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A (–86)-Sphere in the K3 Surface

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Consider the self-intersection number $[\Sigma].[\Sigma]$ of a 2-dimensional sphere Σ embedded into a K3 surface. Since a K3 surface is spin, $[\Sigma].[\Sigma]$ is even and by Gauge theoretical arguments $[\Sigma].[\Sigma] \leq 0$. No other restriction on $[\Sigma].[\Sigma]$ is known. It is a problem 4.105(D) from the Kirby list [2] to determine the possible values of $[\Sigma].[\Sigma]$.

Theorem 1. *There exists a smooth embedding of S^2 into a K3 surface X with the normal Euler number equal to n for any negative even $n \geq -86$.*

To each collection of transverse spheres embedded into X with the normal Euler number (-2) we assign the *intersection graph* putting a vertex for each sphere and an edge for each point of (transverse) intersection of two spheres.

Proof. We deduce the theorem from Proposition 1 below. We equip the 22 spheres of the first intersection graph with such orientations that every point of their intersection is negative (this is possible since every tree is 2-colorable). The result of smoothing of the collection of spheres according to this orientation is a sphere (since the intersection graph is a tree) embedded to X with the normal Euler number $-2\times 22 - 2\times 21 = -86$ (21 is the number of the edges of the tree). For even $n \geq -82$ we use only some of the 22 spheres, we form a sub-tree, if $n = -(4k-2)$, and a 2-component subgraph, if $n = -(4k-4)$, with $k \leq 21$ vertices.

Construction of a (-84)-sphere in the K3 surface was suggested to us by S.Akbulut. We use the second intersection graph of Proposition 1. A connected sum of spheres a and c with orientation chosen properly is a (-4)-sphere not intersecting sphere b. This produces an intersection tree made out of 20 (-2)-spheres and a (-4)-sphere. A smoothing of this collection of spheres results in a (-84)-sphere. □

Remark. The same trick works to produce a (-84)-sphere out of Figure 59 in [1] which contains a handlebody picture of a manifold with boundary embedded in a K3 surface.

Proposition 1. *Both graphs of Figure 1 appear as intersection graphs of (-2)-spheres transversally embedded in a K3 surface.*

Proof. The K3 surface X is the total space of the double covering $X \to \mathbb{C}P^2$ branched along a smooth curve $\mathbb{C}A \subset \mathbb{C}P^2$ of degree 6. We construct the 22 spheres in X as the inverse images of 22 disk membranes in $(\mathbb{C}P^2, \mathbb{C}A)$ (a disk membrane $P \subset \mathbb{C}P^2$ is a surface diffeomorphic to D^2 such that $P \cap \mathbb{C}A = \partial P$ and the tangent planes to P and $\mathbb{C}A$ never

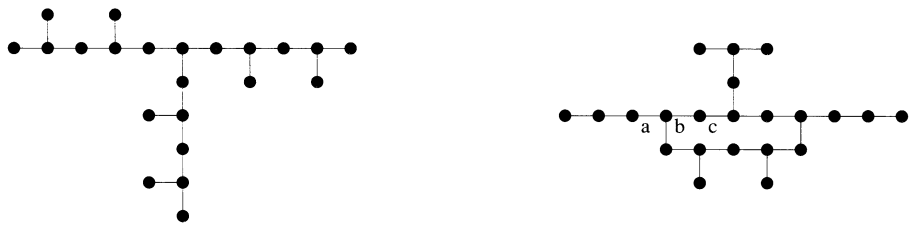

FIGURE 1. Intersection graphs.

coincide). Let $\mathbb{C}C$ be the union of six complex lines in $\mathbb{C}P^2$ defined over $\mathbb{R}$ so that their real parts intersect as it is shown on the left half of Figure 2.

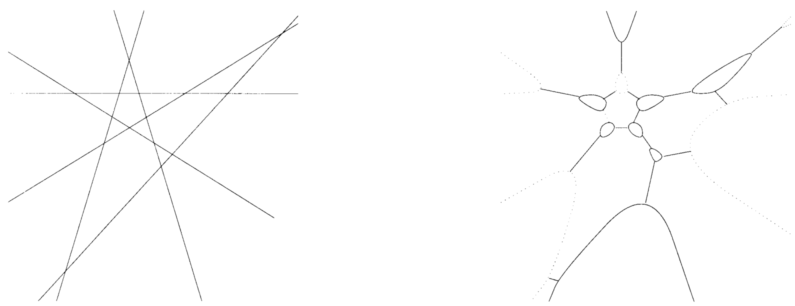

FIGURE 2. Curve $\mathbb{R}C$, curve $\mathbb{R}A$ and imaginary membranes.

Let f be the homogeneous polynomial of degree 6 defining $\mathbb{C}C$, that is a product of six linear forms. Alternating the sign of f if necessary we can make the real zero set, $\mathbb{R}A$, of $f+\epsilon(x^6+y^6+z^6)$, with a small $\epsilon > 0$, look like the right hand part of Figure 2 (x, y and z are homogeneous coordinates in $\mathbb{C}P^2$). Denote the complex zero set of $f+\epsilon(x^6+y^6+z^6)$ by $\mathbb{C}A$.

To construct the first intersection graph we take seven membranes bounded in $\mathbb{R}P^2$ by the seven solid ovals on Figure 2. Note that the components of $\mathbb{R}A$ which look like hyperbolas are also ovals in $\mathbb{R}P^2$ and that we do not use the three dotted ovals to keep the intersection graph a tree after adding the other membranes.

The other 15 membranes come from smoothing of double points of $\mathbb{C}C$. To describe them we choose local coordinates (x, y) near each double point of $\mathbb{C}C$ so that $\mathbb{C}A$ is given by equation $x^2 - y^2 = \epsilon$, $\epsilon > 0$. The disk $\{ (x, iy) : x^2 + y^2 \leq \epsilon,\ x, y \in \mathbb{R} \}$ is the membrane for this double point. This membrane intersects $\mathbb{R}P^2$ along the arc $\{ (x, 0)$, $x^2 \leq \epsilon$, $x \in \mathbb{R}\}$. The 15 arcs of intersection are pictured on the right half of Figure 2).

To construct the second intersection graph we use 8 ovals (we use all but the 2 dotted "hyperbola"-ovals) but do not use the imaginary membrane intersecting $\mathbb{R}P^2$ along the dotted arc.

To check that -2 is the normal Euler number of the spheres in X obtained as the inverse images of the membranes we note that the multiplication by i produces an anti-isomorphism between the normal and the tangent vector bundles to the spheres so that the normal Euler number is equal to $-\chi(S^2)$. □

Remark. If we use all 10 ovals of the curve of Figure 2 and take all 15 imaginary membranes then the resulting intersection graph has 25 vertices. This graph appears as the intersection graph of holomorphic spheres in one of "the two most algebraic K3 surfaces" (see [4]). The intersection graph of the 24 holomorphic spheres of the other most algebraic K3 surface also has a tree with 22 vertices (pictured on Figure 3) as a subgraph. It also gives a (-86)-sphere in a K3 surface. We would like to thank V.Kharlamov for bringing [4] to our attention.

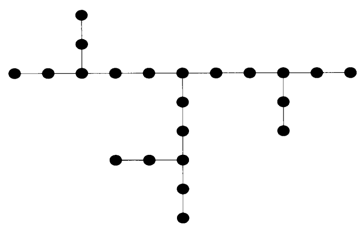

FIGURE 3. The other intersection graph.

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