Then, by Theorem 3.10, $\phi_{|G}$ is right uniformly continuous. Thus, if $\epsilon > 0$, there will be a neighborhood $U \in \mathcal{B}$ of the identity e such that $|\phi(s) - \phi(s')| < \epsilon$ if $s \bullet (s')^{-1} \in U$. There will be a neighborhood $V \in \mathcal{B}$ of the identity e such that, whenever $s \bullet (s')^{-1} \in V, (s \bullet t) \bullet (s' \bullet t)^{-1} \in U$ for every $t \in G$. So, if $s \bullet (s')^{-1} \in V, |\phi(st) - \phi(s't)| < \epsilon$ for every $t \in G$. Now $|\phi(s\eta) - \phi(s'\eta) = \lim_{t \to \eta} |\phi(st) - \phi(s't)|$, and so $|\phi(s\eta) - \phi(s'\eta)| \leq \epsilon$ if $s \bullet (s')^{-1} \in V$. Using the fact that the unique uniform structure on $\widetilde{G}$ can be defined by the functions in $C(\widetilde{G})$, we have shown that the mapping $s \mapsto s \bullet \eta$ from G to $\widetilde{G}$ is right uniformly continuous.

It now follows from Theorem 3.5 that the mapping $s \mapsto s \bullet \eta$ can be extended to a continuous mapping from $\widetilde{G}$ to itself. The image of the element $\xi \in \widetilde{G}$ under this extension will be denoted by $\xi \bullet \eta$.

Thus we have defined a binary operation on $\widetilde{G}$ by a double limit process. If $\xi, \eta \in \widetilde{G}$,

$$\xi \bullet \eta = \lim_{s \to \xi} \lim_{t \to \eta} s \bullet t.$$

We observe that our definitions ensure that, for each $s \in G$, the mapping $\eta \mapsto s \bullet \eta$ is a continuous mapping from $\widetilde{G}$ to itself. Furthermore, for each $\eta \in \widetilde{G}$, the mapping $\xi \mapsto \xi \bullet \eta$ is also a continuous mapping from $\widetilde{G}$ to itself.

The associativity of the operation defined on $\widetilde{G}$ is immediate from the following equations: For every $\xi, \eta, \zeta \in \widetilde{G}$,

$$\xi \bullet (\eta \bullet \zeta) \lim_{s\to\xi} \lim_{t\to\eta} \lim_{k\to\zeta} s \bullet (t \bullet k);$$

$$(\xi \bullet \eta) \bullet \zeta = \lim_{s\to\xi} \lim_{t\to\eta} \lim_{k\to\zeta} (s \bullet t) \bullet k$$

□

Remark 4.2 *Suppose that S is a subgroup of G. We have seen in Theorem 3.15 that $\widetilde{S}$ can be regarded as topologically embedded in $\widetilde{G}$, if S is assumed to have the topological group structure induced by that of G. The embedding is also algebraic, because the inclusion map $i : S \to G$ has an extension $\widetilde{i} : \widetilde{S} \to \widetilde{G}$ which is readily seen to be a homomorphism. Thus $\widetilde{S}$ can be regarded as a subsemigroup of $\widetilde{G}$.*

Lemma 4.3 *Let $s \in G$ and $\xi \in \widetilde{G}$. Then, if $Y \in \xi, s \bullet Y \in s \bullet \xi$.*

Proof. This follows from Lemma 3.6, since the mapping $\lambda_s : \widetilde{G} \to \widetilde{G}$ is continuous. So, if $\xi \in \mathrm{cl}_{\widetilde{G}} Y, s\xi \in \mathrm{cl}_{\widetilde{G}} sY$. □

Lemma 4.4 *For each $s \in G$ and each $\xi \in \widetilde{G}, s \bullet \xi = \{s \bullet Y : Y \in \xi\}$.*

Proof. This is immediate from Lemma 4.3 □

Lemma 4.5 *Let $\xi \in \widetilde{G}$. For each $Y \in \xi$ and each $U \in \mathcal{B}, \mathcal{C}_{U\bullet Y}$ is a neighborhood of ξ in $\widetilde{G}$. Furthermore, the sets of this form provide a basis for the neighborhoods of ξ in $\widetilde{G}$.*

Proof. Since $\xi \in \widetilde{G}\mathcal{C}_{G/(U\bullet Y)} \subseteq \mathcal{C}_{U\bullet Y}, \mathcal{C}_{U\bullet Y}$ is a neighborhood of ξ.

On the other hand, suppose that $T \subseteq G$ and that $\xi \in \widetilde{G}/\mathcal{C}_T$. Then $T \notin \xi$ and so $T \cap (V \bullet Y) = \emptyset$ for some $Y \in \xi$ and some $V \in \mathcal{B}$ (by Lemma 1.4). Let $U \in \mathcal{B}$ be a symmetric neighborhood of the identity e satisfying $U^2 \subseteq V$. Then $\xi \in \mathcal{C}_{U\bullet Y}$ and $\mathcal{C}_{U\bullet Y} \subseteq \widetilde{G}/\mathcal{C}_T$ because $(U \bullet Y) \cap (U \bullet T) = \emptyset$. Thus the sets of the form $\mathcal{C}_{U\bullet Y}$ do provide a basis for the neighborhoods of ξ. □

Theorem 4.6 *Then the mapping $(s, \xi) \mapsto s \bullet \xi$ continuous mapping from $G \times \widetilde{G}$ to $\widetilde{G}$.*

Proof. Let $s \in G, \xi \in \widetilde{G}$ and $U \in \mathcal{B}$. Then $\mathcal{C}_{U\bullet Y}$ is a basic neighborhood of $s\xi$ for each $Y \in s \bullet \xi$ (by Lemma 2.5). Suppose that $V \in \mathcal{B}$ satisfies $V^2 \subseteq U$ and let $W \in \mathcal{B}$ such that $s \bullet Ws^{-1} \subset V$. We claim that, if $t \bullet s^{-1} \in V$ and $\eta \in \mathcal{C}_{Ws^{-1}\bullet Y}$, then $t \bullet \eta \in \mathcal{C}_{U\bullet Y}$.

Since $W \bullet s^{-1} \bullet Y \subset s^{-1} \bullet V \bullet Y$ and $W \bullet s^{-1} \bullet Y \in \eta, s^{-1} \bullet V \bullet Y \in \eta$ which implies that $t^{-1} \bullet (t \bullet s^{-1} \bullet V \bullet Y) \in \eta$. Hence, $t \bullet s^{-1} \bullet V \bullet Y \in t \bullet \eta$ and therefore, $V \bullet V \bullet Y \in t \bullet \eta$ since $t \bullet s^{-1} \in V$. Hence, $U \bullet Y \in t \bullet \eta$. Thus the mapping $(s, \xi) \mapsto s \bullet \xi$ continuous, as claimed. □

In the next theorem, we show that there is a sense in which $\widetilde{G}$ is the largest semigroup compactification of G in which the continuity condition of Theorem 4.6 is satisfied.

Theorem 4.7 *Let $(G, \bullet)$ be a topological group. Suppose that $(T, *)$ is a compact right topological semigroup and that $h : G \to T$ is a continuous homomorphism. Suppose also that the mapping $(s, \eta) \mapsto h(s) \bullet \eta$ is a continuous mapping from $G \times T$ to T. Then there is a continuous homomorphism $\tilde{h} : \widetilde{G} \to T$ for which $h = \tilde{h}_{|G}$.*

Proof. We shall first show that h right uniformly continuous. Let $\phi : T \to [0,1]$ be a continuous function and let $\epsilon > 0$. For each $\eta \in T$ there will be a neighborhood $N(\eta)$ of η in T, and a neighborhood $U(\eta)$ of the identity in G, for which $|\phi(h(s) * \xi) - \phi(\eta)| < \frac{\epsilon}{2}$ whenever $s \in U(\eta)$ and $\xi \in N(\eta)$. Now T will be covered by a finite number of neighborhoods of the form $N(\eta)$, corresponding to points $\eta_1, \eta_2, ..., \eta_n$ in T. Let $U = \bigcap_{i=1}^{n} U(\eta_i)$.

Suppose that $s_1, s_2 \in G$ satisfy $s_1 \bullet s_2^{-1} \in U$. If $h(s_2) \in N(\eta_i)$, then

$$|\phi(h(s_1 \bullet s_2^{-1}) * h(s_2)) - \phi(\eta_i)| < \frac{\epsilon}{2}$$

and

$$|\phi(h(s_2)) - \phi(\eta_i)| < \frac{\epsilon}{2}$$

and so

$$|\phi(h(s_1)) - \phi(h(s_2))| < \epsilon.$$

Thus h is right uniformly continuous.

It follows from Theorem 3.5 that there is a continuous function $\tilde{h} : \widetilde{G} \to T$ for which $h = \tilde{h}_{|G}$. That $\tilde{h}$ is a homomorphism, this can be seen as follows: For any $\xi_1, \xi_2 \in \widetilde{G}$,

$$\begin{aligned} \tilde{h}(\xi_1 \bullet \xi_2) &= \lim_{s \to \xi_1} \lim_{s \to \xi_2} h(s \bullet t) \\ &= \lim_{s \to \xi_1} \lim_{s \to \xi_2} h(s) * h(t) \\ &= \tilde{h}(\xi_1) * \tilde{h}(\xi_2). \end{aligned}$$

□

Corollary 4.8 *If G is a topological group $\widetilde{G}$ can be identified with the compactification G^{LUC}, since G^{LUC} is known to be the largest semigroup compactification of G in which the continuity condition of Theorem 4.6 is satisfied (cf. [1]).*

Theorem 4.9 *Let $\eta, \xi \in \widetilde{G}$. Then $Z \in \xi \bullet \eta$ if and only if*

$$X_W = \{x \in G : x^{-1} \bullet W \bullet Z \in \eta\} \in \xi$$

for every $W \in \mathcal{B}$.

Proof. Suppose that $Z \in \xi \bullet \eta$ and that $X_W \notin \xi$ for some $W \in \mathcal{B}$. Then there exists $Y \in \xi$ and $V \in \mathcal{B}$ for which $X_W \cap (V \bullet Y) = \emptyset$ and hence $X_W \cap Y = \emptyset$. If $y \in Y, y \notin X_W$ and so $y^{-1} \bullet W \bullet Z \notin \eta$. Therefore, $y^{-1} \bullet (W \bullet Z)^* \in \eta$, where $(W \bullet Z)^* = G \setminus (W \bullet Z)$. Hence, $(W \bullet Z)^* \in y \bullet \eta$ which implies that $y \bullet \eta \in \mathrm{cl}_{\widetilde{G}}(W \bullet Z)^*$. We can choose a net $(y_\alpha) \subseteq Y$ converging ξ. Thus, $(y_\alpha \bullet \eta)$ converges to $\xi \bullet \eta$ because for each $\eta \in \widetilde{G}$ the right translation ρ_η is continuous. Hence, $\xi \bullet \eta \in \mathrm{cl}_{\widetilde{G}}(W \bullet Z)^*$ which implies that $(W \bullet Z)^* \in \xi \bullet \eta$. Since $(W \bullet Z) \cap (W \bullet Z)^* = \emptyset -$ contradiction.

Conversely, suppose that for every $W \in \mathcal{B}$,

$$X_W = \{x \in G : x^{-1} \bullet W \bullet Z \in \eta\} \in \xi$$

Let $x \in X_W$ and let $y \in x^{-1} \bullet W \bullet Z$. Then $x \bullet y \in W \bullet Z$. We can choose a net (y_α) in $x^{-1} \bullet W \bullet Z$ converging to η because $\eta \in \mathrm{cl}_{\widetilde{G}}(x^{-1} \bullet W \bullet Z)$. Therefore, $x \bullet \eta \in \mathrm{cl}_{\widetilde{G}}(W \bullet Z)$. Now we can choose a net (x_α) in X_W converging to ξ since $X_W \in \xi$. Therefore, $\xi \bullet \eta \in \mathrm{cl}_{\widetilde{G}}(W \bullet Z)$ because the right translation ρ_η is continuous. Hence, $(W \bullet Z) \in \xi \bullet \eta$. □

Theorem 4.10 *If G is commutative, then $\widetilde{G}$ is commutative.*

Proof. Since G is commutative, $\widehat{G} = \widetilde{G}$ and $\lambda_s = \rho_s$ and therefore $\tilde{\lambda}_s = \tilde{\rho}_s$. From the fact $\lambda_\eta(s) = \rho_s(\eta) = \tilde{\lambda}_s(\eta) = \rho_\eta(s)$, we obtain $\tilde{\lambda}_\eta = \tilde{\rho}_\eta$ which implies that $\widetilde{G}$ is commutative. □

Theorem 4.11 *Let $\eta, \xi \in \widetilde{G}$ and let $x, y \in G$.*

a) $x \bullet \eta = x \bullet \xi$ implies that $\eta = \xi$.

b) $\eta \bullet x = \xi \bullet x$ implies $\eta = \xi$.

Proof. a) Since $x \bullet \eta = x \bullet \xi, x^{-1} \bullet x \bullet \eta = x^{-1} \bullet x \bullet \xi$ and therefore, $\eta = \xi$.

b) Since $\xi \bullet x = \eta \bullet x, \xi \bullet x \bullet x^{-1} = \eta \bullet x \bullet x^{-1}$ which implies that $\eta = \xi$. □

Remark 4.12 *It is clear that for each assertion we made for* $\widetilde{G}$ *in this section we can write similar assertion for* $\widehat{G}$. *Therefore,* $\widehat{G}$ *is a compact left topological compactification of* G.

References

[1] J. F. Berglund, H. D. Junghenn and P. Milnes, *Compact right topological semigroups and generalistion of almost periodicity,* Lecture Notes in Math. **663**, Springer-Verlag, (1978).

[2] N. Bourbaki, *Topologie générale, Herman, Paris* (1971).

[3] L. Gillman and M. Jorison, *Rings of Continuous Functions,* Van Nostrand (1960).

[4] H. J. Kowalsky, *Topological Spaces,* Academic Press (1964).

[5] M. Koçak, *Compactification of a Uniform Space and the LUC-compactification of the Real Numbers in Terms of the Concept of Near Ultrafilters, Ph. D theses,* University of Hull (1995).

[6] M. Koçak and D. Strauss, *Near ultrafilters and Compactifications,* to appear in Semigroup Forum.

[7] P. Samuel, *Ultrafilters and compactification of uniform spaces,* Trans. Am. Math. Soc., **64** (1948), 100-132.

Near Ultrafilterler ve Topolojik Grupların Kompaklaştırılması

Özet

Bu çalışmada yeni bir kavram olan near ultrafiltırlar tanımlanarak bir topolojik grubun LUC-kompaktlaştırılması elde edilmiştir.

Mahmut KOÇAK & Zekeriya ARVASİ
University of Osmangazi
Department of Mathematics,
Eskişehir-TURKEY

Received 10.10.1995